Intuitionistic Fuzzy Approach to Three Player Prisoner's Dilemma Game

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Abstract: ***In this paper an Intuitionistic fuzzy approach is proposed to solve the three player Prisoner's Dilemma game, this approach will use the inequality condition between Triangular Intuitionistic fuzzy numbers to obtain the optimal strategy in the game.***

Keywords: Intuitionistic fuzzy sets, Nash Equilibrium, Prisoner's Dilemma, Bi-Matrix games

1. Introduction

A game is a decision making situation which involves one or more players, where each of them have goals to achieve their outcome. The player may be an individual or an organisation. The concept of game theory was introduced in The theory of games and Economic behavior by Von Neumann and Morgenstern [5]. The players of the game may be cooperative or non-cooperative in a given situation. J.Nash accelerated the concept of non-cooperative games and the equilibrium points [3] [4]. The general methods to find the optimal solution of a game are min-max or using the linear programming method [12]. In real life situation there are other considerations which may affect the process of choosing the optimal strategies such as philosophical motives, moral aesthetics, environmental conditions and lack of information. So to evaluate the outcome of the game in the presence of imprecision or uncertainty we use the concept of fuzzy numbers introduced by L. A. Zadeh [11] [10]. Atanassov [7] proposed the concept of Intuitionistic fuzzy seta which uses the membership and non-membership functions [6]. Using the Triangular Intuitionistic Fuzzy Numbers to solve Bi-matrix games involving two players was introduced [8] [9] in which the inequality conditions between TIFNs are used to find the optimal solution and equilibrium points in the game. Prisoner's Dilemma is a traditional game model in game theory which generally involves two players [12]. The triangular fuzzy membership is used to solve the prisoner's dilemma game in fuzzy game theory [1]. The iterative form of prisoner's dilemma is a game involves more than two players which is called Three player Prisoner's Dilemma (3p-PD) game [2]. In this paper we introduce the concept of solving 3p-PD game using the triangular Intuitionistic fuzzy Numbers and finding the Nash Equilibrium solution of the game.

The paper is organised as follows: In section 2 the inequality relation between two TIFNs is given. In section 3 the concept of prisoner's dilemma game in bi-matrix form is introduced. In section 4the computational procedure is explained using a Numerical example of 3p-PD game with Intuitionistic fuzzy payoffs.

2. Intuitionistic Fuzzy Sets

Definition 2.1

Let $X = \{x_1, x_2, \ldots\ldots, x_n\}$ be a finite universal set. An intuitionistic fuzzy set $\tilde{T}$ in a given universal set X is of the form

$$\tilde{T} = \{\langle x_i, \mu_{\tilde{T}}(x_i), \nu_{\tilde{T}}(x_i)\rangle : x_i \in X\},$$

Where the functions $\mu_{\tilde{T}}(x_i), \nu_{\tilde{T}}(x_i)$ are the degree of membership and degree of non-membership of an element $x_i \in X$, and they satisfy the condition
$0 \le \mu_{\tilde{T}}(x_i) + \nu_{\tilde{T}}(x_i) \le 1, \forall\, x_i \in X$.

Definition 2.2:

A triangular intuitionistic fuzzy number (TIFN) denoted by $\tilde{t} = \langle t, l, r;\ w_t, u_t\rangle$ is a intuitionistic fuzzy set on a real number set R, which has the membership and the non-membership functions of the form

$$\mu_{\tilde{t}}(x) = \begin{cases} \dfrac{x-t+l}{l} w_t; & t-l \le x < t \\ \dfrac{t+r-x}{r} w_t; & t \le x \le t+r \\ 0; & otherwise \end{cases}$$

$$\text{and } \nu_{\tilde{t}}(x) = \begin{cases} \frac{(t-x)+u_t\,(x-t+l)}{l}; & t-l \le x < t \\ \frac{(x-t)+u_t\,(t+r-x)}{r}; & t \le x \le t+r \\ 1; & otherwise \end{cases}$$

where t is the mean value and l, r are called spreads. The maximum degree of membership and the minimum degree of non-membership are denoted by w_t and u_t and they satisfy the condition $0 \le w_t, u_t \le 1$ and $0 \le w_t + u_t \le 1$.

Definition 2.3:

A (α, β)- cut set of a TIFN $\tilde{t} = \langle t, l_t, r_t; w_t, u_t\rangle$is a subset of R, which is defined as
$\tilde{t}_{\alpha,\beta} = \{x: \mu_{\tilde{t}}(x) \ge \alpha, \nu_{\tilde{t}}\,(x) \le \beta\}$, where $0 \le \alpha \le w_t, u_t \le \beta \le 1$ and $0 \le \alpha + \beta \le 1$. A α –cut set of a TIFN $\tilde{t}$ is a subset of R, which is defined as $\tilde{t}_\alpha = \{x: \mu_{\tilde{t}}\,(x) \ge \alpha\}$; where $0 \le \alpha \le w_t$ and $\tilde{t}_\alpha$ is defined by the closed interval $[t_L(\alpha), t_R\,(\alpha)]$ where

$$t_L(\alpha) = (t - l_t) + \frac{l_t \alpha}{w_t}, \qquad t_R(\alpha) = (t + r_t) - \frac{r_t \alpha}{w_t}$$

And the β-cut set of a TIFN $\tilde{t}_\beta = \{x: v_{\tilde{t}}(x) \leq \beta\}$; $\tilde{t}_\beta$ is a closed interval of the form

$\tilde{t}_\beta = [t_L(\beta), t_R(\beta)]$ where $t_L(\beta) = (t - l_t) + \frac{(1-\beta)l_t}{1-u_t}$; and $t_R(\beta) = (t + r_t) - \frac{(1-\beta)r_t}{1-u_t}$.

And $\tilde{t}_{\alpha,\beta} = \tilde{t}_\alpha \wedge \tilde{t}_\beta$ the minimum between $\tilde{t}_\alpha$ and $\tilde{t}_\beta$ is denoted by the symbol "$\wedge$".

2.1 Inequality relation between two TIFNs

Let $\tilde{c} = \langle c, l_c, r_c; w_c, u_c\rangle$ and $\tilde{d} = \langle d, l_d, r_d; w_d, u_d\rangle$ be two TIFNs, $\tilde{c}_\alpha, \tilde{d}_\alpha$ and $\tilde{c}_\beta, \tilde{d}_\beta$ be their α −cuts and β −cuts. we define average ranking index of the membership function and average ranking index of the non-membership functions $S_\mu(\tilde{c}, \tilde{d})$, $S_\vartheta(\tilde{c}, \tilde{d})$ of the form

$$S_\mu(\tilde{c}, \tilde{d}) = \begin{cases} \frac{1}{4}\left[4(d-c) + (r_d - l_d)\left(2 - \frac{w_c}{w_d}\right) - (r_c - l_c)\right] w_c; \ \text{if} \min\{w_c, w_d\} = w_c \\ \frac{1}{4}\left[4(d-c) + (r_d - l_d) - (r_c - l_c)\left(2 - \frac{w_d}{w_c}\right)\right] w_d; \text{if} \min\{w_c, w_d\} = w_d \end{cases}$$

And

$$S_\vartheta(\tilde{c}, \tilde{d}) = \begin{cases} \frac{1}{4}\left[4(d-c) + (r_d - l_d)\left(2 - \frac{1-u_c}{1-u_d}\right) - (r_c - l_c)\right](1-u_c); \text{if} \max\{u_c, u_d\} = u_c \\ \frac{1}{4}\left[4(d-c) + (r_d - l_d) - (r_c - l_c)\left(2 - \frac{1-u_c}{1-u_d}\right)\right](1-u_d); \text{if} \max\{u_c, u_d\} = u_d \end{cases}$$

On the basis of above definition we propose the following inequality relations

1) If $S_\mu(\tilde{c}, \tilde{d}) > 0$ then $\tilde{c}$ is smaller than $\tilde{d}$ and is denoted by $\tilde{c} \prec \tilde{d}$
2) If $S_\mu(\tilde{c}, \tilde{d}) = 0$ then
   a) If $S_\vartheta(\tilde{c}, \tilde{d}) = 0$ then $\tilde{c}$ is equal to $\tilde{d}$, denoted by $\tilde{c} \asymp \tilde{d}$;
   b) If $S_\vartheta(\tilde{c}, \tilde{d}) > 0$ then $\tilde{c}$ is smaller than $\tilde{d}$ and is denoted by $\tilde{c} \prec \tilde{d}$.

The symbol "$\prec$" has the linguistic interpretation as essentially less than.

3. Three- player prisoner's dilemma game (3p-PD)

The prisoner's dilemma game where two players imprisoned for a crime and held in a solitary confinement with no option of interacting with the other. In the absence of enough evidence to convict them the police put them both in imprisonment for a period of time or they offer each prisoner a deal to cooperate with the police. The prisoner who agrees to cooperate with the police will get lesser imprisonment than the one who doesn't. Each player has two strategies in the given situation. Either they choose to cooperate (C) or not to cooperate. The non-cooperation is considered as defect (D).

The traditional form of prisoner's dilemma game possibly extended for involving three players or even more. In this extension form of PD each player has two pure strategies C and D and each round in the game has one of the eight possible outcomes CCC, CCD, CDC, CDD, DCC, DCD, DDC, DDD. In the outcome the first position represents the player under consideration, the second and third position is the opponents. For example CCD represents the strategy combination of player I and player II cooperate and player III defect. This position is generally written as XCD where X could be either C or D.

The possible payoffs are specified by R, K, S, T, L, P which can be numbered by $i = 1, 2, 3, 4, 5, 6$. The payoff matrix for (3P-PD) is of the form

$$\begin{array}{c c} & \begin{array}{ccc} CC & CD & DD \end{array} \\ \begin{array}{c} C \\ D \end{array} & \begin{pmatrix} R & K & S \\ T & L & P \end{pmatrix} \end{array}$$

Where T>R>L>K>P>S.

3.1 Bi-Matrix Games

A bi-matrix game is described as two players I and II have two pure strategy sets available to them let M $= \{1, 2, 3, \dots\dots, m\}$ and N$= \{1, 2, 3, \dots., n\}$ and pay-offs denoted by α_{ij} and β_{ij} which the players I and II receive when they play the pure strategies i and j. The payoff matrix is of the form

$$A = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \cdots & \cdots \alpha_{1n} \\ \alpha_{21} & \alpha_{22} & \cdots & \cdots \alpha_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ \alpha_{m1} & \alpha_{m2} & \cdots & \cdots \alpha_{mn} \end{pmatrix}; B = \begin{pmatrix} \beta_{11} & \beta_{12} & \cdots & \cdots \beta_{1n} \\ \beta_{21} & \beta_{22} & \cdots & \cdots \beta_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ \beta_{m1} & \beta_{m2} & \cdots & \cdots \beta_{mn} \end{pmatrix}$$

The game is denoted as $\hat{G} = \langle\{I, II\}, A.B\rangle$.

3.2 Nash Equilibrium solutions

Two players I and II have pure strategies M$= \{1, 2, \dots\dots m\}$ and N$= \{1, 2, \dots. n\}$. The pair of strategies (row r, column s) is said to have a Nash Equilibrium solution if the bi-matrix game holds the following inequalities

$$\alpha_{is} \leq \alpha_{rs}\ ; \ \beta_{rj} \leq \beta_{rs}$$

For all $i = 1, 2, \dots. m$ and $j = 1, 2, \dots. n$.

For a finite strategy set game $\hat{G} = \langle\{I, II\}, A.B\rangle$ the NES may exist. The resulting pair $(\alpha_{rs}, \beta_{rs})$ is called the Nash Equilibrium outcome of the bi-matrix game. A game can have more than one NES and the equilibrium may have different outcome in each case.

3.3 Triangular Intuitionistic fuzzy Inequality solution to three player Prisoner's Dilemma (3p- PD)

In the three player prisoner's dilemma game the players are named as X, Y and Z. at first the perspective of player I is considered whether he choose to cooperate or defect. It gives two payoff matrix in bi matrix form which are considered as TIFNs. The optimal strategy for player I is determined using the inequality condition defined in section 2. This method is used when the player cannot manipulate the outcome of the game due to the unknowingness of the other players decision. The degree of membership and the degree of non membership values are assigned to each payoff.

The payoff matrix for player X when he chooses the strategy C is denoted as

$$A = \begin{pmatrix} \langle a_{11}, l_{a_{11}}, r_{a_{11}}; w_{a_{11}}, u_{a_{11}} \rangle & \langle a_{12}, l_{a_{12}}, r_{a_{12}}; w_{a_{12}}, u_{a_{12}} \rangle \\ \langle a_{21}, l_{a_{21}}, r_{a_{21}}; w_{a_{21}}, u_{a_{21}} \rangle & \langle a_{22}, l_{a_{22}}, r_{a_{22}}; w_{a_{22}}, u_{a_{22}} \rangle \end{pmatrix}$$

If the player X chooses strategy D the payoff matrix will be

$$B = \begin{pmatrix} \langle b_{11}, l_{b_{11}}, r_{b_{11}}; w_{b_{11}}, u_{b_{11}} \rangle & \langle b_{12}, l_{b_{12}}, r_{b_{12}}; w_{b_{12}}, u_{b_{12}} \rangle \\ \langle b_{21}, l_{b_{21}}, r_{b_{21}}; w_{b_{21}}, u_{b_{21}} \rangle & \langle b_{22}, l_{b_{22}}, r_{b_{22}}; w_{b_{22}}, u_{b_{22}} \rangle \end{pmatrix}$$

Where $w_{a_{ij}}, u_{a_{ij}}$ and $w_{b_{ij}}, u_{b_{ij}}$ where $i, j = 1, 2$ are the confidence and hesitation degrees in the payoff matrices which has values between 0 to 1 and has sum less than or equal to 1.

4. Numerical example

The payoff matrices for player X are A and B given as follows

$$A = \begin{pmatrix} \langle 8, 8, 8; 0.8, 0.1 \rangle & \langle 6, 6, 12; 0.6, 0.3 \rangle \\ \langle 6, 12, 6; 0.4, 0.6 \rangle & \langle 0, 12, 12; 0.6, 0.2 \rangle \end{pmatrix}$$
$$B = \begin{pmatrix} \langle 12, 6, 6; 0.4, 0.6 \rangle & \langle 12, 0, 12; 0.7, 0.1 \rangle \\ \langle 12, 12, 0; 0.5, 0.2 \rangle & \langle 4, 4, 4; 0.5, 0.3 \rangle \end{pmatrix}$$

The payoff matrix A is for X decides to cooperate and B is for X decides to defect. Here the triangular intuitionistic fuzzy inequality conditions are applied in the prospect of player X.

In the payoff matrix A the payoff $\langle 8, 8, 8; 0.8, 0.1 \rangle$denotes player X, Y and Z decides to cooperate they get the sentence for 8 months. The strategy combination for this payoff is $\langle C, C, C \rangle$. The player X decides to cooperate with assumption that the others cooperate as well. The confidence degree is 0.8 and hesitance degree is 0.1 for the payoff. The players aim is to minimize their sentence by choosing the optimal strategy. Similarly the other payoff values are having confidence and hesitation degrees.

Average ranking index of the membership function is $S_\mu(A, B)$ and the average ranking index of the non-membership function is $S_\vartheta\ (A, B)$.

In the given payoff matrix the values are taken as

$$a_{11} = 8, l_{a_{11}} = 8, r_{a_{11}} = 8;\ w_{a_{11}} = 0.8, u_{a_{11}} = 0.1$$
$$a_{12} = 6, l_{a_{12}} = 6, r_{a_{12}} = 12;\ w_{a_{12}} = 0.6, u_{a_{12}} = 0.3$$
$$a_{21} = 6, l_{a_{21}} = 12, r_{a_{21}} = 6;\ w_{a_{21}} = 0.4, u_{a_{21}} = 0.6$$
$$a_{22} = 0, l_{a_{22}} = 12, r_{a_{22}} = 12;\ w_{a_{22}} = 0.6, u_{a_{22}} = 0.2$$

And $b_{11} = 12, l_{b_{11}} = 6, r_{b_{11}} = 6;\ w_{b_{11}} = 0.4, u_{b_{11}} = 0.6$

$$b_{12} = 12, l_{b_{12}} = 0, r_{b_{12}} = 12;\ w_{b_{12}} = 0.7, u_{b_{12}} = 0.1$$
$$b_{21} = 12, l_{b_{21}} = 12, r_{b_{21}} = 0;\ w_{b_{21}} = 0.5, u_{b_{21}} = 0.2$$
$$b_{22} = 4, l_{b_{22}} = 4, r_{b_{22}} = 4;\ w_{b_{22}} = 0.5, u_{b_{22}} = 0.3$$

From () checking the inequality condition between the TIFNs (a_{11}, b_{11})

$$min\{w_{a_{11}}, w_{b_{11}}\} = min\{0.8, 0.4\} = 0.4 = w_{b_{11}}$$

The average ranking index of the membership function between (a_{11}, b_{11}) is

$$S_{\mu_{11}}(a_{11}, b_{11}) = \frac{1}{4}\left[4(b_{11} - a_{11}) + (r_{b_{11}} - l_{b_{11}}) - (r_{a_{11}} - l_{a_{11}})\left(2 - \frac{w_{b_{11}}}{w_{a_{11}}}\right)\right] w_{b_{11}}$$
$$= \frac{1}{4}\left[4(12 - 8) + (6 - 6) - (8 - 8)\left(2 - \frac{0.4}{0.8}\right)\right] 0.4$$
$$= \frac{1}{4}[4\ (4)]0.4$$
$$= 1.6 > 0$$

$S_\mu(a_{11}, b_{11}) > 0$ which implies $a_{11} < b_{11}$.

For (a_{11}, b_{12}),

$$min(w_{a_{11}}, w_{b_{12}}) = \min(0.8, 0.7) = 0.7 = w_{b_{12}}$$
$$S_\mu(a_{11}, b_{12}) = \frac{1}{4}\left[4(b_{12} - a_{11}) + (r_{b_{12}} - l_{b_{12}}) - (r_{a_{11}} - l_{a_{11}})\left(2 - \frac{w_{b_{12}}}{w_{a_{11}}}\right)\right] w_{b_{12}}$$
$$= \frac{1}{4}\left[4(12 - 8) + (12 - 0) - (8 - 8)\left(2 - \frac{0.7}{0.8}\right)\right] 0.7\ = 4.9 > 0$$

Which implies $a_{11} < b_{12}$.

For (a_{11}, b_{21}),

$$min(w_{a_{11}}, w_{b_{21}}) = \min(0.8, 0.5) = 0.5 = w_{b_{21}}$$
$$S_\mu(a_{11}, b_{21}) = \frac{1}{4}\left[4(b_{21} - a_{11}) + (r_{b_{21}} - l_{b_{21}}) - (r_{a_{11}} - l_{a_{11}})\left(2 - \frac{w_{b_{21}}}{w_{a_{11}}}\right)\right] w_{b_{21}}$$
$$= \frac{1}{4}\left[4(12 - 8) + (0 - 12) - (8 - 8)\left(2 - \frac{0.5}{0.8}\right)\right] 0.5\ = 0.5 > 0$$

Which implies $a_{11} < b_{21}$.

For (a_{11}, b_{22}),

$$min(w_{a_{11}}, w_{b_{22}}) = \min(0.8, 0.5) = 0.5 = w_{b_{22}}$$
$$S_\mu(a_{11}, b_{22}) = \frac{1}{4}\left[4(b_{22} - a_{11}) + (r_{b_{22}} - l_{b_{22}}) - (r_{a_{11}} - l_{a_{11}})\left(2 - \frac{w_{b_{22}}}{w_{a_{11}}}\right)\right] w_{b_{22}}$$
$$= \frac{1}{4}\left[4(4 - 8) + (4 - 4) - (8 - 8)\left(2 - \frac{0.5}{0.8}\right)\right] 0.5$$
$$= -2 < 0$$

Which implies $a_{11} > b_{22}$.

Similarly the inequality condition is checked for each payoff values with others for both the payoff matrices A and B. After checking the inequality relations between the TIFNs $\{a_{11}, a_{12}, a_{21}, a_{22},\ b_{11}, b_{12}, b_{21}, b_{22}\}$ the NES solutions occurs at $\langle 0, 12, 12; 0.6, 0.2\rangle$ and $\langle 4, 4, 4; 0.5, 0.3\rangle$. The player X's main goal is to minimize his sentence so by comparison between the degrees of hesitancy $\langle 0, 12, 12; 0.6, 0.2\rangle$ will be the NE solution and $\langle C, D, D\rangle$ will be the best strategy combination for the player. Hence player X choose to cooperate when the other players choose to defect. In the similar way the 3p-PD can be solved in the perspective of players Y and Z as well. And the player's best course of action can be found using the inequality conditions of TIFNs.

5. Conclusion

In this paper we have developed the concept of solving three person prisoner's dilemma game using the triangular intuitionistic fuzzy numbers and it's inequality relations. We also presented a new approach in finding the optimal strategies for players with payoffs of TIFNs by using the Nash- equilibrium strategies for bi-matrix games. The method is illustrated with the numerical example of 3p-Pd game with IF payoffs. This approach is suitable for the decision maker who is unaware about the choices of the opponents and deals with the vagueness and uncertainty of the game problem. This model is suitable for the game models which are assumed to be linear in this paper.

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