Fixed Point Theorem in Controlled Metric - Like Spaces

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Abstract: In this paper, we establish fixed point theorem on rational type contractions in the setting of controlled metric-like spaces, Our result are the extension of several well-known results of literature.

Keywords: fixed point, controlled metric space, controlled metric-like space, extended b-metric space

1. Introduction and Preliminaries

The notion of a b-metric spaces was studied by Bakhtin [1], Czerwik [2] and many fixed point results were obtained for single and multivalued mappings by Czerwik and many other authors. (See [3] - [8]) The generalizations of b-metric spaces Kamran et al. [9] and others (see [10]- [11]) was introduced extended b-metric spaces. Mlaiki et al. [11] introduced controlled metric spaces were obtained many fixed point results and many authors. (see [12]- [15]) Which is generalized form of extended b-metric spaces. Again Mlaiki [11] introduced new type of metric spaces we generalize many results in the literature. New type of metric spaces is metric-like spaces. Das and Gupta [16] established first fixed point theorem for rational contractive type conditions in metric spaces further many authors established fixed point theorems in rational contractive type conditions. (see [17]). Recently Pandey et al. [18] established fixed point theorem on rational type contractions in controlled metric spaces. In this paper, we establish fixed point theorem on rational type contractions in the setting of controlled metric-like spaces. We also provide example to illustrate significance of the established result. Our result are the extension of several well-known results of literature.

Definition 1: [1] Let $M \neq \emptyset$ and $s \geq 1$. A function $Y: M \times M \rightarrow [0, \infty)$ is called b-metric if for all $r, s, t \in M$,

1) $Y(r, s) = 0$ if $r = s$
2) $Y(r, s) = Y(s, r)$
3) $Y(r, t) \leq s [Y(r, s) + Y(s, t)]$

The pair $(M, Y)$ is called a b-metric space.

Definition 2: [9] Let $M \neq \emptyset$ and $\tau: M \times M \rightarrow [1, \infty)$ be a function. A function $Y: M \times M \rightarrow [0, \infty)$ is called an extended b-metric if for all $r, s, t \in M$,

1) $Y(r, s) = 0$ if $r = s$
2) $Y(r, s) = Y(s, r)$
3) $Y(r, t) \leq \tau(r, t) [Y(r, s) + Y(s, t)]$

The pair $(M, Y)$ is called an extended b-metric space.

Definition 3 [11] Let $M \neq \emptyset$ and $\tau: M \times M \rightarrow [1, \infty)$ be a function. A function $Y: M \times M \rightarrow [0, \infty)$ is called controlled metric if for all $r, s, t \in M$,

1) $Y(r, s) = 0$ if $r = s$
2) $Y(r, s) = Y(s, r)$
3) $Y(r, t) \leq \tau(r, t) Y(r, s) + \tau(s, t) Y(s, t)$

The pair $(M, Y)$ is called controlled metric space.

Definition 4 [11] Let $M \neq \emptyset$ and $\tau: M \times M \rightarrow [1, \infty)$ be a function. A function $Y: M \times M \rightarrow [0, \infty)$ is called controlled metric-like space if for all $r, s, t \in M$,

1) $Y(r, s) = 0$ implies $r = s$
2) $Y(r, s) = Y(s, r)$
3) $Y(r, t) \leq \tau(r, s) Y(r, s) + \tau(s, t) Y(s, t)$

The pair $(M, Y)$ is called controlled metric like space.

Example 1 [11] Let $M = \{0, 1, 2\}$. Define the function $Y: M \times M \rightarrow [0, \infty)$ by

$Y(0, 0) = Y(1, 1) = 0, Y(2, 2) = \frac{1}{10}, Y(0, 1) = Y(1, 0) = 1, Y(0, 2) = Y(2, 0) = \frac{1}{2}, Y(1, 2) = Y(2, 1) = \frac{2}{5}\$.

Take $\tau: M \times M \rightarrow [1, \infty)$ by

$\tau(o, o) = \tau(1, 1) = \tau(2, 2) = \tau(0, 2) = 1, \tau(1, 2) = \frac{5}{4}, \tau(0, 1) = \frac{11}{10}$

Hence $Y$ is controlled metric-like on $M$ and $(M, Y)$ is controlled metric-like space.

We have $Y(2, 2) = \frac{1}{10} \neq 0$. Which imply $(M, Y)$ is not a controlled metric type space.

The Cauchy and convergent sequence in controlled metric-like space are defined in this way

Definition 5: [11] Let $(M, Y)$ be a controlled metric-like space and $(r_n)$ be a sequence in $M$. then

1) The sequence $(r_n)$ converges to some $r$ in $M$: if for every $\varepsilon > 0$, their exists $N \in N(\varepsilon) \in N$ such that $Y(r_n, r) < \varepsilon$ for all $n \geq N$. In this case, we write $\lim_{n \to \infty} r_n = r$.  

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2) The sequence \( \{r_n\} \) is Cauchy: if for every \( \varepsilon > 0 \), there exists \( N = N(\varepsilon) \in \mathbb{N} \) such that \( \|Y(r_{m+n}) - Y(r_m)\| < \varepsilon \) for all \( m, n \geq N \). In this case, we write \( \lim_{n \to \infty} (r_n, r_0) = 0 \).

3) The controlled metric - like space \((M, \mathcal{Y})\) is called complete if every Cauchy sequence is convergent.

**Definition 6** [11] Let \((M, \mathcal{Y})\) be a controlled metric - like space. Let \( r \in M \) and \( \varepsilon > 0 \).

1) The open ball \( B_r(\varepsilon, r) \) is
   \[ B_r(\varepsilon, r) = \{ s \in M : \mathcal{Y}(r, s) < \varepsilon \} \]
2) The mapping \( E : M \to M \) is said to be continuous at \( r \in M \); if for all \( \varepsilon > 0 \), there exists \( \delta > 0 \) such that \( E(B_r(\delta, r)) \subseteq B_r(\varepsilon, r) \).

**2. Main Result**

In this part of our main result, we establish fixed point theorem on controlled metric -like spaces. We also provide example to illustrate significance of the established result.

**Theorem 2.1** Let \((M, \mathcal{Y})\) be a complete controlled metric-like space. Let \( \varepsilon : M \to M \) be such that there are \( a_i, \varepsilon_i \) (0,1), for all \( i = 1, 2, 3, 4, 5 \) with \( k = (a_1 + a_2)(1 - a_3, a_4, a_5) < 1 \).

\[
\mathcal{Y}(\varepsilon(r), s) \leq a_1 \mathcal{Y}(r, s) + a_2 \mathcal{Y}(r, \varepsilon r) + a_3 \mathcal{Y}(r, s) + a_4 \mathcal{Y}(r, \varepsilon(s)) + a_5 \mathcal{Y}(r, s) + 1 + \mathcal{Y}(r, s)
\]

(2.1)

For all \( r, s \in M \). For \( \varepsilon \in M \), take \( r = \varepsilon r_0 \) assume that

For \( \varepsilon \geq 0 \), \( \lim_{n \to \infty} \mathcal{Y}(r_{i+1}, r_{i+2}) < 1 \).

Suppose that,

\[
\lim_{n \to \infty} r_{i+1}, r_{i+2} \text{ exist, and } (a_3 + a_4) \lim_{n \to \infty} \mathcal{Y}(r_{i+1}, r_{i+2}) < 1 \text{ for every } r \in M, \text{ then } M \text{ possesses a unique fixed point.}
\]

**Proof:** Let \( r_0 \in M \) be initial point. Consider sequence \( \{r_n\} \) verifies \( r_{n+1} = \varepsilon r_n \) for all \( n \in \mathbb{N} \). Obviously, if \( r_0 \) exists for \( M = M \), then \( \lim_{n \to \infty} \mathcal{Y}(r_{n+1}, r_n) = 0 \), and the proof is finished. Thus, we suppose that \( r_{n+1} = r_n \) for all \( n \in \mathbb{N} \).

Thus, by (2.1), we have

\[
\mathcal{Y}(r_{n+1}, r_{n+2}) = \mathcal{Y}(\varepsilon r_n, \varepsilon r_{n+1}) \leq a_1 \mathcal{Y}(r_n, r_{n+1}) + a_2 \mathcal{Y}(r_n, \varepsilon r_n) + a_3 \mathcal{Y}(r_n, \varepsilon r_{n+1}) + a_4 \mathcal{Y}(r_n, \varepsilon r_{n+2}) + a_5 \mathcal{Y}(r_n, r_{n+2}) + \mathcal{Y}(r_n, \varepsilon r_{n+2}) \leq \mathcal{Y}(r_n, r_{n+2}) \leq (a_1 + a_2) \mathcal{Y}(r_{n+1}, r_n) + (a_3 + a_4 + a_5) \mathcal{Y}(r_{n+1}, r_n)
\]

(2.3)

Thus we have

\[
\mathcal{Y}(r_{n+1}, r_{n+2}) \leq k \mathcal{Y}(r_n, r_{n+1}) \leq k^2 \mathcal{Y}(r_{n-1}, r_{n+1}) \leq \cdots \leq k^n \mathcal{Y}(r_0, r_1)
\]

(2.4)

For all \( n, m \in \mathbb{N} \) and \( n < m \), we have

\[
\mathcal{Y}(r_{m+n}) \leq \mathcal{Y}(r_{m+n}, r_{m+n+1}) + \mathcal{Y}(r_{m+n+1}, r_{m+n+2}) + \cdots + \mathcal{Y}(r_{m+n}, r_{m+1}) \leq (a_1 + a_2) \mathcal{Y}(r_{m+n+1}, r_{m+n+2}) + \cdots + \mathcal{Y}(r_{m+n}, r_{m+1}) \leq k^n \mathcal{Y}(r_0, r_1)
\]

(2.5)

This implies that,

\[
\mathcal{Y}(r_n, r_m) \leq (a_1 + a_2) \mathcal{Y}(r_{m+1}, r_n) + \cdots + \mathcal{Y}(r_{m+n+1}, r_m) + \mathcal{Y}(r_{m+n}, r_{m+1}) \leq k^{m-n} \mathcal{Y}(r_0, r_1) \leq k \mathcal{Y}(r_0, r_1)
\]

(2.6)

Let

\[
u_i = \sum_{j=0}^{n-i} \mathcal{Y}(r_j, r_{j+1})
\]

Thus \( \lim_{n \to \infty} \mathcal{Y}(r_{n+1}, r_n) = 0 \).

Consider

\[
\nu_i = \sum_{j=0}^{n-i} \mathcal{Y}(r_j, r_{j+1}) \leq k \mathcal{Y}(r_0, r_1) (\mathcal{Y}(r_0, r_1) + (u_{m-1} - u_0))
\]

(2.7)

If \( \mathcal{Y}(r_0, r_1) \geq 1 \), Letting \( m, n \to \infty \) in 2.9, we have

\[
\lim_{n \to \infty} \mathcal{Y}(r_{n+1}, r_n) = 0.
\]

(2.10)

Thus, the sequence \( \{r_n\} \) is Cauchy in the complete controlled metric-like space \((M, \mathcal{Y})\). So, there is some \( r_* \in M \) such that

\[
\lim_{n \to \infty} \mathcal{Y}(r_n, r_*) = 0.
\]

(2.11)

Nee, we prove that \( r_* \) is a fixed point of \( M \). By 3.1 and condition (3), we get

\[
\mathcal{Y}(r_*, r_*) \leq \mathcal{Y}(r_*, r_{n+1}) + \mathcal{Y}(r_{n+1}, r_*) \leq \mathcal{Y}(r_*, r_{n+1}) + \mathcal{Y}(r_{n+1}, r_*) \]

(2.12)

Thus we have

\[
\mathcal{Y}(r_{n+1}, r_{n+2}) \leq k \mathcal{Y}(r_n, r_{n+1}) \leq k^2 \mathcal{Y}(r_{n-1}, r_{n+1}) \leq \cdots \leq k^n \mathcal{Y}(r_0, r_1)
\]
We have \( \tau(o,o) = \tau(1,1) = \tau(2,2) = \tau(0,2) = 1, \tau(1,2) = \frac{1}{10} \). Let \( \Upsilon : M \times M \rightarrow [0, \infty) \) be defined by

\[
\Upsilon(r, s) = \frac{1}{10} + a_3 \frac{\Upsilon(r^*, s^*) + a_4 \Upsilon(r, s) + a_5 \Upsilon(r, r^*) + a_6 \Upsilon(r, r^*) + a_7 \Upsilon(r, r^*)}{1 + \Upsilon(r, s)}.
\]

(2.12)

Taking limit \( n \to \infty \) and using 2.10, 2.11 and \( \lim_{n \to \infty} \tau(r_n, r) \) and \( \lim_{n \to \infty} \tau(r_n, r_o) \) exist, finite, hence

\[
\Upsilon(r^*, r^*) \leq \left[ (a_3 + a_5) \lim_{n \to \infty} \tau(r_n, r^*) \Upsilon(r^*, r) \right] \Upsilon(r^*, r^*)
\]

(2.13)

Suppose \( r^* \neq r^* \), having \( (a_3 + a_5) \lim_{n \to \infty} \tau(r_n, r^*) \leq 1 \), so

\[
0 < \Upsilon(r, r^*) \leq \left[ (a_3 + a_5) \lim_{n \to \infty} \tau(r_n, r^*) \right] \Upsilon(r, r^*) \leq \Upsilon(r, r^*)
\]

(2.14)

Contradiction then \( r^* = r^* \).

Now, prove the uniqueness of \( r^* \). Let \( s^* \) be another fixed point of \( f \) in \( M \) then \( f s^* = s^* \).

Now, by 2.1, we have

\[
\Upsilon(r^*, s^*) = \Upsilon(r^*, r^*) \leq a_1 \Upsilon(r, s^*) + a_2 \Upsilon(r^*, r^*) + a_3 \Upsilon(s^*, s^*) + a_4 \frac{\Upsilon(r^*, r^*) + \Upsilon(s^*, s^*)}{1 + \Upsilon(r, s^*)} + a_5 \frac{\Upsilon(r^*, r^*) + \Upsilon(s^*, s^*)}{1 + \Upsilon(r^*, s^*)}
\]

\[
\leq a_1 \Upsilon(r^*, s^*) + a_2 \Upsilon(r^*, r^*) + a_3 \Upsilon(s^*, s^*) + a_4 \frac{\Upsilon(r^*, r^*) \Upsilon(s^*, s^*)}{1 + \Upsilon(r, s^*)} + a_5 \frac{\Upsilon(s^*, s^*)}{1 + \Upsilon(r^*, s^*)}
\]

Contradiction. Hence \( \Upsilon(r^*, s^*) = 0 \) implies \( r^* = s^* \).

**Example 3.1:** Let \( M = \{0, 1, 2\} \). Define the function \( Y : M \times M \rightarrow [0, \infty) \) by

\[
Y(0,0) = Y(1,1) = 0, Y(2,2) = \frac{1}{10}, \quad Y(0,1) = Y(1,0) = 1/2, Y(0,2) = Y(2,0) = \frac{1}{2}, \quad Y(1,2) = Y(2,1) = 1/11
\]

Take \( \tau : M \times M \to [0, \infty) \) by

\[
\tau(o,o) = \tau(1,1) = \tau(2,2) = \tau(0,2) = 1, \quad \tau(2,0) = \frac{1}{10}, \quad \tau(2,2) = \frac{11}{10}
\]

Hence \( Y \) is controlled metric-like on \( M \) and \( (M, Y) \) is controlled metric-like space.

We have \( Y(2,2) = \frac{1}{10} \neq 0 \). Which imply \( (M, Y) \) is not a controlled metric type space.

Given \( f : M \to M \) as \( f(0) = 2, f(1) = 2, f(2) = 1 \).

Let \( a_1 = 1/11, a_2 = a_3 = a_4 = a_5 = 2/11 \). Then

\[
K = (a_1 + a_2)(1 - a_3 a_4 a_5) = \frac{1/11 + 2/11}{1 - 3(2/11)} = 3/5 < 1,
\]

And

\[
\lim_{n \to \infty} \tau(r_{n+1}, r) \tau(r, r_{n+1}) = 1 < 1/k.
\]

Clearly 2.2 satisfied and all the condition of Theorem 2.1 are satisfied, and so \( \ell \) has a unique fixed point, which is \( r^* = 1 \).

**References**


