Fixed Point Theorem in Controlled Metric - Like Spaces

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Abstract: ***In this paper, we establish fixed point theorem on rational type contractions in the setting of controlled metric- like spaces, Our result are the extension of several well- known results of literature.***

Keywords: fixed point, controlled metricspace, controlled metric- like space, extended b- metric space

1. Introduction and Preliminaries

The notion of a b-metric spaces was studied by Bakhtin [1], Czerwik [2] and many fixed point results were obtained for single and multivalued mappings by Czerwik and many other authors. (See [3] - [8]) The generalizations of b- metric spaces Kamran et al. [9] and others (see [10]- [11]) was introduce extended b- metric spaces. Mlaiki et al. [11] introduced controlled metric spaces were obtained many fixed point results and many authors. (see [12]- [15]) Which is generalized form of extended b- metric spaces. Again Mlaiki [11] introduced new type of metric spaces we generalize many results in the literature. New type of metric spaces is metric -like spaces. Das and Gupta [16] established first fixed point theorem for rational contractive type conditions in metric spaces further many authors established fixed point theorems in rational contractive type conditions. (see [17]). Recently Pandey et al. [18] established fixed point theorem on rational type contractions in controlled metric spaces. In this paper, we establish fixed point theorem on rational type contractions in the setting of controlled metric- like spaces. We also provide example to illustrate significance of the established result. Our result are the extension of several well- known results of literature.

Definition 1: [1] Let $M \neq \phi$ and $s \geq 1$. A function Υ: $M \times M \to [0, \infty)$ is called b- metric if for all r, s, t ε M,

1) $\Upsilon(r, s) = 0$ if $r = s$
2) $\Upsilon(r, s) = \Upsilon(s, r)$
3) $\Upsilon(r, t) \leq s\,[\Upsilon(r, s) + \Upsilon(s, t)]$

The pair (M, Υ) is called a b-metric space.

Definition 2: [9] Let $M \neq \phi$ and τ: $M \times M \to [1, \infty)$ be a function. A function Υ: $M \times M \to [0, \infty)$ is called an extended b- metric if for all r, s, t ε M,

1) $\Upsilon(r, s) = 0$ iff $r = s$
2) $\Upsilon(r, s) = \Upsilon(s, r)$
3) $\Upsilon(r, t) \leq \tau(r, t)\,[\Upsilon(r, s) + \Upsilon(s, t)]$

The pair (M, Υ) is called an extended b-metric space.

Definition 3 [11] Let $M \neq \phi$ and $\tau : M \times M \to [1, \infty)$ be a function. A function Υ: $M \times M \to [0, \infty)$ is called controlled metric if for all r, s, t ε M,

1) $\Upsilon(r, s) = 0$ if $r = s$
2) $\Upsilon(r, s) = \Upsilon(s, r)$
3) $\Upsilon(r, t) \leq \tau(r, s)\,\Upsilon(r, s) + \tau(s, t)\,\Upsilon(s, t)$

The pair (M, Υ) is called controlled metric space.

Definition 4 [11] Let $M \neq \phi$ and τ: $M \times M \to [1, \infty)$ be a function. A function Υ: $M \times M \to [0, \infty)$ is called controlled metric- like space if for all r, s, t ε M,

1) $\Upsilon(r, s) = 0$ implies $r = s$
2) $\Upsilon(r, s) = \Upsilon(s, r)$
3) $\Upsilon(r, t) \leq \tau(r, s)\,\Upsilon(r, s) + \tau(s, t)\,\Upsilon(s, t)$

The pair (M, Υ) is called controlled metric like space.

Example 1 [11] Let M = {0, 1, 2}. Define the function Υ: $M \times M \to [0, \infty)$ by

$\Upsilon(0, 0) = \Upsilon(1, 1) = 0$, $\Upsilon(2, 2) = \frac{1}{10}$, $\Upsilon(0,1) = \Upsilon(1, 0) = 1$,
$\Upsilon(0, 2) = \Upsilon(2, 0) = \frac{1}{2}$, $\Upsilon(1, 2) = \Upsilon(2, 1) = \frac{2}{5}$.

Take τ: $M \times M \to [1, \infty)$ by

$\tau(o, o) = \tau(1, 1) = \tau(2, 2) = \tau(0, 2) = 1$, $\tau(1, 2) = \frac{5}{4}$, $\tau(0, 1) = \frac{11}{10}$.

Hence Υ is controlled metric-like on M and (M, Υ) is controlled metric - like space.

We have $\Upsilon(2, 2) = \frac{1}{10} \neq 0$. Which imply (M, Υ) is not a controlled metric type space.

The Cauchy and convergent sequence in controlled metric-like space are defined in this way

Definition 5: [11] Let (M, Υ) be a controlled metric- like space and $\{r_n\}$ be a sequence in M. then

1) The sequence $\{r_n\}$ converges to some r in M: if for every $\mathcal{E} > 0$, their exists $N = N(\mathcal{E})\ \varepsilon\ \mathbf{N}$ such that $\Upsilon(r_n, r) < \mathcal{E}$ for all $n \geq N$. in this case, we write $\lim_{n\to\infty} r_n = r$.

2) The sequence $\{r_n\}$ is Cauchy; if for every $\mathcal{E} > 0$, there exists $N = N(\mathcal{E}) \,\varepsilon\, \mathbf{N}$ such that $\Upsilon(r_m, r_n) < \mathcal{E}$ for all $m, n \geq N$. in this case, we write $\lim_{m,n\to\infty} (r_m, r_n) = 0$.
3) The controlled metric - like space (M, Υ) is called complete if every Cauchy sequence is convergent.

Definition 6 [11] Let (M, Υ) be a controlled metric- like space. Let $r \,\varepsilon\, M$ and $\mathcal{E} > 0$.

1) The open ball $B(r, \mathcal{E})$ is
   $B(r, \mathcal{E}) = \{s \,\varepsilon\, M : \Upsilon(r, s) < \mathcal{E}\}$
2) The mapping $\pounds: M \to M$ is said to be continuous at $r \,\varepsilon\, M$; if for all $\mathcal{E} > 0$, there exists $\delta > 0$ such that $\pounds(B(r, \delta)) \subset B(\pounds r, \mathcal{E})$.

2. Main Result

In this part of our main result, we establish fixed point theorem on controlled metric -like spaces.We also provide example to illustrate significance of the established result.

Theorem 2.1 Let (M, Υ) be a complete controlled metric-like space. Let $\pounds : M \to M$ be so that there are $a_i \,\varepsilon\, (0,1)$, for all $i = 1,2,3,4,5$ with $k = (a_1 + a_2)/(1 - a_3 - a_4 - a_5) < 1$,

$$\Upsilon(\pounds r, \pounds s) \leq a_1 \Upsilon(r, s) + a_2 \Upsilon(r, \pounds r) + a_3 \Upsilon(s, \pounds s) + a_4 \frac{\Upsilon(r,\pounds r)\Upsilon(s,\pounds s)}{1+\Upsilon(r,s)} + a_5 \frac{\Upsilon(r,\pounds r)[1+\Upsilon(s,\pounds s)]}{1+\Upsilon(r,s)} \tag{2.1}$$

For all $r, s \,\varepsilon\, M$. For $r_0 \,\varepsilon\, M$, take $r_n = \pounds^n r_0$, assume that

$$\sup_{m\geq 1} \lim_{i\to\infty} \tau(r_{i+1}, r_{i+2})\, \tau(r_{i+1}, r_m) / \tau(r_i, r_{i+1}) < 1/k. \tag{2.2}$$

Suppose that,
$\lim_{n\to\infty} \tau(r_n, r)$ and $\lim_{n\to\infty} \tau(r, r_n)$ exist are finite, and $(a_3 + a_5) \lim_{n\to\infty} \tau(r, r_n) < 1$ for every $r \,\varepsilon\, M$, then M posses a unique fixed point.

Proof: Let $r_0 \,\varepsilon\, M$ be initial point. Consider sequence (r_n) verifies $r_{n+1} = \pounds r_n$ for all $n \,\varepsilon\, \mathbf{N}$. Obviously, if there exists $n_0 \,\varepsilon\, \mathbf{N}$ for which $r_{no+1} = r_{no}$, then $\pounds r_{no} = r_{no}$, , and the proof is finished. Thus , we suppose that $r_{n+1} \neq r_n$ for every $n \,\varepsilon\, \mathbf{N}$. Thus, by (2.1), we have

$$\Upsilon(r_n, r_{n+1}) = \Upsilon(\pounds r_{n-1}, \pounds r_n) \leq a_1 \Upsilon(r_{n-1}, r_n) + a_2 \Upsilon(r_{n-1}, \pounds r_{n-1}) + a_3 \Upsilon(r_n, \pounds r_n) + a_4 \Upsilon(r_{n-1}, \pounds r_{n-1}) \Upsilon(r_n, \pounds r_n)/1 + \Upsilon(r_{n-1}, r_n) + a_5 \Upsilon(r_n, \pounds r_n)[1 + \Upsilon(r_{n-1}, \pounds r_{n-1})]/1 + \Upsilon(r_{n-1}, r_n)$$
$$= a_1 \Upsilon(r_{n-1}, r_n) + a_2 \Upsilon(r_{n-1}, r_n) + a_3 \Upsilon(r_n, r_{n+1}) + a_4 \Upsilon(r_{n-1}, r_n)\Upsilon(r_n, r_{n+1})/1 + \Upsilon(r_{n-1}, r_n) + a_5 \Upsilon(r_n, r_{n+1})[1 + \Upsilon(r_{n-1}, r_n)]/1 + \Upsilon(r_{n-1}, r_n)$$

$$\leq (a_1 + a_2)\Upsilon(r_{n-1}, r_n) + (a_3 + a_4 + a_5)\Upsilon(r_n, r_{n+1})$$

$$\Upsilon(r_n, r_{n+1}) \leq (a_1 + a_2)\Upsilon(r_{n-1}, r_n) / 1 - (a_3 + a_4 + a_5). \tag{2.3}$$

Thus we have

$$\Upsilon(r_n, r_{n+1}) \leq k\, \Upsilon(r_{n-1}, r_n) \leq k^2)\Upsilon(r_{n-2}, r_{n-1}) \leq \ldots \leq k^n \Upsilon(r_0, r_1) \tag{2.4}$$

For all $n, m \,\varepsilon\, N$ and $n < m$, we have

$$\Upsilon(r_n, r_m) \leq \tau(r_n, r_{n+1})\Upsilon(r_n, r_{n+1}) + \tau(r_{n+1}, r_m)\Upsilon(r_{n+1}, r_m)$$

$$\leq \tau(r_n, r_{n+1})\Upsilon(r_n, r_{n+1}) + \tau(r_{n+1}, r_m)\tau(r_{n+1}, r_{n+2})\Upsilon(r_{n+1}, r_{n+2}) + \tau(r_{n+1}, r_m)\, \tau(r_{n+2}, r_m)\, \Upsilon(r_{n+2}, r_m)$$

$$\leq \tau(r_n, r_{n+1})\Upsilon(r_n, r_{n+1}) + \tau(r_{n+1}, r_m)\tau(r_{n+1}, r_{n+2})\Upsilon(r_{n+1}, r_{n+2}) + \tau(r_{n+1}, r_m)\, \tau(r_{n+2}, r_m)\, \tau(r_{n+2}, r_{n+3})\, \Upsilon(r_{n+2}, r_{n+3}) + \tau(r_{n+1}, r_m)\, \tau(r_{n+2}, r_m)\, \tau(r_{n+2}, r_m)\, \Upsilon(r_{n+3}, r_m)$$

$$\leq \tau(r_n, r_{n+1})\Upsilon(r_n, r_{n+1}) + \sum_{i=n+1}^{m-2}\Big(\prod_{j=n+1}^{i} \tau(r_j, r_m)\Big) \tau(r_i, r_{i+1})\, \Upsilon(r_i, r_{i+1}) + \prod_{j=n+1}^{m-1} \tau(r_j, r_m) \Upsilon(r_{m-1}, r_m). \tag{2.5}$$

This implies that,

$$\Upsilon(r_n, r_m) \leq \tau(r_n, r_{n+1})\, \Upsilon(r_n, r_{n+1}) + \sum_{i=n+1}^{m-2}\Big(\prod_{j=n+1}^{i} \tau(r_j, r_m)\Big)\tau(r_i, r_{i+1})\, \Upsilon(r_i, r_{i+1}) + \prod_{j=n+1}^{m-1} \tau(r_j, r_m) \Upsilon(r_{m-1}, r_m).$$

$$\leq \tau(r_n, r_{n+1})\, k^n \Upsilon(r_0, r_1) + \sum_{i=n+1}^{m-2}\Big(\prod_{j=n+1}^{i} \tau(r_j, r_m)\Big)\tau(r_i, r_{i+1})\, k^i\, \Upsilon(r_0, r_1) + \prod_{j=n+1}^{m-1} \tau(r_j, r_m) k^{m-1} \Upsilon(r_0, r_1).$$

$$\leq \tau(r_n, r_{n+1})\, k^n \Upsilon(r_0, r_1) + \sum_{i=n+1}^{m-1}\Big(\prod_{j=n+1}^{i} \tau(r_j, r_m)\Big)\tau(r_i, r_{i+1})\, k^i\, \Upsilon(r_0, r_1) \tag{2.6}$$

Let

$$u_r = \sum_{i=0}^{r}\Big(\prod_{j=0}^{i} \tau(r_j, r_m)\Big)\tau(r_i, r_{i+1})\, k^i\, \Upsilon(r_0, r_1) \tag{2.7}$$

Consider

$$V_i = \prod_{j=0}^{i} \tau(r_j, r_m) \tau(r_i, r_{i+1})\, k^i \Upsilon(r_0, r_1). \tag{2.8}$$

In view of condition 2.2 and the ratio test the series $\sum_i v_i$ converges. Thus $\lim_{n\to\infty} u_n$ exists. Hence the sequence $\{u_n\}$ is Cauchy. Using 2.6, we get

$$\Upsilon(r_n, r_m) \leq \Upsilon(r_0, r_1)\, [\, k^n\, \Upsilon(r_n, r_{n+1}) + (u_{m-1} - u_n)] \tag{2.9}$$

If $\tau(r,s) \geq 1$. Letting $m,n \to \infty$ in 2.9,we have

$$\lim_{m,n\to\infty} \Upsilon(r_n, r_m) = 0. \tag{2.10}$$

Thus , the sequence $\{r_n\}$ is Cauchy in the complete controlled metric- like space(M, Υ). So. There is some $r^* \,\varepsilon\, M$. So,

$$\lim_{n\to\infty} \Upsilon(r_n, r^*) = 0. \tag{2.11}$$

Noe , we prove that r^* is a fixed point of M. By 3.1 and condition (3), we get

$$\Upsilon(r^*, \pounds r^*) \leq \tau(r^*, r_{n+1})\Upsilon(r^*, r_{n+1}) + \tau(r_{n+1}, \pounds r^*)\Upsilon(r_{n+1}, \pounds r^*)$$

$$= \tau(r^*, r_{n+1})\Upsilon(r^*, r_{n+1}) + \tau(r_{n+1}, \pounds r^*)\Upsilon(\pounds r_n, \pounds r^*)$$

$$\leq \tau(r^*, r_{n+1})\Upsilon(r^*, r_{n+1}) + \tau(r_{n+1}, \pounds r^*)\, [\, a_1 \Upsilon(r_n, r^*) + a_2 \Upsilon(r_n, \pounds r_n) + a_3 \Upsilon(r^*, \pounds r^*)$$

$$+ a_4 \frac{\Upsilon(r_n, \pounds r_n)\Upsilon(r^*, \pounds r^*)}{1+\Upsilon(r_n, r^*)} + a_5 \frac{\Upsilon(r^*, \pounds r^*)[1+\Upsilon(r_n, \pounds r_n)]}{1+\Upsilon(r^*, r_n)}]$$

$$= \tau(r^*, r_{n+1})\Upsilon(r^*, r_{n+1}) + \tau(r_{n+1}, \pounds r^*)\, [\, a_1 \Upsilon(r_n, r^*) + a_2 \Upsilon(r_n, r_{n+1}) + a_3 \Upsilon(r^*, \pounds r^*)$$

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$+a_4 \frac{\Upsilon(r_n,r_{n+1})\Upsilon(r^*,£r^*)}{1+\Upsilon(r_n,r^*)} + a_5 \frac{\Upsilon(r^*,£r^*)[1+\Upsilon(r_n,r_{n+1})]}{1+\Upsilon(r^*,r_n)}]$ (2.12)

Taking , limit $n\to\infty$ and using 2,.10, 2.11 and $\lim_{n\to\infty} \tau(r_n,r)$ and $\lim_{n\to\infty} \tau(r,r_n)$ exist, are finite, hence

$\Upsilon(r^* £r^*) \le [(a_3 + a_5) \lim_{n\to\infty} \tau(r_{n+1},£r^*)]\Upsilon(r^*, £r^*)$ (2.13)

Suppose

$r^* \neq £r^*$, having $(a_3 + a_5) \lim_{n\to\infty} \tau(r_{n+1},£r^*) < 1$, so

$0 < \Upsilon(r^* £r^*) \le [(a_3 + a_5) \lim_{n\to\infty} \tau(r_{n+1},£r^*)]\Upsilon(r^*,£r^*) < \Upsilon(r^* £r^*)$

(2.14)

Contradiction then $r^* = £r^*$.

Now, prove the uniqueness of r^*. Let s^* be another fixed point of £ in M then $£s^* = s^*$.

Now, by 2.1, we have

$\Upsilon(r^*,s^*) = \Upsilon(£r^*, £s^*)$
$\le a_1 \Upsilon(r^*,s^*) + a_2 \Upsilon(r^*,£r^*) + a_3 \Upsilon(s^*,£s^*) + a_4 \frac{\Upsilon(r^*,£r^*)\Upsilon(s^*,£s^*)}{1+\Upsilon(r^*,s^*)}$
$+a_5 \frac{\Upsilon(s^*,£s^*)[1+\Upsilon(r^*,£r^*)]}{1+\Upsilon(r^*,s^*)}$

$= a_1 \Upsilon(r^*,s^*) + a_2 \Upsilon(r^*,r^*) + a_3 \Upsilon(s^*,s^*) + a_4 \frac{\Upsilon(r^*,r^*)\Upsilon(s^*,s^*)}{1+\Upsilon(r^*,s^*)} + a_5 \frac{\Upsilon(s^*,s^*)[1+\Upsilon(r^*,r^*)]}{1+\Upsilon(r^*,s^*)}$

$\le a_1 \Upsilon(r^*,s^*)$.

Contradiction. Hence $\Upsilon(r^*,s^*) = 0$ implies $r^* = s^*$.

Example 3.1: Let M = { 0,1,2}. Define the function $\Upsilon : M \times M \to [0, \infty)$ by
$\Upsilon(0,0) = \Upsilon(1,1) = 0$, $\Upsilon(2,2) = \frac{1}{10}$, $\Upsilon(0,1) = \Upsilon(1,0) = 1/2$, $\Upsilon(0,2) = \Upsilon(2,0) = \frac{1}{2}$, $\Upsilon(1,2) = \Upsilon(2,1) = 1/11$

Take $\tau : M \times M \to [1, \infty)$ by
$\tau(o,o) = \tau(1,1) = \tau(2,2) = \tau(0,2) = 1$, $\tau(1,2) = \frac{5}{4}$, $\tau(0,1) = \frac{11}{10}$.

Hence Υ is controlled metric-like on M and (M, Υ) is controlled metric - like space.
We have $\Upsilon(2,2) = \frac{1}{10} \neq 0$. Which imply (M, Υ) is not a controlled metric type space.

Given $£ : M \to M$ as $£0 = 2$, $£1 = £2 = 1$.
Let $a_1 = 1/11$, $a_2 = a_3 = a_4 = a_5 = 2/11$. Then

$K = (a_1 + a_2)/(1- a_3 - a_4 - a_5) = \frac{1/11 + 2/11}{1-3(2/11)} = 3/5 < 1$,
And

$\text{Sup}_{m\ge1} \lim_{i\to\infty} \tau(r_{i+1},r_{i+2})\, \tau(r_{i+1},r_m)/ \tau(r_i,r_{i+1}) = 1 < 1/k$.

Clearly 2.2 satisfied and all the condition of Theorem 2.1 are satisfied, and so £ has a unique fixed point , which is $r^* = 1$.

References

[1] Bakhtin I.A. "The contrction mapping principle in almost metric spaces". Funct. Anal., Vol. 30, pp. 26-37, 1989.[Google Scholar]

[2] Czerwik S. "Contraction mapping in b-metric spaces". Acta Math. Inform. Univ. , Vol.1, pp. 5-11, 1993.[Google Scholar]

[3] Abdeljawad T.,Abodayeh K. and Mlaiki N, "On fixed point generalizations to partial b- metric spaces", J. Comput. Anal. Appl., Vol. 19, pp. 883-891, 2015. [Google Sholar]

[4] Afshari H., Atapour M. and Aydi H., " Generalized α-Φ Geraghty multivalued mapping on b- metric spaces endowed with a graph". TWNS.J.Appl. Eng. Math., Vol.7, pp. 248-260,2017.[Google Scholar]

[5] Alharbi N., Aydi H., Felhi A., Ozel C. and Sahmim S. "α- contraction mappings n rectangular b- metric space and application to integral equations." J. Math. Anal., Vol. 9,pp. 47-60, 2018. [Google Scholar]

[6] S. S. P. Singh "On some fixed point results in complete b_2-metric spaces". Int. J. of Aquatic Sci. Vol.12 (3), pp. 2584-2596, 2021.

[7] S. S. P. Singh "Common fixed point theorem in b_2-metric-like spaces. Int.J. Resent Sci. Research. Vol.12(8), pp. 1-6, 2021.

[8] Thirunavukarasu P. and Uma M. "Fixed point theorems in b- metric spaces." Adv. Appl. Math. Sci.Vol. 21(8), pp. 4467-4474, 2022.

[9] Kamran T. Samreen M.and lazvic V. "A generaliztion of b- metric space and some fixed point theorems."Mathematics Vol.5, pp. 1-17, 2017.

[10] Huang H., Den G. and radevovic S. "Fixed point theorems in b- metric spaces with applications to differential equations." J. Fixed point Theory Appl., 2018.

[11] Mlaiki N., Aydi H., Souayah N. and Abdeljawad T. "Controlled metric type and the related contractions principle." Mathematics Vol.6, no. 10 p. 194, 2018.

[12] Ahmad J., Mazrooei A., Aydi H. and Sen M. "on fixed point results in controlled metric spaces. J. Func. Space, Vol. 20, article ID 2108167.

[13] Hussain S. "Fixed point theorems for nonlinear contraction in controlled metric type space." Appl. Math. E- Notes, Vol. 21, pp. 53-61, 2021.

[14] S. S. P. Singh "Fixed point theorems on controlled metric spaces." IJAMSS , Vol.10(2),pp. 25-32, 2021.

[15] S. S. P. Singh "on fixed point result in double controlled metric spaces." IJRASET, vOL. 9(VII),pp. 1256-1261, 2021.

[16] Dass B. K, and Gupta S. "An extension of Banach contraction principle through rational expressions." Indian J. Pure Appl. Math. Vol. 6, pp. 1455-1458, 1975.

[17] Nazam M., Aydi H and Arshad M. "A real generalization of the Dass- Gupta fixed point theorem." TWMS J. Pure Appl. Math., Vol. 11, pp. 109-118, 2020.

[18] Pandey B., Pandey A.K. and Ughade M. "Rational type contraction in controlled metric spaces." J. Math. Comput. Sci. Vol. 11(4), pp. 4631-4639, 2021