International Journal of Science and Research (IJSR)
ISSN: 2319-7064
SJIF (2022): 7.942

A Detailed Study of Kirchhoff-type Critical Elliptic Equations and p-Sub-Laplacian Operators within the Heisenberg Group $\mathcal{H}_n$ Framework

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Dated: December 1, 2023

Abstract

This article presents a comprehensive study of *Kirchhoff-type Critical Elliptic Equations* involving p-sub-Laplacian Operators on the *Heisenberg Group* $\mathcal{H}_n$. It delves into the mathematical framework of Heisenberg Group, and explores their Spectral Properties. A significant focus is on the existence and multiplicity of solutions under various conditions, leveraging concepts like the *Mountain Pass Theorem*. This work not only contributes to the theoretical understanding of such groups but also has implications in fields like Quantum Mechanics and Geometric Group Theory.

Keywords and Phrases: Heisenberg Group, sub-Laplacian, Twisted laplacian, Essential Self-Adjointness, Spectrum, Essential Spectra, p-sub-Laplacian, Kirchhoff-type Critical Elliptic Equations, Palais-Smale Condition.

2020 MSC: Primary 35A15, 35A20, 35B38, 35B65, 35J35, 35D30, 35H10, 35J70 .
Secondary 35-02, 35A01, 35A09 .

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SJIF (2022): 7.942

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1 Sub-Laplacian On The Heisenberg Group

1.1 Definition & Construction

Suppose, we consider the identification of $\mathbb{R}^2$ with $\mathbb{C}$ via the map,

$$\mathbb{R}^2 \longrightarrow \mathbb{C}$$
$$(x, y) \mapsto z = x + iy$$

Then, we can interpret, $\mathcal{H}_3 = \mathbb{C} \times \mathbb{R}$, where, $\mathcal{H}_3$ is the *Heisenberg Group* defined on 3 parameters. As observed before, $\mathcal{H}_3$ is a *non-commutative* and a *Unimodular Lie Group* on which the *Haar Measure* is equal to the usual *Lebesgue Measure dzdt*.

A priori denoting the Lie Algebra associated to $\mathcal{H}_3$ as $\mathfrak{h}$, consisting of all left invariant vector fields on the same, we can in fact opt for a basis of $\mathfrak{h}$ as $\{X, Y, T\}$, where,

$$X = \partial_{y_1} - 2y_2\partial_\tau \ , \qquad Y = \partial_{y_2} + 2y_1\partial_\tau \ , \qquad T = 4\partial_\tau \tag{1.1}$$

Definition 1.1.1. (Sub-Laplacian) The **sub-Laplacian** $\mathcal{L}$ on $\mathcal{H}_3$ is defined by,

$$\mathcal{L} = -\left(X^2 + Y^2\right) \tag{1.2}$$

We further introduce the following notations corresponding to the partial differential operators on $\mathbb{C}$ as,

$$\frac{\partial}{\partial z} = \frac{\partial}{\partial y_1} - i\frac{\partial}{\partial y_2}$$

$$\frac{\partial}{\partial \overline{z}} = \frac{\partial}{\partial y_1} + i\frac{\partial}{\partial y_2}$$

Thus it only suffices to study the vector fields Z and $\overline{Z}$ on $\mathcal{H}_3$ given by,

$$Z = X - iY = \frac{\partial}{\partial z} - 2i\overline{z}\frac{\partial}{\partial \tau} \tag{1.3}$$

$$\overline{Z} = X + iY = \frac{\partial}{\partial \overline{z}} + 2iz\frac{\partial}{\partial \tau} \tag{1.4}$$

Important to note that, $\overline{Z}$ is also well-known as the *Hans Lewy Operator* [19], which eventually defies *local solvability* on $\mathbb{R}^3$, and,

$$\mathcal{L} = -\frac{1}{2}\left(Z\overline{Z} + \overline{Z}Z\right) \tag{1.5}$$

We can further compute,

$$\begin{aligned}\mathcal{L} &= -\left(\left(\frac{\partial}{\partial y_1} - 2y_2\frac{\partial}{\partial \tau}\right)^2 + \left(\frac{\partial}{\partial y_2} + 2y_1\frac{\partial}{\partial \tau}\right)^2\right) \\ &= -\Delta - 4\left(y_1^2 + y_2^2\right)\frac{\partial^2}{\partial \tau^2} + 4\left(y_2\frac{\partial}{\partial y_1} - y_1\frac{\partial}{\partial y_2}\right)\frac{\partial}{\partial \tau}\end{aligned}$$

provided, $\Delta = \frac{\partial^2}{\partial y_1^2} + \frac{\partial^2}{\partial y_2^2}$.

Subsequently, the symbol $\sigma(\mathcal{L})$ of $\mathcal{L}$ can be derived as follows,

$$\sigma(\mathcal{L})(y_1, y_2, \tau; \xi, \eta, \gamma) = \left(\xi - 2y_2\gamma\right)^2 + \left(\eta + 2y_1\gamma\right)^2 \tag{1.6}$$

For every (y_1, y_2, τ) , $(\xi, \eta, \gamma) \in \mathcal{H}_3$.

Remark 1.1.1. $\mathcal{L}$ is **Nowhere Elliptic** on $\mathbb{R}^3$.

A priori from the fact that, $[X, Y] = T$, a theorem by *Hörmander* [18, Theorem 1.1] enables us to conclude that, $\mathcal{L}$ is indeed *Hypoelliptic*.

Paper ID: SR231130204147 DOI: https://dx.doi.org/10.21275/SR231130204147

1.2 Twisted Laplacians

For $\tau \in \mathbb{R} \setminus \{0\}$, let, Z_τ and $\overline{Z}_\tau$ be *partial differential operators given by,*

$$Z_\tau = \frac{\partial}{\partial z} - 2\overline{z}\tau$$
$$\overline{Z}_\tau = \frac{\partial}{\partial \overline{z}} + 2z\tau$$

Subsequently, the **Twisted Laplacian** L_τ is defined as,

$$L_\tau = -\frac{1}{2}\left(Z_\tau \overline{Z}_\tau + \overline{Z}_\tau Z_\tau\right) \tag{1.7}$$

To be more explicit, we can write,

$$L_\tau = -\frac{1}{2}\left(\left(\frac{\partial}{\partial z} - 2\overline{z}\tau\right)\left(\frac{\partial}{\partial \overline{z}} + 2z\tau\right) + \left(\frac{\partial}{\partial \overline{z}} + 2z\tau\right)\left(\frac{\partial}{\partial z} - 2\overline{z}\tau\right)\right)$$

$$= -\Delta + 4\left(y_1^2 + y_2^2\right)\tau^2 + 4i\left(y_1\frac{\partial}{\partial y_2} - y_2\frac{\partial}{\partial y_1}\right)\tau \tag{1.8}$$

Remark 1.2.1. The fundamental connection between the *sub-laplacian* and the *twisted laplacian* is given by the following result. (**ref.** [20], [21], [22])

Theorem 1.2.2. *Suppose, $u \in \mathcal{S}'(\mathcal{H}_3) \cap C^\infty(\mathcal{H}_3)$ be such that, $\check{u}(z,\tau)$ is a tempered distribution of τ on $\mathbb{R}$, $\forall \quad z \in \mathbb{C}$, where, $\check{u}$ denotes the Inverse Fourier Transform of u with respect to time t. Then, for almost every $\tau \in \mathbb{R} \setminus \{0\}$,*

$$(\mathcal{L}u)^\tau = L_\tau u^\tau$$

where,

$$(\mathcal{L}u)^\tau(z) = (\mathcal{L}u)\check{}\,(z)\ , \qquad z \in \mathbb{C}$$

and,

$$u^\tau(z) = \check{u}(z,\tau)\ , \qquad z \in \mathbb{C}$$

We recall the definition of **Fourier Transform** $\hat{f}$ of a function $f \in L^1(\mathbb{R})$ as,

$$\hat{f}(\xi) := \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-ix.\xi} f(x)dx\ , \qquad \xi \in \mathbb{R}. \tag{1.9}$$

Since, our primary intention is to study the *spectral properties* of L_τ, we introduce the following.

Definition 1.2.1. The **Fourier-Wigner Transform** $V_\tau(f,g)$ of the functions $f,g \in \mathcal{S}(\mathbb{R})$ is defined by,

$$V_\tau(f,g)(q,p) = \frac{1}{\sqrt{2\pi}}|\tau|^{1/2}\int_{-\infty}^{\infty} e^{i\tau q.y} f(y-2p)\overline{g(y+2p)}dy \ , \qquad \forall \quad q,p \in \mathbb{R}$$

If, $\tau = 1$, then, $V_1(f,g) = V(f,g)$, which in fact, defines the *Classical Fourier-Wigner Transform*. (**ref.** [25],[26], [27])

It can be further established that,

$$V_\tau(f,g)(q,p) = |\tau|^{1/2} V(f,g)(\tau q,p)$$

For $\tau \in \mathbb{R}\setminus\{0\}$ and, $k = 0,1,2,\cdots$, we define the function $e_{k,\tau}$ on $\mathbb{R}$ by,

$$e_{k,\tau}(x) = |\tau|^{1/4} e_k(\sqrt{|\tau|}x) \ , \qquad x \in \mathbb{R}$$

Where, e_k denotes the **Hermite Function** defined as follows,

$$e_k(x) := \frac{1}{(2^k k! \sqrt{\pi})^{1/2}} e^{-\frac{x^2}{2}} H_k(x) \ , \qquad x \in \mathbb{R}. \tag{1.10}$$

Such that,

$$H_k(x) := (-1)^k e^{x^2} \left(\frac{d}{dx}\right)^k \left(e^{-x^2}\right) \ , \qquad x \in \mathbb{R}. \tag{1.11}$$

Given $j,k = 0,1,2,...$, we define the function $e_{j,k,\tau}$ on $\mathbb{C}$ as follows,

$$e_{j,k,\tau} = V_\tau(e_{j,\tau} \ , e_{k,\tau}) \tag{1.12}$$

Further computation yields,

$$e_{j,k,1} = V_1(e_{j,1} \ , e_{k,1}) = V(e_j \ , e_k)$$

Where, $V(e_j \ , e_k)$ is the *Classical Hermite Function* on $\mathbb{C}$. [**ref.** [25]]

Remark 1.2.3. The above result can also be interpreted as an analogue of [25, Proposition 21.1].

Proposition 1.2.4. *The set,* $\{e_{j,k,\tau} \mid j,k = 0,1,2,....\}$ *is an orthonormal basis for* $L^2(\mathbb{C})$.

Indeed we can learn about the respective *spectral properties* of L_τ , $\tau \in \mathbb{R}\setminus\{0\}$.

Theorem 1.2.5. *For* $j,k = 0,1,2,....$, *the following holds true,*

$$L_\tau e_{j,k,\tau} = (2k+1)|\tau| e_{j,k,\tau}$$

Proof. Proof is similar to the derivation cited in [25, Theorem 22.2].

A priori from the statement of [25, Theorem 22.1], it follows that, for $j = 0, 1, 2, ...$ and $k = 0, 1, 2, ...$;

$$Z\overline{Z}e_{j,k} = i(2k+2)^{1/2}Ze_{j,k-1} = -(2k+2)e_{j,k}$$

and,

$$\overline{Z}Ze_{j,k} = i(2k)^{1/2}Ze_{j,k-1} = -(2k)e_{j,k}$$

Thus,

$$L_\tau e_{j,k} = -\frac{1}{2}\left(Z\overline{Z} + \overline{Z}Z\right)|\tau|e_{j,k,\tau}$$

Important to observe that, the above identity also holds for $e_{j,0,\tau}$, where, $j = 0, 1, 2, ...$ due to the fact that,

$$Ze_{j,0} = 0\ , \qquad \forall \quad j = 0, 1, 2, ...$$

□

1.3 Essential Self-Adjointness Property

Our aim in this section is to study the *Sub-Laplacian* $\mathcal{L}$ as an *unbounded linear operator* from $L^2(\mathcal{H}_3)$ to $L^2(\mathcal{H}_3)$ with dense domain denoted as, $\mathcal{S}(\mathcal{H}_3)$.

Proposition 1.3.1. *$\mathcal{L}$ is an injective symmetric operator from $L^2(\mathcal{H}_3)$ to $L^2(\mathcal{H}_3)$ with dense domain $\mathcal{S}(\mathcal{H}_3)$. Furthermore, it is strictly positive.*

Proof follows from integration by parts.

Remark 1.3.2. The above proposition implies that, $\mathcal{L}$ is closed. Suppose we denote its *closure* by $\mathcal{L}_0$. Hence, $\mathcal{L}_0$ is closed, symmetric and positive operator from $L^2(\mathcal{H}_3)$ to $L^2(\mathcal{H}_3)$.

In fact, $\mathcal{L}$ is **Essentially Self-Adjoint** (**ref.** [23, Section 4, pp. 1603]) in the following sense that, it has a unique *self-adjoint extension*, which, subsequently equals to $\mathcal{L}_0$. Further details on *essential self-adjointness* can be found in [24, Theorem X.23].

Remark 1.3.3. The results and derivations in this article are also valid for the *Sub-Laplacian* on the n-dimensional *Heisenberg Group* $\mathcal{H}_n$, $n > 1$, having an underlying space as $\mathbb{C}^n \times \mathbb{R}$, although, we have only explored the case for $n = 1$, which is $\mathcal{H}_3$ for the sake of lucidity.

Paper ID: SR231130204147 DOI: https://dx.doi.org/10.21275/SR231130204147

International Journal of Science and Research (IJSR)
ISSN: 2319-7064
SJIF (2022): 7.942

2 The Spectrum of the sub-Laplacian

A priori given a *closed linear operator* $\mathcal{T}$ from a complex Banach Space X with dense domain $\mathcal{D}(\mathcal{T})$, we provide the following definitions,

Definition 2.0.1. (Spectrum) The **Resolvent Set** $\rho(\mathcal{T})$ of $\mathcal{T}$ is defined as follows,

$$\rho(\mathcal{T}) \ := \ \{\lambda \in \mathbb{C} \mid \mathcal{T} - \lambda I \ : \ \mathcal{D}(\mathcal{T}) \longrightarrow X \text{ is bijective}\}$$

Where, I denotes the **identity operator** on X.
The **Spectrum**, denoted by $\Sigma(\mathcal{T})$ is defined to be the complement of $\rho(\mathcal{T})$ in $\mathbb{C}$.

Definition 2.0.2. (Point Spectrum) The **point spectrum** [28] of $\mathcal{T}$, denoted by $\Sigma_p(\mathcal{T})$ is defined as,

$$\Sigma_p(\mathcal{T}) \ := \ \{\lambda \in \mathbb{C} \mid \mathcal{T} - \lambda I \ : \ \mathcal{D}(\mathcal{T}) \longrightarrow X \text{ is not injective}\}$$

Definition 2.0.3. (Continuous Spectrum) The **Continuous Spectrum** of $\mathcal{T}$, denoted by $\Sigma_c(\mathcal{T})$, is defined as,

$$\Sigma_c(\mathcal{T}) \ := \ \left\{\lambda \in \mathbb{C} \mid Range(\mathcal{T} - \lambda I) \ \text{ is dense in } X \ , (\mathcal{T} - \lambda I)^{-1} \text{ exists, but is unbounded}\right\}$$

Definition 2.0.4. (Residual Spectrum) The **Residual Spectrum** of $\mathcal{T}$, denoted by $\Sigma_r(\mathcal{T})$, is defined as,

$$\Sigma_r(\mathcal{T}) \ := \ \left\{\lambda \in \mathbb{C} \mid Range(\mathcal{T} - \lambda I) \ \text{ is not dense in } X \ , (\mathcal{T} - \lambda I)^{-1} \text{ exists and is bounded}\right\}$$

We can indeed deduce that, $\Sigma_p(\mathcal{T})$, $\Sigma_c(\mathcal{T})$ and $\Sigma_r(\mathcal{T})$ are *mutually disjoint*. Furthermore,

$$\Sigma(\mathcal{T}) = \Sigma_p(\mathcal{T}) + \Sigma_c(\mathcal{T}) + \Sigma_r(\mathcal{T})$$

Proposition 2.0.1. *A priori given a complex and separable Hilbert Space X, if $\mathcal{T}$ is a self-adjoint operator , then,*

$$\Sigma_r(\mathcal{T}) = \phi$$

With the above notations and concepts, we thus delineate a more precise illustration of the Spectrum of the *sub-Laplacian* on the *Heisenberg Group*.

Theorem 2.0.2. *We have,*

$$\Sigma(\mathcal{L}_0) = \Sigma_c(\mathcal{L}_0) = [0, \infty)$$

Proof. We first intend to show that, no eigenvalue of $\mathcal{L}_0$ lies in the interval $[0, \infty)$. We know for a fact that, 0 is not an eigenvalue of $\mathcal{L}_0$ (Proof mentioned in [22]). Suppose, λ be a positive number such that, $\exists$ a function $u \in L^2(\mathcal{H}_3)$ satisfying,

$$\mathcal{L}_0 u = \lambda u$$

Consequently,

$$L_\tau u^\tau = \lambda u^\tau$$

Although, the above relation implies that, $u^\tau = 0, \quad \forall\ \tau \in \mathbb{R} \setminus \{0\}$ and,

$$|\tau| \neq \frac{\lambda}{(2k+1)}\ , \qquad k = 0, 1, 2, \cdots \tag{2.1}$$

This helps us conclude that, $u = 0$, a contradiction. Moreover, $\mathcal{L}_0$ being *self-adjoint*, it implies,

$$\Sigma(\mathcal{L}_0) = \Sigma_c(\mathcal{L}_0)$$

Thus, it only suffices to establish that, $(\mathcal{L}_0 - \lambda I)$ is not *surjective* $\forall \quad \lambda \in [0, \infty)$.

Assuming $(\mathcal{L}_0 - \lambda I)$ to be surjective for some $\lambda_0 \in [0, \infty)$, we can infer that, $\lambda_0 \in \rho(\mathcal{L}_0)$. Hence, $\exists$ an open interval $I_{\lambda_0} \subset \rho(\mathcal{L}_0)$ containing λ_0.

Define f on $\mathcal{H}$ as,

$$f(x, y, t) = h(x, y)e^{-\frac{t^2}{2}}\ , \quad x, y, t \in \mathbb{R}$$

Where, $h \in L^2(\mathbb{R}^2)$.

Therefore, for every $\lambda \in I_{\lambda_0}$, $\exists$ a function $u_\lambda \in L^2(\mathcal{H}_3)$, such that,

$$(\mathcal{L}_0 - \lambda I)u_\lambda = f$$

Computing the *Inverse Transform* with respect to t helps us conclude,

$$(L_\tau - \lambda I)u_\lambda^\tau = he^{-\frac{\tau^2}{2}}\ , \qquad \text{for almost every } \tau \in \mathbb{R} \setminus \{0\} \tag{2.2}$$

As a consequence, $(L_\tau - \lambda I)$ is *surjective* $\forall \quad \tau \in S_\lambda$ for which the *lebesgue measure*,

$$m(\mathbb{R} \setminus S_\lambda) = 0$$

Consider, $\tau \in \bigcap_{r \in I_{\lambda_0} \cap \mathbb{Q}} S_r$, $\mathbb{Q}$ being the set of all *rationals*. Hence, $(L_\tau - \lambda I)$ is surjective, and subsequently *injective* for every $\lambda \in I_{\lambda_0} \cap \mathbb{Q}$.

International Journal of Science and Research (IJSR)
ISSN: 2319-7064
SJIF (2022): 7.942

Thus, $L_\tau - \lambda I$ is *bijective*, $\forall \lambda \in I_{\lambda_0}$. (Use the fact that, the *Resolvent Set* of L_τ is *open*) Furthermore, we can also observe that, $(L_\tau - \lambda I)$ is *injective* iff,

$$\lambda \neq (2k+1)|\tau| \ , \qquad k = 0, 1, 2, \cdots$$

A contradiction to the fact that, if we assume $\tau \in \bigcap_{r \in I_{\lambda_0}} S_r$ to be sufficiently small, such that, $(2k+1)|\tau| \in I_{\lambda_0}$ for some $k = 0, 1, 2, \cdots$. Hence, the proof is done. □

Remark 2.0.3. As an application to *Theorem* (2.0.2), in the next section, we shall introoduce various *Essential Spectra* which'll be extremely helpful to us.

3 Essential Spectra of sub-Laplacian

Given a *closed* linear operator $\mathcal{T}$ densly defined on a complex Banach Space X.

Definition 3.0.1. The **Essential Spectrum** of $\mathcal{T}$, denoted as $\Sigma_{DS}(\mathcal{T})$ [*Dunford* and Schwartz [29]] is defined as follows,

$$\Sigma_{DS}(\mathcal{T}) \ := \ \{\lambda \in \mathbb{C} \mid Range(\mathcal{T} - \lambda I) \ \text{ is not closed in } X\}$$

Let us recall the following concept from *Functional Analysis*.

Definition 3.0.2. (Fredholm Operator) Given any two Banach Spaces X and Y, and a bounded linear operator $\mathcal{T} : X \longrightarrow Y$, we define $\mathcal{T}$ to be **Fredholm** if, the following conditions hold true:

1. $ker(\mathcal{T})$ is of finite dimension.
2. $Range(\mathcal{T})$ is closed.
3. $Coker(\mathcal{T})$ is of finite dimension.

If $\mathcal{T}$ is *Fredholm*, then, **Index of** $\mathcal{T}$ is defined to be equal to $\{dim(ker(\mathcal{T})) - dim(Coker(\mathcal{T}))\}$.

Denote $\Phi_W(\mathcal{T})$ to be the set of all $\lambda \in \mathbb{C}$, such that, $\mathcal{T} - \lambda I$ is a *Fredholm Operator*. Furthermore, suppose, $\Phi_S(\mathcal{T})$ be the set of all complex numbers λ satisfying, $\mathcal{T} - \lambda I$ is *Fredholm* with index 0.

Then, the essential spectrums $\Sigma_W(\mathcal{T})$ [Wolf [30] [31]] and, $\Sigma_S(\mathcal{T})$ [Schechter [32]] of $\mathcal{T}$ having the following definitions,

$$\Sigma_W(\mathcal{T}) = \mathbb{C} \setminus \Phi_W(\mathcal{T})$$

and,

$$\Sigma_S(\mathcal{T}) = \mathbb{C} \setminus \Phi_S(\mathcal{T})$$

It is obvious from the respective definitions that,

$$\Sigma_{DS}(\mathcal{T}) \subseteq \Sigma_W(\mathcal{T}) \subseteq \Sigma_S(\mathcal{T}) \tag{3.1}$$

In the particular case when, $\mathcal{T}$ denotes the *sub-laplacian* on the *Heisenberg Group* $\mathcal{H}_3$, we can indeed deduce the following important result.

Theorem 3.0.1. *We shall have,*

$$\Sigma_{DS}(\mathcal{L}_0) = \Sigma_W(\mathcal{L}_0) = \Sigma_S(\mathcal{L}_0) = [0, \infty) \tag{3.2}$$

Proof. It only requires us to verify that,

$$[0, \infty) \subseteq \Sigma_{DS}(\mathcal{L}_0)$$

Assume, $\lambda \in [0, \infty)$, although, $\lambda \notin \Sigma_{DS}(\mathcal{L}_0)$. Thus, $Range(\mathcal{L}_0 - \lambda I)$ is *closed* in $L^2(\mathcal{H}_3)$. This in turn helps us conclude that, $\mathcal{L}_0 - \lambda I$ is *bijective*, i.e., $\lambda \in \mathcal{L}_0$, a contradiction. Hence the proof is complete. □

Remark 3.0.2. Similar technique can be implemented to compute the *Spectrum* of the unique **self-adjoint extension** $\Delta_{\mathcal{H}_3,0}$ of the *Laplacian* $\Delta_{\mathcal{H}_3}$ on the *Heisenberg Group* $\mathcal{H}_3$ defined as,

$$\Delta_{\mathcal{H}_3} = -\left(X^2 + Y^2 + T^2\right)$$

In fact, we have [33],

$$\Sigma\left(\Delta_{\mathcal{H}_3,0}\right) = \Sigma_c\left(\Delta_{\mathcal{H}_3,0}\right) = [0, \infty)$$

Therefore, we can infer,

$$\Sigma_{DS}(\Delta_{\mathcal{H}_3,0}) = \Sigma_W(\Delta_{\mathcal{H}_3,0}) = \Sigma_S(\Delta_{\mathcal{H}_3,0}) = [0, \infty) \tag{3.3}$$

4 Kirchhoff-type Critical Elliptic Equations involving p-sub-Laplacians on $\mathcal{H}_n$

4.1 Some Important Concepts

A priori given a generalized *Heisenberg Group* $\mathcal{H}_n$, a *lie group* of topological dimension $(2n+1)$, having $\mathbb{R}^{2n+1}$ as a background manifold, endowd with the non-Abelian group law,

$$\tau \;:\; \mathcal{H}_n \longrightarrow \mathcal{H}_n \;, \quad \tau_\xi(\xi') = \xi \circ \xi'$$

where,

$$\xi \circ \xi' = \left(x + x', y + y', t + t' + 2\sum_{i=1}^{n}(y_i x_i' - x_i y_i') \right) \ , \quad \forall \quad \xi, \xi' \in \mathcal{H}_n$$

Subsequently, inverse of this operation can be deduced as, $\xi^{-1} = -\xi$, thus,

$$(\xi \circ \xi')^{-1} = (\xi')^{-1} \circ (\xi)^{-1}$$

Applying similar concepts as in (1.1) the corresponding Lie Algebra of *left-invariant vector fields* is generated by,

$$X_j = \partial_{x_j} + 2y_j\partial_t \ , \qquad Y_j = \partial_{y_j} - 2x_j\partial_t \ , \qquad T = 4\partial_t \tag{4.1}$$

For every $j = 1, 2, 3, \cdots, n$. As a consequence, the basis $\beta = \{X_j, Y_j, T\}_{j=1(1)n}$ satisfies the *Heisenberg Canonical Communication Relations* for position and momentum,

$$[X_j, Y_j] = -\delta_{jk}T$$

And all other commutators are *zero*.

Remark 4.1.1. A vector field in the span of β is called **Horizontal**.

Definition 4.1.1. (Korányi Norm) It can be observed that, the anisotropic dilation structure on the Heisenberg Group $\mathcal{H}_n$ induces the **Korányi Norm** defined as follows:

$$r(\xi) := r(z,t) = (|z|^4 + t^2)^{\frac{1}{4}} \ , \quad \forall \ \xi = (z,t) \in \mathcal{H}_n$$

Some properties of Korányi Norm include that, its *homogeneous degree* with resopect to dilations is equal to 1.

Subsequently, the **Korányi Distance** is defined as :

$$d_H(\xi, \xi') = r(\xi^{-1} \circ \xi') \ , \quad \forall \ (\xi, \xi') \in \mathcal{H}_n \times \mathcal{H}_n$$

And, the **Korányi Open Ball** of radius R centered at ξ_0 is,

$$B_R(\xi_0) = \{\xi \in \mathcal{H}_n \mid d_H(\xi, \xi_0) < R\}$$

Remark 4.1.2. We can indeed infer that, the *Haar Measure* on $\mathcal{H}_n$ is consistent with the *Lebesgue Measure* on $\mathbb{R}^{2n+1}$, and is invariant under the left translations of $\mathcal{H}_n$. Moreover, it is Q-Homogeneous with respect to dilations (Q denotes the *Hausdorff Dimension*).

Thus, the *topological dimension* of $\mathcal{H}_n$ (is equal to $2n+1$) is strictly less than $Q = 2n+2$.

Definition 4.1.2. We define the **Horizontal Gradient** of a C^1-function $u : \mathcal{H}_n \longrightarrow \mathbb{R}$ as:

$$D_H(u) = \sum_{j=1}^{n} ((X_j u)X_j + (Y_j u)Y_j) \tag{4.2}$$

An important observation is that, $D_H u$ is in fact an element of $span(\beta)$. Thus, we can define the natural inner product in $span(\beta)$ as :

$$(X, Y)_H := \sum_{j=1}^{n} \left(x^j y^j + \tilde{x}^j \tilde{y}^j\right)$$

For every $X = \{x^j X_j + \tilde{x}^j Y_j\}_{j=1(1)n}$, $Y = \{y^j X_j + \tilde{y}^j Y_j\}_{j=1(1)n}$. This eventually helps us define the **Hilbertian Norm**,

$$|D_H(u)| := \sqrt{(D_H(u), D_H(u))_H} \tag{4.3}$$

for any horizontal vector field $D_H(u)$.

Definition 4.1.3. Given any horizontal vector field function, $X = X(\xi)$, $X = \{x^j X_j + \tilde{x}^j Y_j\}_{j=1(1)n}$ of class $C^1(\mathcal{H}_n, \mathbb{R}^{2n})$, the **Horizontal Divergence** of X is defined as,

$$div_H X = \sum_{j=1}^{n} \left(X_j(x^j) + Y_j(y^j)\right)$$

We can generalize the notion of *sub-Laplacians* in the case for $\mathcal{H}_3$ to a generalized Heisenberg Group $\mathcal{H}_n$.

Definition 4.1.4. (sub-Laplacian) For every $u \in C^2(\mathcal{H}_n)$, the **sub-Laplacian** or, *Kohn-Spencer Laplacian* of u is defined as:

$$\Delta_H(u) = \sum_{j=1}^{n} \left(X_j^2 + Y_j^2\right) u$$

$$= \sum_{j=1}^{n} \left(\frac{\partial^2}{\partial x_j^2} + \frac{\partial^2}{\partial y_j^2} + 4y_j \frac{\partial^2}{\partial x_j \partial t} - 4x_j \frac{\partial^2}{\partial y_j \partial t} \right) u + 4|z|^2 \frac{\partial^2 u}{\partial^2 t} \tag{4.4}$$

Hörmander [18] established the fact that, Δ_H is *Hypoelliptic*. To be more precise,

$$\Delta_H(u) = div_H D_H(u) \ , \quad \forall \quad u \in C^2(\mathcal{H}_n)$$

Paper ID: SR231130204147 DOI: https://dx.doi.org/10.21275/SR231130204147

We can in fact further generalize the *Kohn-Spencer Laplacian* to obtain the so called **p-Laplacian** on $\mathcal{H}_n$, having the following expression:

$$\Delta_{H,p}(\varphi) = div_H\left(|D_H(\varphi)|_H^{p-2} D_H(\varphi)\right) \tag{4.5}$$

for every $\varphi \in C_c^{\infty}(\mathcal{H}_n)$.For further study, interested readers can refer to [34], [35], [36], [37].

let us recall some significant properties of *classic Sobolev Spaces* on $\mathcal{H}_n$.

Definition 4.1.5. The standard L^p**-norm** is defined as,

$$||u||_p^p = \int_{\omega} |u|^p d\xi \ , \quad \forall \ u \in \Omega$$

A priori given Ω to be a *bounded Lipschitz Domain* in $\mathcal{H}_n$ or, $\Omega = \mathcal{H}_n$. Then we denote $W^{1,p}(\Omega)$ as the *Horizontal Sobolev Space* of the functions $u \in L^p(\Omega)$, provided, $D_H(u)$ exists in the sense of *distributions*, and furthermore, $|D_H(u)| \in L^p(\Omega)$, endowed with the norm,

$$||u||_{W^{1,p}(\Omega)} = \left(||u||_p^p + ||D_H(u)||_p^p\right)^{\frac{1}{p}}$$

Consider the function space,

$$HW_V^{1,p}(\mathcal{H}_n) := \left\{ u \in W^{1,p}(\mathcal{H}_n) : \int_{\mathcal{H}_n} V(\xi)|u(\xi)|^p d\xi < \infty \right\}$$

with the followig norm defined on it,

$$||u|| = ||u||_{HW_V^{1,p}(\mathcal{H}_n)} := \left(||D_H(u)||_p^p + ||u||_{p,V}^p\right)^{\frac{1}{p}} \tag{4.6}$$

and,

$$||u||_{p,V}^p = \int_{\mathcal{H}_n} V(\xi)|u(\xi)|^p d\xi \tag{4.7}$$

Where, V denotes the **potenial function**. Furthermore, under the assumption that, $V(\xi) \geq V_0 > 0$, we can in fact conclude that, $HW_V^{1,p}(\mathcal{H}_n)$ is a *reflexive Banach Space*. For proof involving Euclidean setting, it can be found in [38, Lemma 10], whereas, in case for $\mathcal{H}_n$, we shall be needing few minor alterations. The continuous embedding of $HW_V^{1,p}(\mathcal{H}_n) \hookrightarrow W^{1,p}(\mathcal{H}_n) \hookrightarrow L^t(\mathcal{H}_n) \quad \forall$ $p \leq t < p^*$, where, $p^* := \frac{Qp}{Q-p}$ is the *Critical Sobolev Exponent* on $\mathcal{H}_n$.

Remark 4.1.3. In fact, one can establish that, the best value of the constant V_0, denoted by C_{p^*} is attained in the *Folland-Stein Spaces* $S^{1,p}(\mathcal{H}_n)$, which, also can be interpreted as the completion of $C_c^\infty(\mathcal{H}_n)$ in terms of the norm,

$$||D_H(u)||_p = \left(\int\limits_{\mathcal{H}_n} |D_H(u)|_H^p d\xi \right)^{\frac{1}{p}}$$

Thus, we can obtain the following estimate of C_{p^*} of the *Folland-Stein Inequality* as,

$$C_{p^*} = \inf_{u \in S^{1,p}(\mathcal{H}_n), u \neq 0} \frac{||D_H(u)||_p^p}{||u||_{p^*}^p}$$

For further details, see [39].

4.2 Introduction to Critical Kirchhoff Equations

In this section, we shall deal with a class of *Kirchhoff*-type Critical *Elliptic Equations* (Kirchhoff, 1883) as a generalization of *D'Alembert*'s Wave Equation for free vibrations of elastic strings, involving p-sub-Laplacians, having the following representation,

$$M\left(||D_H(u)||_p^p + ||u||_{p,V}^p\right)\left\{-\Delta_{H,p}(u) + V(\xi)|u|^{p-2}u\right\} = \lambda f(\xi, u) + |u|^{p^*-2}u \tag{4.8}$$

$$\xi \in \mathcal{H}_n\ , \qquad u \in HW_V^{1,p}(\mathcal{H}_n)$$

in both non-degenerate and degenerate cases separately.

Remark 4.2.1. In (4.8), λ is a real parameter, and, M denotes the *Kirchhoff Function*.

A priori, for pre-determined constants ρ, P_0, h, E, L having some physical interpretation, Kirchhoff thus established a model given by the equation,

$$\rho \frac{\partial^2 u}{\partial t^2} - \left(\frac{P_0}{h} + \frac{E}{2L} \int\limits_0^L \left| \frac{\partial u}{\partial x} \right|^2 dx \right) \frac{\partial^2 u}{\partial x^2} = 0$$

In particular, the study of critical Kirchhoff-type problems were first initially studied in the seminal paper of *Brézis* & *Nirenberg* [ref. [40]], in which their intention was to study the Laplacian equations.

Over the years, there have been many generalizations of [40] in various directions. For instance, *Liao et al.* [41] studied the following non-local problem with Critical Sobolev Exponent

International Journal of Science and Research (IJSR)
ISSN: 2319-7064
SJIF (2022): 7.942

of the form,

$$-\left\{a+b\int_{\Omega}|\nabla u|^2\,dx\right\}\Delta u=\mu|u|^{2^*-2}u+\lambda|u|^{q-2}u\;,\qquad x\in\Omega \tag{4.9}$$

$$u=0\;,\quad x\in\partial\Omega$$

where, $\Omega\subseteq\mathbb{R}^N$ $(N\geq 4)$ is a smooth bounded domain, $2^*=\frac{2N}{N-2}$ is the *Critical Sobolev Exponent*. The existence an multiplicity of the solutions of (4.9) are obtained by applying the Variational Methods and the *Critical Point Theorem*. *Liang et al.* [42] effectively followed the similar approach to derive the solutions to the fractional Schrödinger-Kirchhoff equations with electro-magnetic fields and critical non-linearity in the no-degenerate Kirchhoff case by using the fractional versions of the *concentration compactness principle* and*variational methods*. Results related to the existence of solution in case of non-degenerate Kirchhoff problems are illustrated, for example, in [43], [44], [45] and [38].

Furthermore, there are extensive amount of research which are currently going on for the *degenerate* case. Interested readers can look at the findings of *wang et al.* [46] and even *Song & Shi* [47] in this regard.

The motivation behind studying the problem (4.8) primarily originates from the significant applications of the Heisenberg Group. *Liang & Pucci* [48] considered a class of critical Kirchhoff-Poisson systems in the Heisenberg group under suitable assumptions. On the contrary, the existence of multiple solutions is obtained by using the symmetric **Mountain Pass Theorem**. A priori applying this result along with *Singular Trudinger-Moser Inequality*, *Deng & Tian* [49] discussed the existence of solutions for Kirchhoff-type systems involving Q-laplacian operator in the Heisenberg Group,

$$-K\left(\int_{\Omega}|\nabla_{\mathcal{H}_n}u|^Q\,d\xi\right)\Delta_Q(u)=\lambda\frac{G_u(\xi,u,v)}{\rho(\xi)^{\wp}}\qquad\text{in }\Omega$$

$$-K\left(\int_{\Omega}|\nabla_{\mathcal{H}_n}v|^Q\,d\xi\right)\Delta_Q(v)=\lambda\frac{G_v(\xi,u,v)}{\rho(\xi)^{\wp}}\qquad\text{in }\Omega$$

$$u=v=0\qquad\text{on }\partial\Omega$$

Where, Ω is an open, smooth and bounded subset of $\mathcal{H}_n$, K is Kirchhoff-type function & non-linear terms : G_u and G_v have critical exponent growth.

Whereas, *Pucci & Temeprini* [50] studied the (p, q) critical systems on the Heisenberg Group,

$$-div_H\left(A(|D_H(u)|_H)\right) + B\left(|u|\right)u = \lambda H_u(u, v) + \frac{\alpha}{\wp^*}|v|^{\beta}|u|^{\alpha-2}u$$

$$-div_H\left(A(|D_H(v)|_H)\right) + B\left(|v|\right)v = \lambda H_v(u, v) + \frac{\beta}{\wp^*}|u|^{\alpha}|v|^{\beta-2}v$$

Subsequently, the existence of entire non-trivial solutions are obtained by applying the concentration-compactness principle in the vectorial Heisenberg context and variational methods.

Remark 4.2.2. Further details on these results can be explored from [51].

5 Existence and Multiplicity of Solutions

A priori having the concepts discussed in detail in the previous section, we now proceed towards proving the existence and multiplicity for a special class of Kirchhoff-type critical elliptic equations as mentioned in (4.8) involving the p-sub-Laplacian operators on $\mathcal{H}_n$ for both the *degenerate* and the *non-degenerate* case separately.

5.1 Some Important Assumptions

In order to establish our main results, a priori we shall assume that, $M : \mathbb{R}_0^+ \longrightarrow \mathbb{R}_0^+$ is a continuous and non-decreasing function, the potential function V and M. Therefore, they will satisfy the following properties :

- $V : \mathcal{H}_n \longrightarrow \mathbb{R}_+$ is *continuous*, and $\exists\ V_0 > 0$ such that, $V \geq V_0 > 0$ in $\mathcal{H}_n$.

1. $\exists\ m_0 > 0$ such that, $\inf_{t\geq 0} M(t) = m_0$.
2. $\exists\ \tau \in \left[1, \frac{p^*}{p}\right)$ satisfying,

$$\tau\mathcal{M}(t) \geq M(t)t\ , \qquad \forall \quad t \geq 0$$

Where,

$$\mathcal{M}(t) := \int_0^t M(s)ds.$$

International Journal of Science and Research (IJSR)
ISSN: 2319-7064
SJIF (2022): 7.942

3. $\exists\ m_1 > 0$ such that, $M(t) \geq m_1 t^{\tau-1}$, $\quad \forall\ t \geq 0$ and $M(0) = 0$.

Furthermore, we impose the following hypotheses on the non-linearity of f:

- $f : \mathcal{H}_n \times \mathbb{R} \longrightarrow \mathbb{R}$is a *Carathéodory Function* such that, f is odd with respect to the second variable.
- $\exists$ a constant r satisfying, $p^* > r > p\tau$ such that,

$$|f(\xi,t)| \leq a(\xi)\,|t|^{r-2}\,t\ , \qquad \text{for a.e. } \xi \in \mathcal{H}_n \text{ and } t \in \mathbb{R}$$

  where, $0 \leq a(\xi) \in L^{\eta}(\mathcal{H}_n) \cap L^{\infty}(\mathcal{H}_n)$, and, $\eta := \frac{p^*}{p^*-r}$, $\xi \in \mathcal{H}_n$.
- $\exists$ a constant θ satisfying $p\tau < \theta < p^*$ such that, $0 < \theta F(\xi,t) \leq f(\xi,t)t$, $\forall \quad t \in \mathbb{R}_+$. We define,

$$F(\xi,t) := \int_0^t f(\xi,s)ds.$$

5.2 Palais-Smale Condition $(PS)_c$

A priori we adhere to some standard notations, where, $\mathcal{N}(\mathcal{H}_n)$ denotes the space of all signed finite *Radon Measures* on $\mathcal{H}_n$ equipped with the norm. In other words, we identify $\mathcal{N}(\mathcal{H}_n)$ with the dual of $C_0(\mathcal{H}_n)$, the completion of all continuous functions $u : \mathcal{H}_n \longrightarrow \mathbb{R}$, having con=mpact support, and also is connected to the supremum norm $||.||_\infty$.

Important observation one can make here is that, the problem (4.8) has a variational structure. The *Euler-Lagrange Functional*, $\mathcal{J}_\lambda : HW_V^{1,p}(\mathcal{H}_n) \longrightarrow \mathbb{R}$ associated to this problem is defined as follows :

$$\mathcal{J}_\lambda(u) = \frac{1}{p}\mathcal{M}\left(||D_H(u)||_p^p + ||u||_{p,V}^p\right) - \lambda\int_{\mathcal{H}_n} F(\xi,u)d\xi - \frac{1}{p^*}\int_{\mathcal{H}_n} |u|^{p^*}d\xi \tag{5.1}$$

It implies that, under the conditions described in (5.1), $\mathcal{J}_\lambda$ is of class $C^1\left(HW_V^{1,p}(\mathcal{H}_n)\right)$. Moreover, for every $u,v \in HW_V^{1,p}(\mathcal{H}_n)$, we define the *Fréchet Derivative* of $\mathcal{J}_\lambda$ is given as:

$$\begin{aligned}\langle \mathcal{J}_\lambda'(u), v\rangle = M\left(||D_H(u)||_p^p + ||u||_{p,V}^p\right)&\left(\langle \mathcal{A}(u), v\rangle + \int_{\mathcal{H}_n} V(\xi)|u|^{p-2}uvd\xi\right)\\ &-\lambda\int_{\mathcal{H}_n} f(\xi,u)uvd\xi - \int_{\mathcal{H}_n} |u|^{p^*-2}uvd\xi\end{aligned}$$

International Journal of Science and Research (IJSR)
ISSN: 2319-7064
SJIF (2022): 7.942

Where,

$$\langle \mathcal{A}_p(u), v\rangle = \int_{\mathcal{H}_n} |D_H(u)|_H^{p-2} .D_H(u).D_H(v)d\xi$$

Remark 5.2.1. It can be verified that the weak solutions for problem (4.8) indeed coincide with the *critical points* of $\mathcal{J}_\lambda$.

With all these notations and definitions, we define,

Definition 5.2.1. A sequence $\{u_n\}_{n\in\mathbb{N}} \subset X$ is termed as a **Palais-Smale Sequence** for the functional $\mathcal{J}_\lambda$ at level c if,

$$\mathcal{J}_\lambda(u_n) \longrightarrow c\ , \qquad \mathcal{J}'_\lambda(u_n) \longrightarrow 0 \quad \text{in } X' \text{ as } n \to +\infty \tag{5.2}$$

If (5.2) implies the existence of a subsequence of $\{u_n\}_n$ which converges in X, we assert that, $\mathcal{J}_\lambda$ satisfies the **Palais-Smale Condition** $(PS)_c$. Moreover, if this strongly convergent subsequence exists only for some c values, we comment that, $\mathcal{J}_\lambda$ satisfies a Local *Palais-Smale Condition.*

5.3 The Non-Degenerate Case

In this section, we shall state and prove two theorems which best illustrates our purpose of analyzing existence and multiplicity of solutions for the problem (4.8) in the *Non-Degenerate case.*

Theorem 5.3.1. *A priori we assume that,* (5.1) *holds true. If M satisfies conditions* (1) *and* (2) *of* (5.1)*, and f verifies* (5.1)*, then* $\exists \quad \lambda_1 > 0$ *such that, for any $\lambda \geq \lambda_1$, the problem* (4.8) *has a non-trivial solution in $HW_V^{1,p}(\mathcal{H}_n)$.*

Theorem 5.3.2. *A priori we assume that,* (5.1) *holds true. If M satisfies conditions* (1) *and* (2) *of* (5.1)*, and f verifies* (5.1)*. Additionally, suppose we consider that, one of the following condition also holds :*

1. *$\exists$ a positive constant $m^* > 0$ for every $m_0 > m^*$ and $\lambda > 0$.*
2. *$\exists$ a positive constant $\lambda_2 > 0$ for every $\lambda > \lambda_2$ and, $m_0 > 0$.*

Then, the problem (4.8) *admits of at least n pairs of non-trivial weak solutions in $HW_V^{1,p}(\mathcal{H}_n)$.*

Before we indulge ourselves into the proof of these statements, it is imperative to study all the necessary results required for justification of the same.

Theorem 5.3.3. *(Mountain Pass Theorem) For a given real Banach Space E and $\mathcal{J} \in C^1(E)$, satisfying $\mathcal{J}(0) = 0$. We further assume that,*

1. $\exists\ \rho, \alpha > 0$ *such that,* $\mathcal{J}(u) \geq \alpha \quad \forall\ u \in E$, *with* $||u||_E = \rho$;
2. $\exists\ e \in E$ *satisfying* $||e||_E > \rho$ *such that,* $\mathcal{J}(e) < 0$

If we denote, $\gamma := \{\gamma \in C([0,1], E) \mid \gamma(0) = 1\ ,\ \gamma(1) = e\}$. *Then,*

$$c = \inf_{\gamma \in \Gamma} \max_{0 \leq t \leq 1} \mathcal{J}(\gamma(t)) \geq \alpha \tag{5.3}$$

and there exists a $(PS)_c$ *sequence* $\{u_n\}_n \subset E$.

One can in fact verify that, $\mathcal{J}_\lambda$ satisfies the geometric properties (1) and (2) of *Mountain Pass Theorem* (5.3.3).

5.3.1 Proof of Theorem (5.3.1)

First, we intend to establish the following:

Claim 5.3.4. We shall have,

$$0 < c_\lambda = \inf_{\gamma \in \Gamma} \max_{0 \leq t \leq 1} \mathcal{J}_\lambda(\gamma(t)) < \left(\frac{1}{\theta} - \frac{1}{p^*}\right)(m_0 C_{p^*})^{\frac{p^*}{p^*-p}} \tag{5.4}$$

for large enough λ.

Proof. Choose $v_0 \in HW_V^{1,p}(\mathcal{H}_n)$ such that,

$$||v_0|| = 1\ ,\ \text{and,} \qquad \lim_{t \to \infty} \mathcal{J}_\lambda(tv_0) = -\infty$$

Therefore, $\sup_{t \geq 0} \mathcal{J}_\lambda(tv_0) = \mathcal{J}_\lambda(t_\lambda v_0)$ for some $t_\lambda > 0$. Thus, t_λ satisfies,

$$M\left(||D_H(t_\lambda v_0)||_p^p + ||t_\lambda v_0||_{p,V}^p\right) \cdot \left(||D_H(t_\lambda v_0)||_p^p + ||t_\lambda v_0||_{p,V}^p\right)$$

$$= \lambda \int_{\mathcal{H}_n} f(\xi, t_\lambda v_0)|t_\lambda v_0|^2 d\xi + \int_{\mathcal{H}_n} |t_\lambda v_0|^{p^*} d\xi \tag{5.5}$$

It suffices to prove that, $\{t_\lambda\}_{\lambda > 0}$ is bounded. Since, $t_\lambda \geq 1\ \forall\ \lambda > 0$. Thus, by property (2) of (5.1) and (5.5), we deuce that,

$$\tau \mathcal{M}(1) t_\lambda^{2p\tau} \geq \tau \mathcal{M}(1)(||t_\lambda v_0||^p)^\tau$$

$$\geq M\left(||D_H(t_\lambda v_0)||_p^p + ||t_\lambda v_0||_{p,V}^p\right) \cdot \left(||D_H(t_\lambda v_0)||_p^p + ||t_\lambda v_0||_{p,V}^p\right)$$
$$= \lambda \int_{\mathcal{H}_n} f(\xi, t_\lambda v_0)|t_\lambda v_0|^2 d\xi + \int_{\mathcal{H}_n} |t_\lambda v_0|^{p^*} d\xi$$

$$\geq t_\lambda^{p^*} \int_{\mathcal{H}_n} |v_0|^{p^*} d\xi \tag{5.6}$$

It follows from (5.6) that, $\{t_\lambda\}_{\lambda>0}$ is bounded, since, $p\tau < p^*$. Next, we shall prove that, $t_\lambda \longrightarrow 0$ as, $\lambda \longrightarrow \infty$. By contradiction, suppose, $\exists\ t_0 > 0$ and a sequence $\{\lambda_n\}_n$ with $\lambda_n \longrightarrow \infty$ as $n \to \infty$, and consequently, $t_{\lambda_n} \longrightarrow t_0$ as $n \to \infty$.

A simple application of the *Lebesgue Dominated Convergence Theorem* yields,

$$\int_{\mathcal{H}_n} f(\xi, t_{\lambda_n} v_0) \left|t_{\lambda_n} v_0\right|^2 d\xi \longrightarrow \int_{\mathcal{H}_n} f(\xi, t_\lambda v_0) \left|t_\lambda v_0\right|^2 d\xi$$

as $n \to \infty$. Thus, we can conclude that,

$$\lambda_n \int_{\mathcal{H}_n} f(\xi, t_\lambda v_0) \left|t_\lambda v_0\right|^2 d\xi \longrightarrow \infty \text{ , as,} \quad n \to \infty$$

A contradiction to the fact that, (5.5) holds true. Thus, $t_\lambda \longrightarrow 0$ as, $\lambda \to \infty$. Another deduction which can indeed be made from (5.5) is,

$$\lim_{\lambda \to \infty} \lambda \int_{\mathcal{H}_n} f(\xi, t_\lambda v_0) \left|t_\lambda v_0\right|^2 d\xi = 0$$

and,

$$\lim_{\lambda \to \infty} \int_{\mathcal{H}_n} \left|t_\lambda v_0\right|^{p^*} d\xi = 0$$

A priori, using the deduction, $t_\lambda \longrightarrow 0$ as, $\lambda \to \infty$ and the definition of $\mathcal{J}_\lambda$, we obtain,

$$\lim_{\lambda \to \infty} \left(\sup_{t \geq 0} \mathcal{J}_\lambda(tv_0)\right) = \lim_{\lambda \to \infty} \mathcal{J}_\lambda(tv_0) = 0$$

Hence, $\exists\ \lambda_1 > 0$ satisfying for every $\lambda \geq \lambda_1$,

$$\sup_{t \geq 0} \mathcal{J}_\lambda(tv_0) < \left(\frac{1}{\theta} - \frac{1}{p^*}\right) (m_0 C_{p^*})^{\frac{p^*}{p^*-p}}$$

Choosing $e = t_1 v_0$, for large enough t_1, it can be verified that,

$$\mathcal{J}_\lambda(e) < 0$$

$$\implies 0 < c_\lambda \leq \max_{t\in[0,1]} \mathcal{J}_\lambda(\gamma(t)) \qquad \text{, by setting, } \gamma(t) = tt_1v_0$$

Therefore, we conclude,

$$0 < c_\lambda = \sup_{t\geq 0} \mathcal{J}_\lambda(tv_0) < \left(\frac{1}{\theta} - \frac{1}{p^*}\right)(m_0 C_{p^*})^{\frac{p^*}{p^*-p}}$$

for λ large enough, which completes the proof. □

5.3.2 Proof of Theorem (5.3.2)

To prove the theorem, we shall in fact use the concept related to *Krasnoselskii*'s Genus Theory [52]. Given a *Banach Space* X, and let, Λ denotes the class of all *closed subsets* $A \subset X \setminus \{0\}$ that are symmetric with respect to the origin (in other words, $u \in A$ implies, $-u \in A$).

Theorem 5.3.5. *For an infinite dimensional Banach Space X and $\mathcal{J} \in C^1(X)$ be an even functional, such that, $\mathcal{J}(0) = 0$. Further, we assume that, $X = Y \oplus Z$, Y being finite dimensional. Then, $\mathcal{J}$ satisfies the following conditions,*

1. *$\exists$ constants $\rho, \alpha > 0$ satisfying, $\mathcal{J}(u) \geq \alpha$ for every $u \in \partial B_\rho \cap Z$.*
2. *$\exists\ \Theta > 0$ such that, $\mathcal{J}$ satisfies the $(PS)_c$ condition $\forall\ c$, with $c \in (0, \Theta)$.*
3. *Given any finite dimensional subspace $\tilde{X} \subset X$, $\exists\ R = R(\tilde{X}) > 0$ satisfying $\mathcal{J}(u) \leq 0$ on $\tilde{X} \setminus B_R$*

In addition to the above, we further consider, $dim(Y) = k$ and that, $Y = Span\{v_1, v_2, \cdots, v_k\}$. For $n \geq k$, we inductively choose that, $v_{n+1} \notin E_n = Span\{v_1, v_2, \cdots, v_n\}$. Suppose, $R_n = R(E_n)$, and, $\Omega_n = B_{R_n} \cap E_n$. Define,

$$G_n := \left\{\psi \in C(\Omega_n, X) \mid \psi|_{\partial B_{R_n \cap E_n}} = id \text{ and, } \psi \text{ is odd}\right\}$$

and,

$$G_n := \left\{\psi(\overline{\Omega_n \setminus V}) \mid \psi \in G_n \ , n \geq j \ , V \in \Lambda \ , \gamma(V) \leq n - j\right\}$$

Where, $0 \leq c_j \leq c_{j+1}$ and, $c_j < \Theta$ for $j > k$, then, we assert that, c_j is a *critical value* of $\mathcal{J}$.

Furthermore, if $c_j = c_{j+l} = c < \Theta$ for every $1 \leq l \leq m$ and for $j > k$. Thus, $\gamma(K_c) \geq m + 1$, provided,

$$K_c := \left\{u \in X \mid \mathcal{J}(u) = c \text{ and, } \mathcal{J}'(u) = 0\right\}$$

Now that, we have introduced all the terminologies required for the proof of Theorem (5.3.2), we proceed to the proof of the statement of the same.

Proof. We shall apply Theorem (5.3.5) to $\mathcal{J}_\lambda$. A priori using the fact that, $HW_V^{1,p}(\mathcal{H}_n)$ is indeed a **reflexive** *Banach Space* and, $\mathcal{J}_\lambda \in C^1\left(HW_V^{1,p}(\mathcal{H}_n)\right)$.

Applying (5.1), we can infer that, the functional $\mathcal{J}_\lambda$ satisfies, $\mathcal{J}_\lambda(0) = 0$. We shall thus prove the theorem in three steps.

Step 1 :

We can in fact obtain that, $\mathcal{J}_\lambda$ indeed satisfies (1) and (3) of Theorem (5.3.5).

Step 2 :

We claim that, $\exists$ a monotone increasing sequeence $\{\Phi_n\}_n$ in $\mathbb{R}_+$, such that,

$$c_n^\lambda = \inf_{E\in\Gamma_n} \max_{u\in E} \mathcal{J}_\lambda(u) < \Phi_n$$

The above definition of c_n^λ allows us to infer,

$$c_n^\lambda = \inf_{E\in\Gamma_n} \max_{u\in E} \mathcal{J}_\lambda(u) \leq \inf_{E\in\Gamma_n} \max_{u\in E} \left\{ \mathcal{M}\left(||D_H(u)||_p^p + ||u||_{p,V}^p \right) - \frac{1}{p^*} \int_{\mathcal{H}_n} |u|^{p^*} d\xi \right\}$$

Set,

$$\Phi_n = \inf_{E\in\Gamma_n} \max_{u\in E} \left\{ \mathcal{M}\left(||D_H(u)||_p^p + ||u||_{p,V}^p \right) - \frac{1}{p^*} \int_{\mathcal{H}_n} |u|^{p^*} d\xi \right\}$$

Therefore, $\Phi_n < \infty$ and, $\Phi_n \leq \Phi_{n+1} \quad \forall\, n \in \mathbb{N}$ (Follows from the definition of Γ_n).

Step 3 :

We claim that, the problem (4.8) has at least k pairs of weak solutions. To this end, we distingish two cases:

Case I :

Fix $\lambda > 0$. We set m_0 very large such that,

$$\sup_n \Phi_n < \left(\frac{1}{\theta} - \frac{1}{p^*}\right) (m_0 C_{p^*})^{\frac{p^*}{p^*-p}}$$

Case II :

using similar discussions as in Theorem (5.3.5), $\exists\ \lambda_2 > 0$ satisfyng,

$$c_n^\lambda \leq \Phi_n < \left(\frac{1}{\theta} - \frac{1}{p^*}\right) (m_0 C_{p^*})^{\frac{p^*}{p^*-p}} \qquad \forall\ \lambda > \lambda_2.$$

International Journal of Science and Research (IJSR)
ISSN: 2319-7064
SJIF (2022): 7.942

Thus, in any case, we must obtain,

$$0 < c_1^\lambda \leq c_2^\lambda \leq \cdots \leq c_n^\lambda < \Phi_n < \left(\frac{1}{\theta} - \frac{1}{p^*}\right)(m_0 C_{p^*})^{\frac{p^*}{p^*-p}}$$

Applying similar arguments as described in [52], we can in fact guarantee that, the levels, $c_1^\lambda \leq c_2^\lambda \leq \cdots \leq c_n^\lambda$ are indeed the *critical values* of $\mathcal{J}_\lambda$.

If $c_j^\lambda = c_{j+1}^\lambda$ for some $j = 1, 2, \cdots k-1$, therefore, by [53, Theorem 4.2, Remark 2.12], the set $K_{c_j^\lambda}$ contains infinitely many distinct points, and hence, the problem (4.8) has **infinitely many weak solutions**.

Finally thus we conclude that, the problem (4.8) has at least k pairs of solutions, and the proof is done.

□

5.4 The Degenerate Case

In this section, we shall investigate the degradation of the problem (4.8). We shall always assume that, M satisfies conditions (2) and (3) of (5.1), and f verifies the cooonditions mentioned in (5.1). We state the main results which helps us comment on the existence and the multiplicity of the solutions to problem (4.8) in the *Degenerate Case.*

Theorem 5.4.1. *A priori we assume that,* (5.1) *holds true. If M satisfies conditions* (2) *and* (3) *of* (5.1)*, and f verifies* (5.1)*, then $\exists \quad \lambda_3 > 0$ such that, for any $\lambda \geq \lambda_3$, the problem* (4.8) *has a non-trivial solution in $HW_V^{1,p}(\mathcal{H}_n)$.*

Theorem 5.4.2. *A priori we assume that,* (5.1) *holds true. If M satisfies conditions* (2) *and* (3) *of* (5.1)*, and f verifies* (5.1)*. Additionally, suppose we consider that, one of the following condition also holds :*

1. *$\exists$ a positive constant $m_* > 0$ for every $m_1 > m_*$ and $\lambda > 0$.*

2. *$\exists$ a positive constant $\lambda_4 > 0$ for every $\lambda > \lambda_4$ and, $m_1 > 0$.*

Then, the problem (4.8) *admits of at least n pairs of non-trivial weak solutions in $HW_V^{1,p}(\mathcal{H}_n)$.*

Before indulging ourselves into the proofs of these statements above, we must state the following lemma which apparently plays a major role in determining the existence of solution to problem (4.8).

Lemma 5.4.3. *Under the assumptions on M and f mentioned above, if suppose, $\{u_n\}_n \subset HW_V^{1,p}(\mathcal{H}_n)$ be a Palais-Smale Sequence of functional $\mathcal{J}_\lambda$ (as defined in* (5.2) *).*

Furthermore, if,

$$0 < c_\lambda < \left(\frac{1}{\theta} - \frac{1}{p^*}\right)(m_0 C_{p^*})^{\frac{p^*}{p^*-p}}$$

Then, $\exists$ *a subsequence of* $\{u_n\}_n$ ***Strongly Convergent*** *in* $HW_V^{1,p}(\mathcal{H}_n)$.

For the proof of the Lemma, interested readers can look up [51].

Remark 5.4.4. In fact, one can further infer that, the functional $\mathcal{J}_\lambda$ satisfies all the assumptions of the **Mountain Pass Theorem** (Theorem (5.3.3)).

5.4.1 Proof of Theorem (5.4.1)

Proof. A priori using similar argments as in the proof of the *Mountain Pass Theorem* (Theorem (5.3.3)), we get,

$$0 < c_\lambda = \inf_{\gamma\in\Gamma} \max_{0\le t\le 1} \mathcal{J}_\lambda(\gamma(t)) < \left(\frac{1}{\theta} - \frac{1}{p^*}\right)\left(m_0 C_{p^*}^{\tau}\right)^{\frac{p^*}{p^*-p\tau}}$$

The rest of the proof follows in a similar manner to that described in Theorem (5.3.1). □

5.4.2 Proof of Theorem (5.4.2)

Proof. The proof of the theorem is similar to that of Theorem (5.3.2). □

Statements and Declarations

Conflicts of Interest Statement

I as the sole author of this article certify that I have no affiliations with or involvement in any organization or entity with any financial interest (such as honoraria; educational grants; participation in speakers' bureaus; membership, employment, consultancies, stock ownership, or other equity interest; and expert testimony or patent-licensing arrangements), or non-financial interest (such as personal or professional relationships, affi liations, knowledge or beliefs) in the subject matter or materials discussed in this manuscript.

Data Availability Statement

I as the sole author of this aarticle confirm that the data supporting the findings of this study are available within the article [and/or] its supplementary materials.

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