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Exploring Novel Edge Connectivity in Graph Theory and its Impact on Eulerian Line Graphs

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Abstract: ***This study introduces a groundbreaking concept in graph theory: Edge Connectivity, which aids in predicting the Eulerity of a line graph without the need to draw the line graph $L(G)$of a connected graph G. The paper not only delineates this novel concept but also discusses its salient features and applications, emphasizing its importance in graph theory. Through rigorous theoretical analysis, the study demonstrates the potential of Edge Connectivity in simplifying and enhancing the understanding of complex graph relationships, thereby opening new avenues for research in this field.***

Keywords: Line Graph, Edge Connectivity, Dual Graph, Link Vertex, Eulerian Graphs, Graph Theory Applications

Abbreviations:
1) G: – The given connected finite graph.
2) $L(G)$: – The corresponding line graph.
3) $E_c(e_i)$: – External connectivity of an Edge e_i
4) $D(G)$: – Dual of a given graph

1. Introduction

We are already acquainted with the new concepts of (1) Link vertex (2) Linear graph L(G) of the given graph

These concepts help tracing the given Euler graph with all basic properties known (1) Evenness of all the vertices (2) Union of finite number of cycles.

Once we have the graph G, then we construct its line graph. The question whether the line graph is Euler. In fact, the process of drawing line graph of a given graph is not very quick and needs much attention and then arises the question of Eulerity check.

2. External Connectivity of an Edge:

External connectivity of an edge is a novel concept and shall prove very important in further development of graph theory. It helps clarify the structure of the given graph and its associated graphs like line graph and dual graph.

Let $G = < V, E >$ be a given connected graph with n vertices and k number of edges; $(m, k \in N)$ External connectivity of an edge e_i of the set E $[\, i = 1, 2 \dots k]$Is the sum of number of edges incident on end vertices of the edge e_i.This is denoted as $E_c(e_i) = m_1 + m_2 = m$ where m_1 and m_2 are the number of edges from the end vertices of the said edge;$m_1 + m_2 \in N \cup \{0\}$

Case – 1

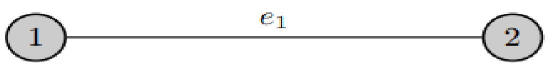

Figure 1: $E_c(e_1) = 0$

Case – 2

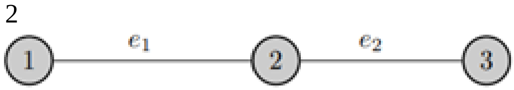

Figure 2: $E_c(e_1) = 1$, $E_c(e_2) = 1$

Case – 3

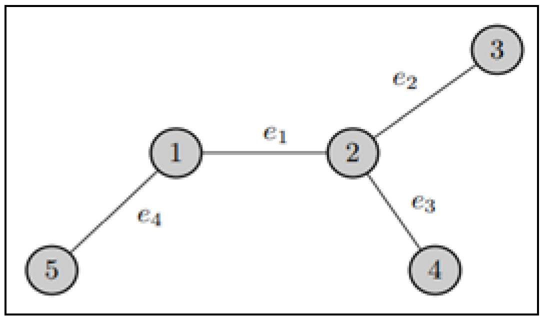

Figure 3: $E_c(e_1) = 1 + 2 = 3$, $E_c(e_2) = 0 + 2 = 2$

It is time to discuss some features which requires some principles in the form of theorems.

Theorem 1: A closed connected graph with n vertices and n edges such that each edge has external connectivity equal to two (2) then it is an Euler graph as well as a Hamiltonian graph.

Proof: Let $G = < V, E >$ be a closed connected graph with n vertices and n edges. As given, each edge, say an edge e_j is incident on two vertices v_{j-1} and v_j for all $j = 1, 2, \dots \dots, n$ has external edge connectivity = 2 implies that there are two distinct edges $e_{j-1}$and e_{j+1} which are correspondingly connected to the vertices v_{j-1} and v_{j+1}.

This in turn establishes that degree of each vertex is two. This proves that it is an Euler graph.

In the light of external edge connectivity and closeness property of the graph, there is a Hamiltonian circuit. This proves the statement.

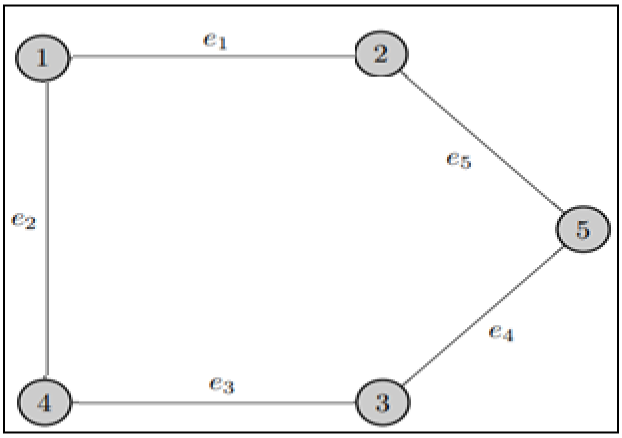

Figure 4: Euler + Hamiltonian Graph

The graph G with above mentioned feature is both an EULER as well as HAMILTONIAN

2.1 Lemma 1: In addition to the status of the above properties, if we join any two non- adjacent vertices by an edge then, edge connectivity becomes odd [one more than two, violating the rim conditions] which jeopardizes Eulerity. At the same time Hamiltonian property is preserved as the new graph becomes the super graph of the original graph G

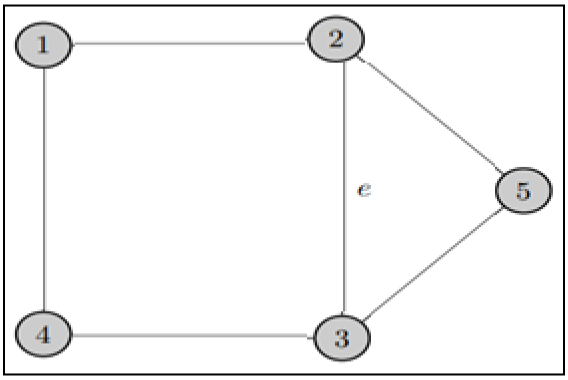

Figure 5: G + 2-4: Hamiltonian but not Euler Graph

Note: continuing this way constructing super graphs, so long as the fundamental theme of preserving evenness of edge connectivity of each edge is observed the graph continues to remain Euler

1) Line Graph and Euler Property:

In the following unit, we define the line graph of the given graph G and try to apply the tenets of external edge connectivity to check Eulerity of the line graph.

Definition- Line Graph of the given Graph:

Let $G = < V, E >$ be the given graph with vertex set $V = \{v_1, v_2, \dots \dots, v_n\}$ and the set of edges $E = \{e_1, e_2, \dots \dots, e_m\}$ — n vertices and m edges. Corresponding to each edge $e_j, j = 1$ *to* 4 there is a corresponding set $X = \{x_1, x_2, \dots \dots, x_n\}$ of vertices of the said line graph, denoted as $L(G)$, such that if any two edges e_i and e_j of the graph G are incident on a vertex then in accordance to that there are vertices x_i and x_j which are adjacent to each other in the graph $L(G)$.

i.e. (1) The number of edges of G equals number of vertices of the line graph $L(G)$.

(2) There is a strict one –one sequential correspondence between the set $E = \{e_1, e_2, \dots \dots, e_n\}$ of G and the set of vertices $X = \{x_1, x_2, \dots \dots, x_m\}$ of the graph $L(G)$; so that if any two edges e_i and e_j are incident on a vertex in G parallels that there is an edge connecting the vertices x_i and x_j of the line graph $L(G)$.

We give two illustrations

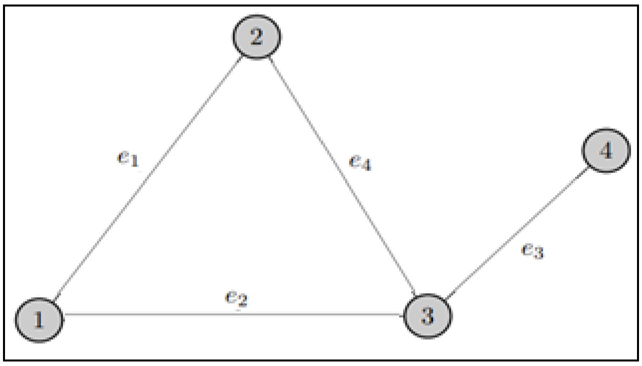

Figure 6: Graph

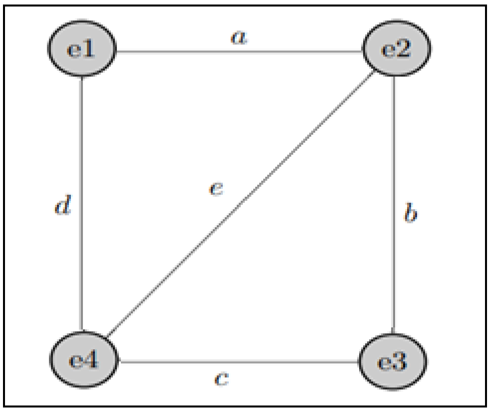

Figure 7: Line Graph

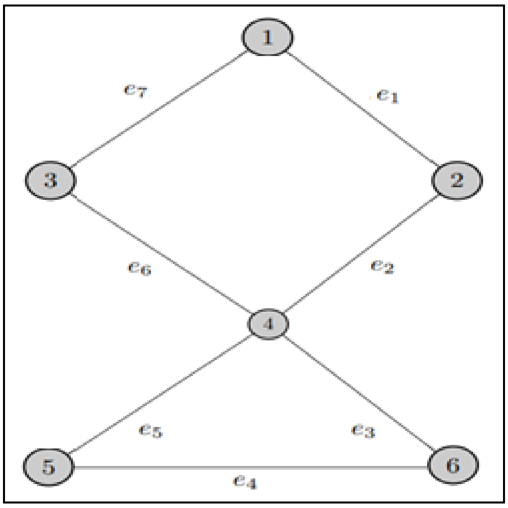

Figure 8: Graph

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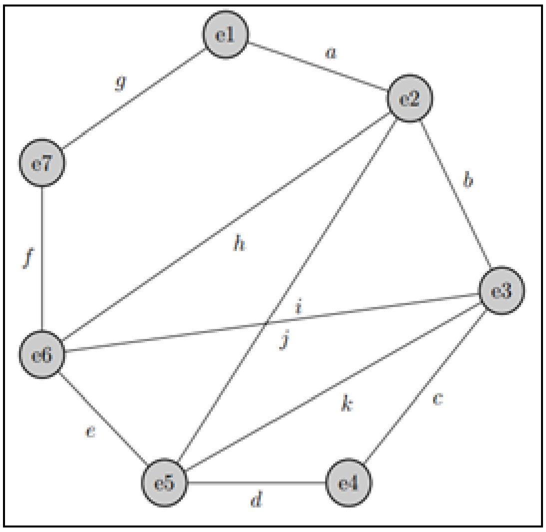

Figure 9: Line Graph

2.1 Eulerian Property of the line graph

In this section we will prove important deduction of external connectivity of edges of a graph. this helps predict Eulerity of the line graph **without drawing** the line graph $L(G)$ of the given graph G.

Theorem -2: For the given connected graph G [may not be a simple graph] if the external edge connectivity of every edge is even then its corresponding line graph is an Euler graph.

Proof: The proof depends on definition of (1) definition of external connectivity of an edge, and (2) definition of line graph of the given graph.

Consider a connected graph $G = < V, E >$ with vertex set $V = \{v_1, v_2, \dots \dots \dots, v_n\}$ and the set of edges $E = \{e_1, e_2, \dots \dots, e_m\} - n$ vertices and m edges. Let $L(G)$ be its line graph with vertex set, say $X = \{x_1, x_2, \dots \dots, x_m\}$ – each vertex of the set X of $L(G)$ has sequentially ordered correspondence with set of edges E of the set G.

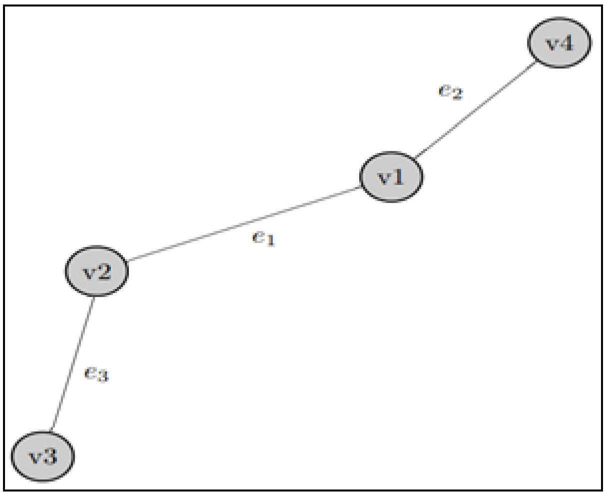

Figure 10: Graph

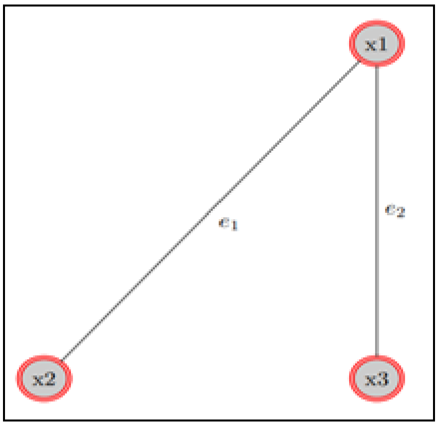

Figure 11: Line Graph

Here the edges e_1 and e_2 are incident on the vertex v_1 and accordingly the vertices x_1 and x_2 corresponding to edges e_1 and e_2 become adjacent in $L(G)$.

Theorem 3: Degree of each vertex of the line graph $L(G)$ of the given graph G equals external
Connectivity of the corresponding edge of G.

Proof: Consider a connected graph $G = < V, E >$ with vertex set $V = \{v_1, v_2, \dots \dots \dots, v_n\}$ and the set of edges $E = \{e_1, e_2, \dots \dots, e_m\} - n$ vertices and m edges. Let $L(G)$ be its line graph with vertex set, say $X = \{x_1, x_2, \dots \dots, x_m\}$ – each vertex of the set X of $L(G)$ has sequentially ordered correspondence with set of edges E of the set G.

Consider an edge e_1 in G, between the vertices v_1 *and* v_2. Also consider that the vertex v_1 is connected with different n_1 number of edges and the vertex v_2 with n_2 number of edges. It implies that deg $(v_1) = n_1 + 1$ and deg $(v_2) = n_2 + 1$

It is known that there is a corresponding vertex x_1 in $L(G)$ which is connected to $n_1 + n_2$ number of vertices in correspondence to the edges of G.

We conclude that external connectivity of the edge $e_1 = E_c(e_1) = n_1 + n_2 = \deg(x_1)$

2.2 Lemma: If the external connectivity of each edge of the given graph G is even then its corresponding line graph is an Euler graph.

[Proof of the above lemma is an immediate outcome of the above theorem. External connectivity of each edge equals the degree of the corresponding vertex of the line graph $L(G)$ and degree of each vertex being even the claim is justified.]

Note: If any edge of the given graph G has odd External connectivity, then its line graph is not an Euler graph.

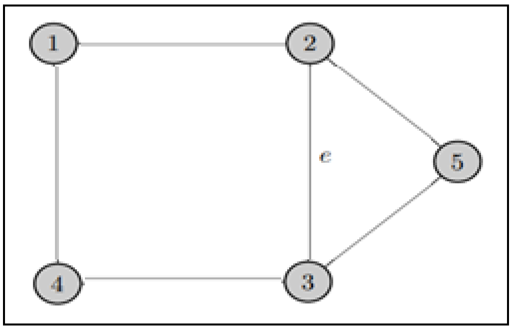

Figure 12: Line Graph

[Comment: Edge e has external connectivity is 3 and so its corresponding line graph will not be an Euler graph.

Deductions: We have certain deductions on the above-mentioned conclusion regarding external connectivity of edges of the graph.

(1) Line graph of $k_n$denoted as $L(k_n)$ for 'n' an odd integer are Euler graphs.

Complete graphs like $k_3, k_5, \dots \dots \dots, k_{2n+1}$ for $n \in N$ have $n(n-1)/2$ number of edges [n sides and $n(n-3)/2$ number of diagonals] Each edge has external connectivity $= 2 + (n-1) = n+1$It implies that for n an odd integer, external connectivity is even and claim for Eulerian property of the line graph is derived.

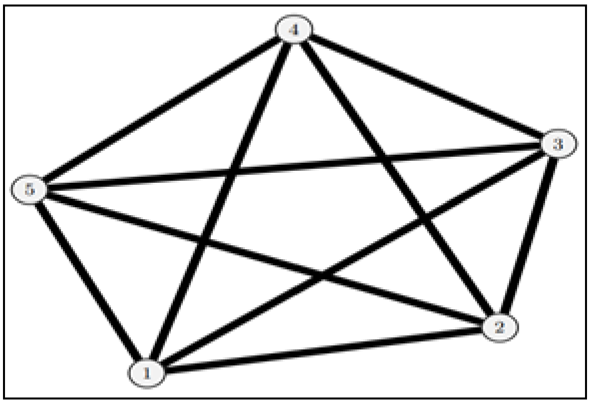

Figure 13: Complete Graph (k_5)

(2) The graphs W_{2n} for $n = 2, 3, 4, \dots \dots \dots$are anot Euler but their line graphs areEuler.

In this case, the external connectivity of each edge of$W_{2n} = 2n, \dots \dots \dots$ an even integer and so, the theorem supports the claim that the line graph is Euler.

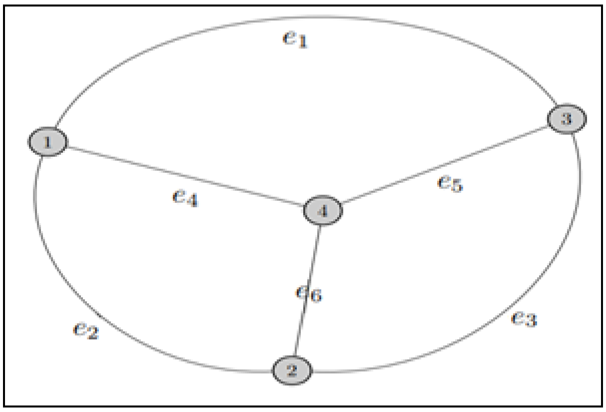

Figure 14: Wheel Graph (W_4)

[Comment: W_4 is not an Euler graph but its line graph L(W_4) is Euler. In general,

$E_c(w_{2n}) = 2n$ which is even and so $L(w_{2n})$ is an Euler graph]

(3) A regular closed graph on n vertices and degree of each vertex being equal to 2, is self-reflexive with property of its line graph.

A complete Bipartite graph $G = K_{m,n}$ is an Euler graph $(m, n\ both\ are\ even\ integer)$

Its line graph $L(K_{m,n})$ is also an Euler graph.

$$E_c(e) = n = even$$

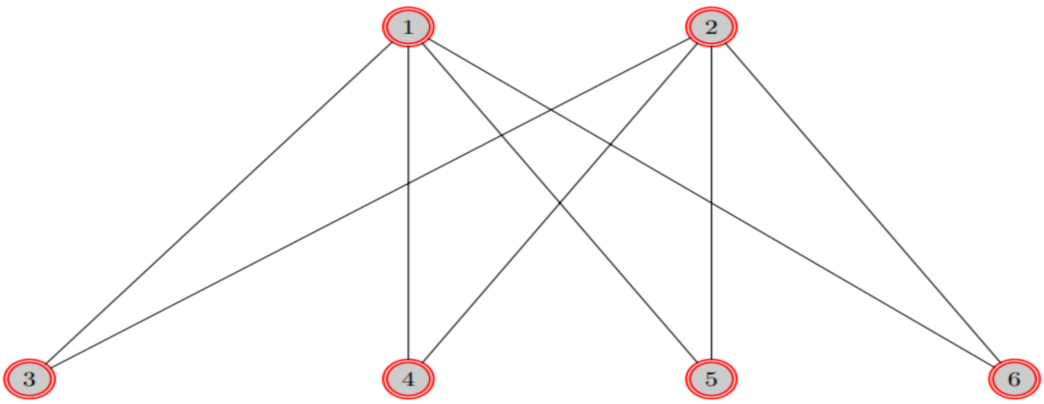

Figure 15: Complete Bipartite Graph $(K_{2,4})$

$K_{2,4}$ is an Euler graph. $E_c(e_i) = 4 \therefore L(K_{2,4})$ is an Euler graph.

3. Conclusion

This study exploration of Edge Connectivity in graph theory presents a significant advancement in understanding Eulerian properties of line graphs. The findings demonstrate how this novel concept simplifies the analysis of complex graph structures, offering new insights and potential applications in graph theory. This research opens new horizons for further exploration, promising advancements in both theoretical and applied aspects of the field.

References

[1] Rajeshri Prajapati, Amit Parikh, Pradeep Jha, "Special Graphs of Euler's Family* and Tracing Algorithm-(A New Approach)",The Ciência & Engenharia - Science & Engineering Journal – ISSN: 0103-944X, Volume 11 Issue 1, pp: 2243 – 2251, 2023

[2] John Clark Derek Allan Holton, "A FIRST LOOK AT GRAPH THEORY", First Edition, Allied Publishers LTD, ISBN 81-7023-463-8

[3] M. Venkatachalapathy, K. Kokila and B. Abar, "Some Trends In Line Graphs", Advances in Theoretical and Applied Mathematics – ISSN 0973-4554, Volume 11, Number 2, pp. 171-178, 2016

[4] Linda Eroh, Cong X. Kang, and Eunjeong Yi, "Metric Dimension and Zero Forcing Number of Two Families of Line Graphs", Mathematica, April 16, 2013

[5] Aruna R , Madhu N.R & Shashidhar S.N,"EULERIAN GRAPHS AND ITS APPLICATIONS",International Journal of Advance Research in Science and Engineering Vol. No. 6, February 2017

[6] Dr Sudhir Prakash Srivastava, "Study of Different algorithm in Euler Graph", International Journal on Recent and Innovation Trends in Computing and Communication – Volume: 4 Issue: 12, December 2016,

[7] Ashish Kumar," A study on Euler Graph and it's applications", International Journal of Mathematics Trends and Technology (IJMTT) – Volume 43 Number 1- March 2017

[8] Pandey, J. K., Veeraiah, V., Talukdar, S. B., Talukdar, V. B., Rathod, V. M., & Dhabliya, D. (2023). Smart City Approaches Using Machine Learning and the IoT. In Handbook of Research on Data-Driven Mathematical Modeling in Smart Cities (pp. 345–362). IGI Global.

[9] Anand, R., Ahamad, S., Veeraiah, V., Janardan, S. K., Dhabliya, D., Sindhwani, N., & Gupta, A. (2023). Optimizing 6G Wireless Network Security for Effective Communication. In Innovative Smart Materials Used in Wireless Communication Technology (pp. 1–20). IGI Global.

[10] Abdul Samad Ismail , Roslan Hasni and K. G. Subramanian", SOME APPLICATIONS OF EULERIAN GRAPHS", International Journal of Mathematical Science Education © Technomathematics Research Foundation Vol. 2, No. 2, 1 – 10, 2009

[11] Kumar, A., Dhabliya, D., Agarwal, P., Aneja, N., Dadheech, P., Jamal, S. S., & Antwi, O. A. (2022). Research Article Cyber-Internet Security Framework to Conquer Energy-Related Attacks on the Internet of Things with Machine Learning Techniques.

[12] Talukdar, V., Dhabliya, D., Kumar, B., Talukdar, S. B., Ahamad, S., & Gupta, A. (2022). Suspicious Activity Detection and Classification in IoT Environment Using Machine Learning Approach. 2022 Seventh International Conference on Parallel, Distributed and Grid Computing (PDGC), 531–535. IEEE