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Degree of Approximation in Generalized Zygmund Class Using (N, Pn) (E, 1) Means of Fourier Series

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Abstract: ***This article delves into the theory of summability, specifically focusing on the generalization of the limit concept for sequences and series influenced by linear means of sequence or series. Researchers have shown significant interest in exploring the degree of approximation of functions in the Lipschitz and Zygmund classes using various means of Fourier series and conjugate Fourier series. The paper introduces the generalized Zygmund class and investigates the degree of approximation of functions within it using the (N, Pn) (E, 1) means of Fourier series. The article presents a theorem that provides an expression for the degree of approximation in this context. The proof of the theorem involves several lemmas and mathematical techniques. This work contributes to the understanding of approximation theory and the application of Fourier series in generalized Zygmund classes.***

Keywords: Degree of approximation, Generalized Zygmund class, (N, Pn) mean , (E, 1) mean, (N, Pn) (E, 1) mean . summability method

MSC: 41A24, 41A25, 42B05, 42B08

1. Introduction

The theory of summability is concern about the generalisation of concept of the limit of a sequence of series which is affected by an auxiliary of linear means of sequence or series The degree of approximation of function in Lipschitz and Zygmund class using different means of Fourier series ans conjugate Fourier series have been great interest among the researcher.The generalized Zygmund class was introduced by Leindler [3] Moricz [4], moricz and Nemeth [5] etc. Recently Singh et. al. [7] Mishra et. al. [6], Kim [2] find the results in Zygmund class by using different summability Means. In this paper we find the degree of approximation of function in the generalized Zygmund class by (N, Pn) (E, 1) means of Fourier series.

2. Definition

Let f be a periodic function of period 2π integrable in the sense of Lebesgue over [π, - π]. Then the Fourier series of f given by

$$f(t) \approx \frac{a_o}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx) \quad \text{.......(2.1)}$$

Zygmund class z is defined as

$$Z = \{f \in C[-\pi,\pi] \; |f(x+t) + f(x-t) - 2f(x)| = O(t).$$

In this paper , we introduce a generalized Zygmund $Z^w(\alpha,\gamma)$ defined as

$$Z^w(\alpha,\gamma) = \left\{ f \in C[-\pi,\pi] \left(\int_{-\pi}^{\pi} |f(x+t) + f(x-t) - 2f(x)|^{\gamma} dx \right)^{\frac{1}{\gamma}} = O(|t|^{\alpha}\omega(t)) \right\}$$

...........(2.2)

Where $\alpha \geq 0$, $\gamma \geq 1$ and ω is a continuous non negative and non decreasing function. If we take $\alpha = 1$, $\omega = constant\ and\ \gamma \to \infty$, then $Z^w(\alpha,\gamma)$ class reduces to the z class.

We write $through\ the\ paper$

$$\emptyset_x(t) = f(x+t) - 2f(x) + f(x-t) \quad \text{....... (2.3)}$$

$$K_n(t) = \frac{1}{2\pi P_n} \sum_{k=o}^{n} \frac{p_{n-k}}{(1+q)^k} \left\{ \sum_{v=0}^{k} \binom{k}{v} q^{k-v} \frac{\sin(v+\frac{1}{2})t}{\sin(\frac{t}{2})} \right\} \quad (2.4)$$

3. Main Result

In this paper we prove the following theorem.

Theorem - Let f be a 2π periodic function, Lebesgue integrable in $[0, 2\pi]$ and belonging to generalized Zygmund class $Z_r^{(w)}$ $(r \geq 1)$. Then the degree of approximation of function f by (N, Pn) (E,1) product mean of Fourier series is given by

$$E_n(f) = inf\ \|t_n^{NE} - f\|_r^v = o\left(\int_{\frac{1}{n+1}}^{\pi} \frac{w(t)}{tv(t)} dt \right)$$

Where $\omega(t)\ and\ v(t)$ denotes the Zygmund modulai of continuity such that $\frac{w(t)}{v(t)}$ is positive and increasing.

4. Lemma

To prove the theorem we need the following lemma.

Lemma 4(a) - For $0 \leq t \leq \frac{\pi}{n+1}$ we have $\sin nt = n\ sint$

$$|K_n(t)| = o(n) \quad \text{.......(4.1)}$$

Proof - For $0 \leq t \leq \frac{\pi}{n+1}$ and $\sin nt = n\ sint$ then

$$|K_n(t)| = \left| \frac{1}{2\pi P_n} \sum_{k=o}^{n} \frac{p_{n-k}}{(2)^k} \left\{ \sum_{v=0}^{k} \binom{k}{v} \frac{\sin(v+\frac{1}{2})t}{\sin(\frac{t}{2})} \right\} \right|$$

$$\leq \frac{1}{2\pi P_n} \left| \sum_{k=o}^{n} \frac{p_{n-k}}{(2)^k} \left\{ \sum_{v=0}^{k} \binom{k}{v} \frac{(2v+1)\sin(\frac{t}{2})}{\sin(\frac{t}{2})} \right\} \right|$$

$\leq \frac{1}{2\pi P_n} \left| \sum_{k=o}^{n} \frac{p_{n-k}}{(2)^k} (2k+1) \{ \sum_{v=0}^{k} \binom{k}{v} \} \right|$
$\leq \frac{1}{2\pi P_n} |\sum_{k=o}^{n} p_{n-k} (2k+1)|$
$= \frac{(2n+1)}{2\pi P_n} |\sum_{k=o}^{n} p_{n-k} |$
$= o(n)$

Lemma 4(b) - For $\frac{\pi}{n+1} \leq t \leq \pi , sin\frac{t}{2} \geq \frac{t}{\pi}$ $and\ sin\ nt \leq 1$ we have
$| K_n(t)| = o\left(\frac{1}{t}\right)$ ....(4.2)
Proof - For $\frac{\pi}{n} \leq t \leq \pi , sin\frac{t}{2} \geq \frac{t}{\pi}$ $and\ sin\ nt \leq 1$

$$| K_n(t)| = \left| \frac{1}{2\pi P_n} \sum_{k=o}^{n} \frac{p_{n-k}}{(2)^k} \left\{ \sum_{v=0}^{k} \binom{k}{v} \frac{\sin(v+\frac{1}{2})t}{\sin(\frac{t}{2})} \right\} \right|$$

$$\leq \frac{1}{2\pi P_n} \left| \sum_{k=o}^{n} \frac{p_{n-k}}{(2)^k} \left\{ \sum_{v=0}^{k} \binom{k}{v} \frac{\pi}{t} \right\} \right|$$
$$\leq \frac{1}{2tP_n} \left| \sum_{k=o}^{n} p_{n-k} \right|$$
$$= o\left(\frac{1}{t}\right)$$

Lemma 4(c) – Let $f \in Z_p^{(w)}$ then for $0 < t \leq \pi$
(i) $\|\phi(.,t)\|_p = o(w(t))$
(ii) $\|\phi(.+y,t) + \phi(.-y,t) - 2\phi(.,t)\|_p = \begin{cases} o(w(t) \\ o(w(y) \end{cases}$
(iii) If $\omega(t)$ and $v(t)$ are defined as in theorem then
$$\|\phi(.+y,t) + \phi(.-y,t) - 2\phi(.,t)\|_p = \left\{ v(y) \frac{\omega(t)}{v(t)} \right.$$

Where $\phi(x,t) = f(x+t) + f(x-t) - 2f(x)$.

5. Proof of Theorem 3

Let $S_n(x)$ denotes the partial sum of Fourier series given in (2.1) then we have

$S_n(x) - f(x) = \frac{1}{2\pi} \int_0^\pi \emptyset(t) \frac{\sin(n+\frac{1}{2})t}{sin\frac{t}{2}} dt$ ....(5.1)

The (E, q) transform E_n^q of S_n is given by

$E_n^1 - f(x) = \frac{1}{2\pi(2)^n} \int_0^\pi \emptyset(t) \left\{ \sum_{k=0}^{n} \binom{n}{k} \frac{sin\left(k+\frac{1}{2}\right)t}{\sin(\frac{t}{2})} \right\} dt$ ......(5.2)

The (N, Pn) (E,q) transform of $S_n(x)$ is given by

$$t_n^{NE}(f) - f(x) = \frac{1}{2\pi P_n} \sum_{k=0}^{n} \left[\frac{p_{n-k}}{(2)^k} \int_0^\pi \emptyset(t) \left\{ \sum_{v=0}^{k} \binom{k}{v} \frac{sin\left(v+\frac{1}{2}\right)t}{\sin(\frac{t}{2})} \right\} dt \right]$$ ......(5.3)

$= \int_0^\pi \emptyset(t)\ k_n(t)$ ....(5.4)

Let $l_n(x) = t_n^{NE} - f(x) = \int_0^\pi \emptyset(x,t)\ k_n(t)dt$ then

$$l_n(x+y) + l_n(x-y) - 2\, l_n(x) = \int_0^\pi [\phi(x+y,t) + \phi(x-y,t) - 2\phi(x,t)]\ k_n(t)\, dt$$

Using the generalized Minkowaski's inequality we get

$$\|\phi(.+y,t) + \phi(.-y,t) - 2\phi(.,t)\|_p = \left\{ \frac{1}{2\pi} \int_0^{2\pi} |l_n(x+y) + l_n(x-y) - 2\, l_n(x)|^p\, dx \right\}^{\frac{1}{p}}$$

$$= \left\{ \frac{1}{2\pi} \int_0^{2\pi} \left| \int_0^\pi [\phi(x+y,t) + \phi(x-y,t) - 2\phi(x,t)]\ k_n(t)\, dt \right|^p dx \right\}^{\frac{1}{p}}$$

$$\leq \int_0^\pi \left\{ \frac{1}{2\pi} \int_0^{2\pi} |[\phi(x+y,t) + \phi(x-y,t) - 2\phi(x,t)]\ k_n(t)|^p\, dx \right\}^{\frac{1}{p}} dt$$

$$= \int_0^\pi (|\ k_n(t)|^p)^{\frac{1}{p}} \left\{ \frac{1}{2\pi} \int_0^{2\pi} |[\phi(x+y,t) + \phi(x-y,t) - 2\phi(x,t)]|^p\, dx \right\}^{\frac{1}{p}} dt$$

$$= \int_0^\pi \|\phi(.+y,t) + \phi(.-y,t) - 2\phi(.,t)\|_p |\ k_n(t)|\, dt$$

$$= \int_0^{\frac{1}{n+1}} \|\phi(.+y,t) + \phi(.-y,t) - 2\phi(.,t)\|_p |\ k_n(t)|\, dt + \int_{\frac{1}{n+1}}^{\pi} \|\phi(.+y,t) + \phi(.-y,t) - 2\phi(.,t)\|_p |\ k_n(t)|\, dt$$

$= I_1 + I_2\ (say)$ ....(5.5)

Using lemma 4(a) and 4(c) and the monotonically of $\frac{\omega(t)}{v(t)}$ with respect to t we have

$$I_1 = \int_0^{\frac{1}{n+1}} \|\phi(.+y,t) + \phi(.-y,t) - 2\phi(.,t)\|_p |\ k_n(t)|\, dt$$

$$= \int_0^{\frac{1}{n+1}} o\left(v(y) \frac{\omega(t)}{v(t)} \right) o(n)\, dt$$

$$= o\left(nv(y) \int_0^{\frac{1}{n+1}} \frac{\omega(t)}{v(t)}\, dt \right)$$

Using second mean value theorem of integral we have

$$I_1 \le o\left(nv(y) \int_0^{\frac{1}{n+1}} \frac{\omega(t)}{v(t)}\, dt \right)$$

$$= o\left(\frac{n}{n+1}\, v(y)\, \frac{\omega\left(\frac{1}{n+1}\right)}{v\left(\frac{1}{n+1}\right)} \right)$$

$$= o\left(v(y)\, \frac{\omega\left(\frac{1}{n+1}\right)}{v\left(\frac{1}{n+1}\right)} \right) \qquad \ldots\ldots\ldots\ldots(5.6)$$

For I_2 using lemma 4(b) and 4(c) we have

$$I_2 = \int_{\frac{1}{n+1}}^{\pi} \|\phi(.+y,t) + \phi(.-y,t) - 2\phi(.,t)\|_p |\, k_n(t)|\, dt$$

$$= o\left(\int_{\frac{1}{n+1}}^{\pi} \left(v(y) \frac{\omega(t)}{v(t)} \right) \frac{1}{t}\, dt \right)$$

$$= o\left(v(y) \int_{\frac{1}{n+1}}^{\pi} \left(\frac{\omega(t)}{t\, v(t)} \right) dt \right) \qquad \ldots\ldots\ldots\ldots(5.7)$$

From (5.5) (5.6) and (5.7) we get

$$\|l_n(.+y) + l_n(.-y) - 2l_n(.)\|_p = o\left(v(y)\, \frac{\omega\left(\frac{1}{n+1}\right)}{v\left(\frac{1}{n+1}\right)} \right) + o\left(v(y) \int_{\frac{1}{n+1}}^{\pi} \left(\frac{\omega(t)}{t\, v(t)} \right) dt \right)$$

$$\sup_{y \ne 0} \frac{\|l_n(.+y) + l_n(.-y) - 2l_n(.)\|_p}{v(y)} = o\left(\frac{\omega\left(\frac{1}{n+1}\right)}{v\left(\frac{1}{n+1}\right)} \right) + o\left(\int_{\frac{1}{n+1}}^{\pi} \left(\frac{\omega(t)}{t\, v(t)} \right) \right). \qquad \ldots\ldots\ldots(5.8)$$

Again using lemma we have

$$\|l_n(.)\|_p \le \left(\int_0^{\frac{1}{n+1}} + \int_{\frac{1}{n+1}}^{\pi} . \right) \|\phi(.,t)\| |K_n(t)|\, dt$$

$$= o\left(n \int_0^{\frac{1}{n+1}} \omega(t) dt \right) + o\left(\int_{\frac{1}{n+1}}^{\pi} \frac{\omega(t)}{t}\, dt \right)$$

$$= o\left(\frac{n}{n+1}\, \omega\left(\frac{1}{n+1}\right) \right) + o\left(\int_{\frac{1}{n+1}}^{\pi} \frac{\omega(t)}{t}\, dt \right)$$

$$= o\left(\omega\left(\frac{1}{n+1}\right) \right) + o\left(\int_{\frac{1}{n+1}}^{\pi} \frac{\omega(t)}{t}\, dt \right) \qquad \ldots\ldots\ldots\ldots(5.9)$$

From (5.8) and (5.9) we have

$$\|l_n(.)\|_p^v = \|l_n(.)\|_p + \sup_{y \ne 0} \frac{\|l_n(.+y) + l_n(.-y) - 2l_n(.)\|_p}{v(y)}$$

$$= o\left(\omega\left(\frac{1}{n+1}\right) \right) + o\left(\int_{\frac{1}{n+1}}^{\pi} \frac{\omega(t)}{t}\, dt \right) + o\left(\frac{\omega\left(\frac{1}{n+1}\right)}{v\left(\frac{1}{n+1}\right)} \right) + o\left(\int_{\frac{1}{n+1}}^{\pi} \left(\frac{\omega(t)}{t\, v(t)} \right) \right)$$

$$= \sum_{i=1}^{4} J_i$$

Now we write J_1 in terms of J_3 *and* J_2 , J_3 *in* term of J_4.

In view of the monotonicity of v(t) we have

$$\omega(t) = \left(\frac{\omega(t)}{v(t)} \right), v(t) \le v(\pi) \left(\frac{\omega(t)}{v(t)} \right) = o\left(\frac{\omega(t)}{v(t)} \right) \text{ for } 0 < t \le \pi$$

Therefore we can write

$$J_1 = o(J_3)$$

Again using monotonicity of v(t)

$$J_2 = \int_{\frac{1}{n+1}}^{\pi} \frac{\omega(t)}{t}\, dt = \int_{\frac{1}{n+1}}^{\pi} \left(\frac{\omega(t)}{t\, v(t)} \right) dt$$

$$\le v(\pi) \int_{\frac{1}{n+1}}^{\pi} \left(\frac{\omega(t)}{t\, v(t)} \right) dt = o(J_4)$$

$$\ldots\ldots\ldots\ldots (5.10)$$

Using the fact $\frac{\omega(t)}{v(t)}$ is positive and non decreasing, we have

$$J_4 = \int_{\frac{1}{n+1}}^{\pi} \left(\frac{\omega(t)}{t\, v(t)} \right) dt$$

$$= \frac{\omega\left(\frac{1}{n+1}\right)}{v\left(\frac{1}{n+1}\right)} \int_{\frac{1}{n+1}}^{\pi} \left(\frac{1}{t} \right) dt \ge \frac{\omega\left(\frac{1}{n+1}\right)}{v\left(\frac{1}{n+1}\right)}$$

Therefore we can write

$$J_3 = o(J_4)$$

So we have

$$\|l_n(.)\|_p^v = o(J_4) = o\left(\int_{\frac{1}{n+1}}^{\pi} \left(\frac{\omega(t)}{t\, v(t)} \right) dt \right)$$

Hence

$$E_n(f) = \inf \|l_n(.)\|_p^v = o\left(\int_{\frac{1}{n+1}}^{\pi} \left(\frac{\omega(t)}{t\, v(t)} \right) dt \right)$$

This complete the proof.

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