# Exploring the Fascinating World of Kneser Graphs: Characteristics, Construction, and Applications

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**Abstract:** ***Kneser graphs are an intriguing class of combinatorial structures in graph theory. They bear the name of the German mathematician Martin Kneser. Numerous disciplines, including as combinatorics, topology, and algebraic geometry, have found uses for these graphs. An overview of Kneser graphs, their characteristics, construction techniques, and some of its famous applications are the goals of this study.***

**Keywords:** Kneser graph

## 1. Introduction

In the field of graph theory, there is an intriguing class of combinatorial structures known as Kneser graphs, which bear the name of the German mathematician Martin Kneser. Numerous disciplines, including as combinatorics, topology, and algebraic geometry, have found uses for these graphs.

Years of mathematical research were spurred on by the following questions and potential generalizations: “How big may $C$ be if $C$ is a collection of mutually overlapping $k$-subsets of a given $n$-set?What is the structure of $C$ in the worst-case scenario?” Although Erdos, Ko, and Rado claimed to have the answer to this question in 1938 [14], they did not publish their findings until 1961 [15]. If equality holds, then $C$ is the collection of all $k$-subsets having some fixed member of the specified $n$-set for $2k < n$. One must have $|C| \leq \binom{n-1}{k-1}$ for $2k \leq n$. This bound can be established for $2k = n$ by selecting any one $k$-set from each complementary pair.

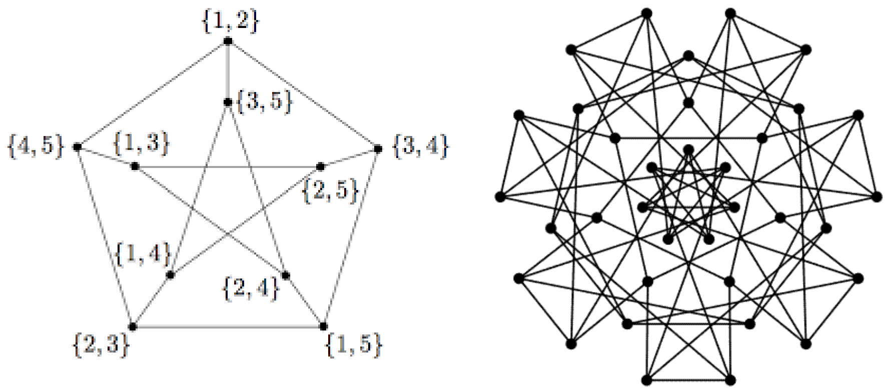
In his renowned proof of Kneser's conjecture, Lovász [1] introduced Kneser graphs. He established the existence of the field of topological combinatorics by demonstrating that the chromatic number of $K(n,k) = n - 2k + 2$ using the Borsuk-Ulam theorem. Some crucial Kneser graph features are outlined in this paper. According to the well-known Erdos-Ko-Rado [2] theorem, the maximum independent set in $K(n,k)$ has size $\binom{n-1}{k-1}$. Also, the graph $K(n,k)$ is vertex-transitive, meaning that from the perspective of any vertex, it "looks the same," and all vertices have degree $\binom{n-k}{k}$. Finally, observe that the Kneser graph $K(n,k)$ does not include cliques of size c, but when $n \geq ck$, it contains such cliques.

This interesting class of graphs have been studied by a lot many authors as evident from [5]. Different authors have studied different invariants of Kneser graphs. Hamiltonicity of Kneser graphs was studied by Bellmann et.al.[3], Chen et.al.[4] and Merino et.al [5]. Park et.al [6] and Balakrishnan et. al [7] studied the toughness and Weiner index of Kneser graphs respectively. The diameter of Kneser was studied by Valencia et.al [10] and extremal problems concerning Kneser graphs was studied by Frankl et.al [8]. Chromatic number and its variants of Kneser graphs were discussed by Zhu et.al [9] and Stahl [11]. In this paper, we review the structure and some properties of Kneser available in literature. For all basic definitions not mentioned here we refer to [12]

## 2. Definition and Preliminaries

- **Kneser graph**: For integers $k \geq 1$ and $n \geq 2k + 1$, the *Kneser graph*$K(n,k)$ has as vertices all $k$-element subsets of $[n]$ = {1, 2, . . ., n}, and an edge between any two sets A and B that are disjoint, i.e.,$A \cap B = \varphi$.
- **Generalized Kneser graphs: [13]** A generalized Kneser graph is a graph that we denote here by$GK(n: k_1,\dots,k_l)$. Each vertex correspond to a series of sets, $(S_1,\dots,S_l)$ where ; $\varphi \subset S_1 \subset \cdots . S_l$and $0 < k_1 < \cdots < k_l < n$, $|X| = n$, $|S_i| = k_i$for all $i, (1 \leq i \leq l) k_0 = 0$ and $k_{l+1} = |X|$.Two vertices are adjacent if either the condition $S_i \cap T_j = \varphi$ or $S_i \cap T_j = X$ holds.
- **Bipartite Kneser graphs:** For integers $k \geq 1$ and $n \geq 2k + 1$, the bipartite Kneser graph$H(n, k)$ has as vertices all $k$- element and $(n - k)$-element subsets of $[n]$, and an edge between any two sets $A$ and $B$ that satisfy $A \subseteq B$.

**Figure 1:** (a) is$K(5,2)$which is the Petersen graph and Fig: 1(b) is $K(7,3)$.




(a) (b)

**Figure 1: (a)** K (5, 2) **(b)** K (7, 3)

Some results on the connectivity and diameter of Kneser graphs and generalized Kneser graphs are as follows.

**Theorem 1**:[13]The generalized Kneser graph $\Gamma = GK(n : k_1, \ldots, k_l\ )$ is connected unless there exists $i, j$ such that

$$k_i + k_j = n \quad \text{and}\ n > 2.$$

**Theorem 2**:[13] For two vertices $A$, $B$ in $K(n, k)$, if they are joined by a path of length 2p,then $|A \cap B| \geq k - (n - 2k)p$.

**Theorem 3**:[13] Let $A$, $B$ be two vertices of $K(n, k)$ joined by a path of length $2p+1$.Then, $|A \cap B| \leq (n - 2k)p$.

**Theorem 4**: [13] Let n, k be positive integers where $n > 2k$. Let $k \geq 2$, and $n - 3k \geq -1$. Then the diameter of the Kneser graph is equal to 2.

**Theorem 5**:[13] For a Kneser graph $K(n, k)$, let $n > 2k$.Then, the diameter of the Kneser graph is $\left\lceil \frac{k-1}{n-2k} \right\rceil + 1$.

It was Kneser who conjectured the following result about the chromatic number of Kneser graph and it was proved by Lovasz in [1].

**Theorem 6:** [1]The chromatic number of Kneser graph, $\chi(K(n; k) = n - 2k + 2$.

There was a lot of work concerning the hamiltonicity of Kneser graphs. First it was proved for subcases like odd graphs, bipartite Kneser graphs. Recently, Merino et.al. [5] settled the conjecture on Hamiltonian cycles of Kneser graphs positively and in a general manner as follows. In [5, they illustrated a method for construction of Hamiltonian cycle in Kneser graphs, generalized Kneser graphs and also bipartite Kneser graphs.

**Theorem 7**: [5]For all $k \geq 1$ and $n \geq 2k + 1$, the Kneser graph $K(n, k)$ has a Hamilton cycle, unless it is the Petersen graph, i.e., $(n, k) = (5,2)$.

**Theorem 8**:[5] If $K(n, k)$ admits a Hamilton cycle, then $H(n, k)$ admits a Hamilton cycle or path.

- **Toughness of a graph: [6]** Toughness of a graph is a measure of connectivity. It is defined as $t(G) = min_S \frac{|S|}{c(G \backslash S)}$, where $S$ ranges over all vertex cuts of $G$ and $c(G \backslash S)$ the number of components in $G \backslash S$. A graph is *t-tough if* $t(G) \geq t$.

In [6], Park et.al, investigated the toughness of Kneser graphs and obtained the following results.

**Theorem 9**: [6] Let $k \in \{3, 4\}$. The toughness of the Kneser graph $K(n, k)$ equals

$$t\big(K(n, k)\big) = \frac{n}{k} - 1$$

for any $n \geq 2k + 1$. Moreover, any subset of vertices $S$ satisfying $t(K(n, k)) = \frac{|S|}{c(K(n,k) \backslash S)}$ must be the complement of a maximum independent set in $K(n, k)$.

They also obtained the spectrum of adjacency matrix of Kneser graph as follows.

**Theorem 10**: [6] The eigen values of the adjacency matrix of $K(n, k)$ are $(-1)^j \binom{n-k-j}{k-j}$ with multiplicities $\binom{n}{j} - \binom{n}{j-1}$ for $j = 0, 1, \ldots, k$.

Due to its vertextransitive and edge-transitive nature, Kneser graph exhibits several special features. An edge-transitive graph has vertex-connectivity equal to its minimal degree, according to Tindell's [30] proof. This implies that the vertex-connectivity for a connected, $l$-regular edge-transitive graph is $l$. Given that Kneser graph $K(n, k)$has these characteristics for $n \geq 2k + 1$ and k $\geq$2, the following result is proved in [6].

**Theorem 11**: [6] For any vertex $u$ in $K(n, k)$, the graph obtained by deleting $u$ and its neighborhood is connected.

**Theorem 12:** [6] Let $G$ be a non-bipartite $l$-regular graph. Assume the edge-connectivity of $G$ is, and the only disconnecting sets of $l$ edges are the $l$ edges incident to a vertex. Then for any vertex cut $S$, $|S| > c(G \setminus S)$. In particular, $t(K(n, k)) > 1$ for any $n \geq 2k + 1$.

**Theorem 13**: [6] Let S be a vertex cutset achieving the toughness of $K(n,k)$. Then every component in $K(n,k) \setminus S$ is either a singleton, $K_2$, or is biconnected.

**Theorem 14**:[16] For $n \geq 9$, $t(K(n,3)) = \frac{n}{3} - 1$. Moreover, if $S$ is a disconnecting set of $K(n,3)$ such that $\frac{|S|}{c(K(n.3)\setminus S)} = \frac{n}{3} - 1$, then $S$ is the complement of a maximum independent set of $K(n,3)$.

**Theorem 15**: [16] $t(K(8,3)) = \frac{5}{3}$. Moreover, if $S$ is a disconnecting set of $K(n,3)$ such that $\frac{|S|}{c(K(n.3)\setminus S)} = \frac{5}{3}$, then $S$ is the complement of a maximum independent set of $K(8,3)$.

In [7], Balakrishnan et.al computed the Weiner index of Kneser graph. The Weiner index in important topological index of a graph which is given by the formula $W = \frac{1}{2}\sum_{u,v\in V(G)} d(u,v)$.

**Theorem 16**: [7] The Wiener number $W$ of the Kneser graph $KG(m,n)$is given by

$$W = \frac{1}{2}\binom{2n+k}{n}\left[\sum_{i=0}^{\lfloor\frac{D-1}{2}\rfloor}(2i)\sum_{j=n-ki}^{\min\{n-k(i-1)-1,n\}}\binom{n}{j}\binom{n+k}{n-j} + \sum_{i=0}^{\lceil\frac{D-1}{2}\rceil-1}(2i+1)\sum_{j=\max\{k(i-1)+1,0\}}^{ki}\binom{n}{j}\binom{n+k}{n-j} + S\right],$$

*where*

$$S = \begin{cases} D\displaystyle\sum_{j=k(\frac{D}{2}-1)+1}^{n-k(\frac{D}{2}-1)-1}\binom{n}{j}\binom{n+k}{n-j} & \textit{if } D \textit{ is even,} \\ D\displaystyle\sum_{j=k(\frac{D-1}{2}-1)+1}^{n-k(\frac{D-1}{2})-1}\binom{n}{j}\binom{n+k}{n-j} & \textit{if } D \textit{ is odd.} \end{cases}$$

**Methods of Construction**: Several methods can be employed to construct Kneser graphs. Two of them are as follows.

1) **Lovász Construction** : This recursive construction method for Kneser graphs was introduced by Lovász and it provides forms the basis for many algorithms and proofs related to these graphs.
2) **Sperner's Theorem**: Sperner's theorem givesan algorithm to construct a Kneser graph by using antichains in the Boolean lattice. This construction is particularly useful for understanding the chromatic number of Kneser graphs

## 1. Applications of Kneser graphs and Open Problems

Applications for Kneser graphs can be found in many disciplines, including combinatorics, topology, coding theory, and combinatorial optimization. They are basic elements of combinatorial theory and frequently give rise to intriguing puzzles and hypotheses. Research in this field is still driven by queries concerning their chromatic number and other graph-theoretic features. Understanding the homotopy type of certain spaces is one way that Kneser graphs relate to topological issues. In coding theory, Kneser graphs are utilized to create codes with effective error-correcting capabilities. In optimization problems like the largest clique problem and the maximum independent set problem, Kneser graphs occur and offer good bounds.

Though there is a rich literature on Kneser graphs, there are some areas which are unexplored. Two of them are as follows:

1) Develop an efficient algorithm to find a Hamiltonian cycle in a Kneser graph.
2) Compute other topological indices like Steiner-Weiner index, Centrality measures of a Kneser graph.

## 2. Conclusion

In the fields of combinatorics and graph theory, Kneser graphs provide a rich and fascinating topic of study. Mathematicians and academics have been fascinated by their properties, techniques of construction, and applications for many years. Even though there are still many unanswered concerns regarding Kneser graphs, further research into them continues to provide insightful knowledge about the underlying combinatorial structures and their importance in numerous mathematical fields.

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