Dispersion of a chemically reacting solute in a three – dimensional Newtonian fluid flow through a Darcy - Brinkman porous medium

Srihari M Subodha[1], S Manjunath[2], and N P Chandrashekara[3]

[1]Department of Mathematics, RNS Institute of Technology, RR Nagar post, Channasandra, Bengaluru 560098 and affiliated to Visvesvaraya Technological University, Belagavi 590018, India
[2]RV University, RV Vidyanikethan 8th mile, Mysuru Road, Bangaluru 560059, India
[3]Department of Mathematics, BNM Institute of Technology, 12th Main Road, 27th Cross, Banashankari II Stage, Bengaluru 560070 and affiliated to Visvesvaraya Technological University, Belagavi 590018, India

Abstract

In this paper, the dispersion of a chemically active solute in a sparsely packed porous medium is studied. The aim of this work is to study the effects permeability of the porous medium and a homogeneous chemical reaction on the dispersion coefficient. A mathematical model is obtained using the Gill – Sankarasubramanian(1970) approach for the three-dimensional flow of a Newtonian liquid. The velocity profile is obtained using Darcy – Brinkmann momentum equation. The effect of the chemical reaction rate parameter on the convection coefficient and dispersion coefficient is discussed. The mean concentration distribution is also computed for various values of the Darcy number and reaction rate parameter and the results are represented graphically. This problem finds its applications in the fields of waste water management, chemical engineering, and biomechanical problems.

Keywords: Dispersion, Newtonian liquid, Chemical reaction, Porous Medium, Darcy number, Brinkman number.

Introduction

The topic of fluid flow in porous medium has drawn the interest of fluid mechanics community for a long time. This due to the availability of porous materials in most of the useful phenomena such as filtering and storage of water, cogeneration systems etc. Further, Dispersion in porous medium is one of the very useful studies in this respect. Early literature in this field have been found in the works of Taylor (1953) who studied small time dispersion, and Aris (1956) who studied long time dispersion. But a more generalized approach was later proposed by Gill and Sanakarasubramanian (1970). Soundalgekar and Gupta (1977) extended Taylor's approach to electrically conducting walls. Shivakumar et al. (1987) studied dispersion porous channel flows to obtain the closed form solution. Later, Pal (1998) extended the same approach to study the effect of first order chemical reaction on such a flow. Recently, Madie et al, (2022) studied dispersion in porous medium with varying dispersion coeffcients.

The above mentioned works have considered two - dimensional flow of fluids. One can note that, an abundant literature is available with respect to three - dimensional flows in this regard too. It was Doshi et al (1978) who initially conducted such a research for studying the effect of the walls on the three - dimensional flow of the Newtonian liquids. Recently, Manjunath and Siddheshwar (2011) considered the couple stress effect on the three - dimensional laminar dispersion in the rectangular channel flows. In the present work, the unsteady dispersion is studied for a fully developed flow of the Newtonian liquid through the rectangular Darcy - Brinkman porous channel, and thereby the effects of the first order homogeneous chemical reaction on the dispersion coefficient and on the mean concentration profile are discussed.

1 Mathematical Formulation of the Problem

Figure 1 shows a physical system consisting of an infinitely sparsely packed porous medium with a cross section of breadth 2b and height 2l. It is assumed that the flow is unidirectional and influenced by the porous matrix. The drag force is neglected. Under these conditions a fully developed flow can be governed by the following momentum equation.

$$\mu^{'}(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2})-\frac{\mu}{k}w=\frac{dp}{dz} \tag{1.1}$$

The equation (1.1) is called the Darcy – Brinkmann momentum equation. In this equation, ρ is the density of the fluid, w represents the filter velocity, μ is dynamic viscosity, $\mu^{'}$ is the effective viscosity, k is the permeability of the porous medium.

The no - slip boundary condition to solve the equation (1.1) is as follows:

$$w=0 \quad at \quad x=\pm b \quad and \quad w=0 \quad at \quad y=\pm h \tag{1.2}$$

In the equation (1.2), we have assumed the no-slip boundary condition at both horizontal and vertical walls. In order to obtain the dimensionless form of the equation (1.1), we apply the following definitions:

$$X=\frac{x}{b},Y=\frac{y}{h},W=\frac{w}{w^*(-\frac{dp}{dz})},P=\frac{p}{p^*},Z=\frac{z}{hPe} \tag{1.3}$$

Paper ID: SR231007111819 DOI: 10.21275/SR231007111819

where w^* and p^* are some reference velocity and reference pressure respectively, $Pe = (w^*h)/D$ and D is the solutal diffusivity. Using (1.3), the equation (1.1) takes the form:

$$\frac{1}{\alpha^2}\frac{\partial^2 W}{\partial X^2} + \frac{\partial^2 W}{\partial Y^2} - AW = -\Lambda \tag{1.4}$$

where $a = \frac{b}{h}$ is the aspect ratio, $Da = \frac{h}{\sqrt{K}}, Re = \frac{\rho u h}{\mu}(-\frac{dp}{dz}), A = \Lambda Da^2, \Lambda = \frac{\mu}{\mu'}$. Here, Λ is is the Brinkman number, Da is the Darcy number and Re is the Reynolds number.

The equation (1.2) in non - dimensional form is given by

$$\begin{aligned} W &= 0 \quad at \quad X = \pm 1, \quad -1 < Y < 1 \\ W &= 0 \quad at \quad Y = \pm 1, \quad -1 < X < 1 \end{aligned} \tag{1.5}$$

Due to the symmetry of W about X and Y, the equation (1.5) can be replaced by:

$$\begin{aligned} W(X,1) &= 0, \quad 0 \le X \le 1 \\ W(1,Y) &= 0, \quad 0 \le Y \le 1 \end{aligned} \tag{1.6}$$

$$\frac{\partial W}{\partial Y}(X,0) = 0 \tag{1.7}$$

$$\frac{\partial W}{\partial X}(0,Y) = 0 \tag{1.8}$$

We also consider centre-line distribution along the X and Y directions as:

$$W(X,0) = \eta(1 - X^2), \quad W(0,Y) = \eta(1 - Y^2) \tag{1.9}$$

where η to be determined later. Let us assume the Maclaurin series solution of the equation (1.4) in the following form:

$$W(X,Y) = \sum_{k=0}^{\infty}\sum_{h=0}^{\infty} S(k,h)X^k Y^h \tag{1.10}$$

On using the equation (1.10) in the equation (1.4) we get,

$$\begin{aligned} \frac{1}{\eta^2}(k+1)(k+2)S(k+2,h) + (h+1)(h+2)S(k,h+2) - A.S(k,h) \\ = -\Lambda\delta(k-0,h-0) \end{aligned} \tag{1.11}$$

along with certain specific coefficients as follows:

$$S(k,1) = 0, \quad S(1,h) = 0 \tag{1.12}$$

A four - term solution of this equation is given by

$$W(X,Y) = \eta - \eta X^2 - \eta Y^2 - BX^2Y^2 \tag{1.13}$$

where

$$B = \frac{a^2}{2}[A\eta + \Lambda]$$

For a fully developed flow which satisfies the above equation, an initial concentration is introduced to the source at $z = 0$ and assuming the solute undergoes first order homogeneous chemical reaction, the corresponding mass - balance equation is given by

$$\frac{\partial C}{\partial t} + w\frac{\partial c}{\partial z} = D(\frac{\partial^2 c}{\partial x^2} + \frac{\partial^2 c}{\partial y^2} + \frac{\partial^2 c}{\partial z^2}) - R_1 C \tag{1.14}$$

where D is the solutal diffusion coefficient and R_1 is the reaction rate constant.

To solve the equation (1.15), we use the following initial and boundary conditions:

$$C(x, y, z, 0) = \phi(x, y).\delta(z) \tag{1.15}$$

$$(\frac{\partial C}{\partial x})_{x=0,b} = 0 \tag{1.16}$$

$$(\frac{\partial C}{\partial y})_{y=0,h} = 0 \tag{1.17}$$

$$\lim_{z\to\infty} C = \lim_{z\to\infty}(\frac{\partial C}{\partial z}) = 0 \tag{1.18}$$

In the equation (1.16), $\delta(z)$ is the Dirac delta function. The strength of the initial slug input is given by the following equation.

$$\phi(x, y) = \begin{cases} C_0, & \text{for } \frac{-y_s}{2} \leq y \leq \frac{y_s}{2}, \frac{-x_s}{2} \leq y \leq \frac{x_s}{2}. \\ 0, & \text{otherwise.} \end{cases} \tag{1.19}$$

2 The Solute Dispersion Model

A dispersion model for the concentration involving the cross – sectional variations in terms the derivatives of concentraion was proposed by Taylor (1953). Aris (1956), then introduced the model in which the dispersion is asymptotically valid for small values of time. But, at large times after the release of the slug input, we have $C - C_m$ directly proportional to $(\frac{\partial C_m}{\partial z})$. From such a postulate, Gill – Sankarasubramanian (1970) proposed a generalised dispersion model given by $C - C_m = \sum_{n=1}^{\infty} f_n(x, y, t)\frac{\partial^n C_m}{\partial z^n}$. Hence, let us assume the dispersion model to be

$$C(x, y, z, t) = C_m f_0(x, y, t) + \sum_{n=1}^{\infty} f_n(x, y, t)\frac{\partial^n C_m}{\partial z^n} \tag{2.1}$$

where

$$C_m = \frac{1}{bh}\int_0^h \int_0^b C dx dy \tag{2.2}$$

Equation (2.2) represents the mean concentration.

Paper ID: SR231007111819 DOI: 10.21275/SR231007111819

Integrating the equation (1.14) with respect to x and y as required by the equation (2.2) and applying the boundary conditions (1.16) and (1.17), we get

$$\frac{\partial C_m}{\partial t}+\frac{1}{bh}\frac{\partial}{\partial z}\int_0^h\int_0^b wCdxdy=D\frac{\partial^2 C_m}{\partial z^2}-R_1C_m. \tag{2.3}$$

By using the equation (2.1) in the equation (2.3) we get,

$$\frac{\partial C_m}{\partial t}=K_1(t)\frac{\partial C_m}{\partial z}+K_2(t)\frac{\partial^2 C_m}{\partial z^2}+K_3(t)\frac{\partial^3 C_m}{\partial z^3}+...-R_1C_m \tag{2.4}$$

where

$$K_n(t)=-\frac{1}{bh}\int_0^h\int_0^b wf_{n-1}dxdy+\delta_{n2}D \tag{2.5}$$

and δ_{n2} is Kronecker delta.

By substituting the equation (2.1) in the equation (1.14) and by equating the coefficients of each of $(\frac{\partial^n z}{\partial z^n})$ terms to zero and by simplifying we get the following equations for f_n

$$\frac{\partial f_0}{\partial t}=D\nabla^2 f_0-R_1f_0 \tag{2.6}$$

$$\frac{\partial f_1}{\partial t}=D\nabla^2 f_1-(w+K_1)f_0-R_1f_1 \tag{2.7}$$

$$\frac{\partial f_n}{\partial t}=D\nabla^2 f_n-wf_{n-1}+Df_{n-2}-\sum_{i=1}^{\infty}K_if_{n-i}-R_1f_n \tag{2.8}$$

where $\nabla^2=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}$.

On solving the equation (2.4) with suitable boundary conditions and expressing it in its dimensionless form we get,

$$f_0=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}A_{mn}cos(m\pi Y)cos(n\pi X)exp[-(m^2+\frac{n^2}{a^2})\pi^2\tau]exp[-\alpha^2\tau] \tag{2.9}$$

where $\tau=\frac{tD}{h^2},\ w^*=\frac{\mu Pe}{h\rho},\ \alpha^2=\frac{h^2R_1}{D}$ and A_{mn} are given by

$$A_{mn}=\frac{16}{\pi^2 mnY_sX_s}sin(\frac{\pi mY_s}{2})sin(\frac{\pi mX_s}{2}). \tag{2.10}$$

Similarly on solving the equation (2.5) we get its solution in dimensionless form as follows:

$$f_1(X,Y,\tau)=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}B'_{mn}cos(m\pi Y)cos(n\pi X)[1-exp[-(m^2+\frac{n^2}{a^2})\pi^2\tau-\alpha^2\tau]] \tag{2.11}$$

Here, B'_{mn} are given by the following equation:

$$B'_{mn}=\frac{(-1)^{m+n}a^2}{(a^2m^2+n^4)}[\frac{16a^2}{n^2m^2\pi^6}(\Lambda Da^2\eta+2F\eta^2+\Lambda],\forall m,n\in N. \tag{2.12}$$

Now, by putting $n = 1$ in the equation (2.5) and non dimensionalizing it with suitable dimensionless ratios we get

$$K_1(\tau) = \int_0^1 \int_0^1 W f_0(X, Y, \tau) dX dY \tag{2.13}$$

where $K_1(\tau) = -\frac{K_1(t)}{w^*}$.

Now using the equations (1.13) and (2.9) in the equation (2.13) we obtain the expression for the dimensionless convection coefficient as follows:

$$K_1(\tau) = \frac{2a^2}{m^2n^2\pi^2}[\Lambda(Da)^2\eta + \Lambda]$$
$$\times \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} A_{mn} cos(m\pi) cos(n\pi) exp[-(m^2 + \frac{n^2}{a^2})\pi^2 - \alpha^2]\tau. \tag{2.14}$$

Similarly by putting $n = 2$ in the equation (2.5) and simplifying it to express in its dimensionless form we obtain the following equation:

$$K_2(\tau) = -\int_0^1 \int_0^1 W f_1(X, Y, \tau) dX dY + \frac{1}{Pe^2}, \tag{2.15}$$

where

$$K_2(\tau) = -\frac{K_2(t)}{w^*.h.Pe}$$

On using the equation (2.11) in the equation (2.15) and simplifying, we get

$$K_2(\tau) = \frac{4B}{\pi^4} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{B'_{mn}}{m^2n^2} cos(m\pi) cos(n\pi)[1 - exp[-(m^2 + \frac{n^2}{a^2})\pi^2\tau - \alpha^2\tau]] + \frac{1}{Pe^2} \tag{2.16}$$

where B'_{mn}'s are given by the equation (2.12).

To find the mean concentration

We observe that $K_3, K_4...$ have numerically too small magnitude to be neglected. Hence the truncated form of the equation (2.4) is given by:

$$\frac{\partial C_m}{\partial t} = K_1(t)\frac{\partial C_m}{\partial z} + K_2(t)\frac{\partial^2 C_m}{\partial z^2} - R_1 C_m \tag{2.17}$$

Equation (2.17) can be further reduced to get the unsteady diffusion equation by using the following transformations.

$$z_1 = z + t\overline{K_1(t)}, \tag{2.18}$$

$$t_1 = \frac{t\overline{K_2(t)}}{D}, \tag{2.19}$$

where

$$\overline{K_i(t)} = \frac{1}{t}\int_0^t K_i(\gamma)d\gamma \quad (i = 1, 2, 3, ...) \tag{2.20}$$

By solving the equation (2.17) using the method of infinite Fourier transforms and by applying the suitable initial and boundary conditions we obtained the solution in its simplified, dimensionless form as follows:

$$\theta_m = \frac{e^{-\alpha^2\tau}}{\sqrt{4\pi\tau\overline{K_2(\tau)}}} exp[\frac{-(Z+\tau\overline{K_1(\tau)})^2}{4\tau\overline{K_2(\tau)}}] \tag{2.21}$$

3 Results and Discussions

In the present research, we have studied the three – dimensional unsteady convective diffusion of a solute that undergoes a first - order homogeneous chemical reaction in a porous medium using the generalized miscible theory of dispersion of Gill – Sankarasubramanian (1970) and Doshi et al. (1978).

From the figure 2, we note that the filter velocity is symmetric about the two axes of X and Y. To see the symmetry more clearly, we have plotted the figures 3 and 4. From this, it is evident that the velocity does have symmetry about y axis. Similarly, one can see its symmetry about the x axis. We also observe that as there is an increase in Brinkman number there is an increase in the velocity. This is due to the increase in the viscous dissipation causing the rise in the temperature. However, increase is in the Darcy number is creating a quite opposite effect. This is due to an increse in the permeability of porous medium.

Figures 5 - 7 shows that the dimensionless convection co efficient grows exponentially with the value of the aspect ratio a. We have taken three different values of the Darcy number, $Da = 3, 4, 5...$ in the figure. As there is a decrease in the value of Da, there is an increase in the value of $K_1(\tau)$. Similarly, we note that the value of $K_1(\tau)$ increases in the same fashion with the increase in the value of a for the chemical reaction rate, $\alpha = 0.4, 0.8, 1.2$. The increase of $K_1(\tau)$ with a is steep for large values of both α and Da. This shows that the chemical reaction reduces the convection coefficient. This result holds true for a uniform source and when the reaction at the sidewalls is ignored.

Figures 8 and 9 shows that the dispersion coefficient increases with increase in dimensionless time for small values of time and levels off later. On this occasion too, we observe that the Darcy number and Brinkman number are creating opposite effects on the dispersion. we also observed from calculations that the difference in chemical reaction rate does not make a huge difference in the values of the dispersion coefficient.

Figure 10 clearly indicates that the increase in the chemical reaction causes the increase in the dispersion and hence resulting in the decrease in the mean concentration. Figure 11 shows that the differences in Darcy number causes no huge difference in attaining the peak of the mean concentration. Figure 12, obviously indicate the same effect as in figure 10, but the value of the concentration is more as the point close to the axis. Figures 13 and 14 shows the Gaussian nature of the mean concentration against the axial distance which clearly indicates that the concentration distribution on either sides of the axis is similar. Apart from that, the effects of other parameters remains the same.

Figures

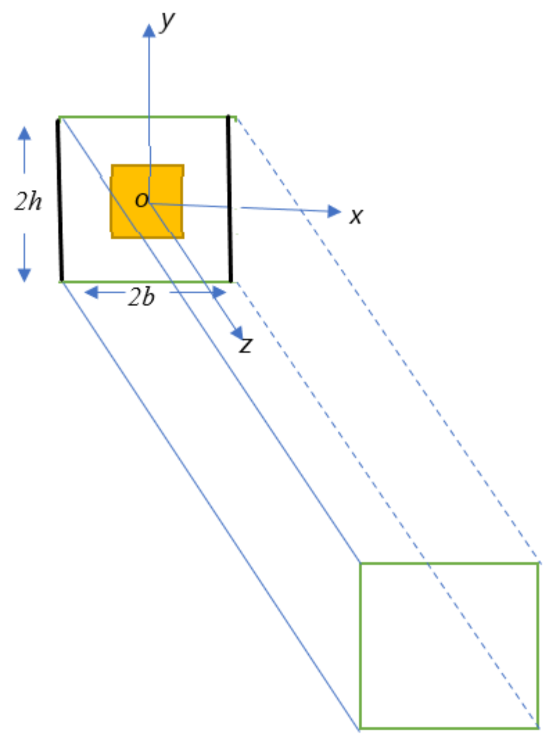

Figure 1: Representation of the rectangular duct and the slug input

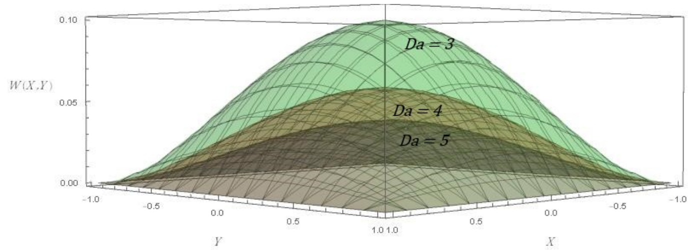

Figure 2: Three - Dimensional plots of the filter velocity distribution, $W(X,Y)$ for various values of the Darcy number Da, with the Brinkmann number $\Lambda = 2$, and the Aspect ratio $a = 1$.

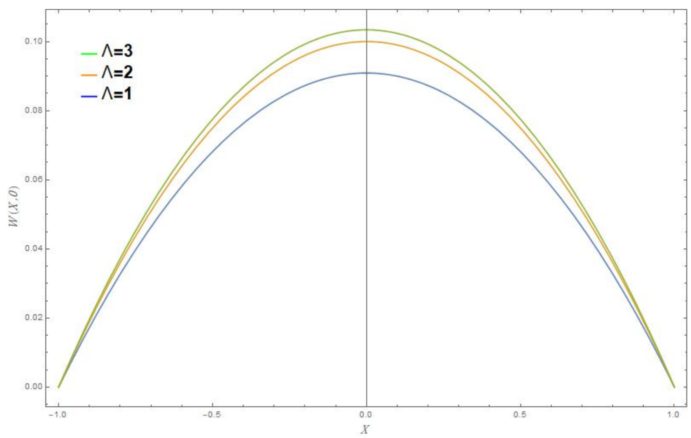

Figure 3: Plots of the mid-plane filter velocity, $W(X,0)$, for various values of Λ with $Da = 3, a = 1$.

International Journal of Science and Research (IJSR)
ISSN: 2319-7064
SJIF (2022): 7.942

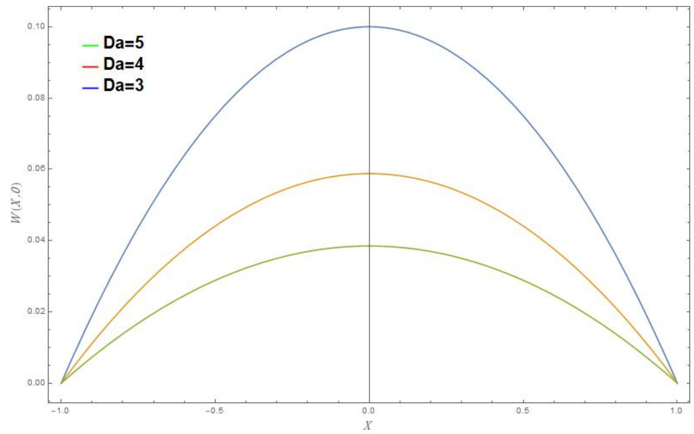

Figure 4: Plots of the dimensionless convection coefficient, $K_1(\tau)$ vs a, for various values of Da with $\Lambda = 2, a = 1.$ and the chemical reaction rate parameter, $\alpha = 0.8$.

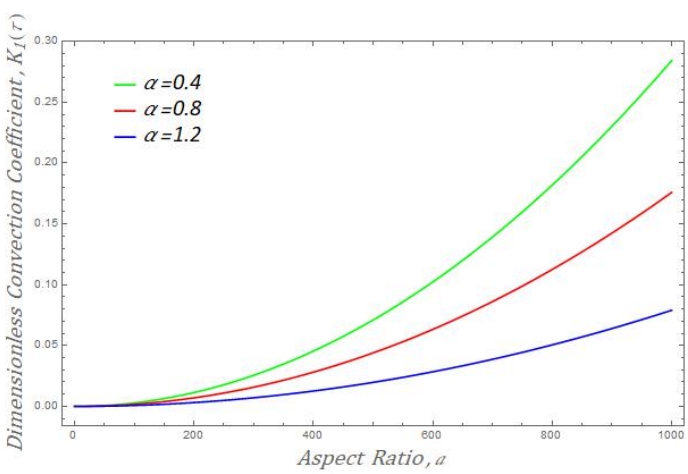

Figure 5: Plots of $K_1(\tau)$ vs a, for various values of α with $\Lambda = 2, a = 1$ and $Da = 3$

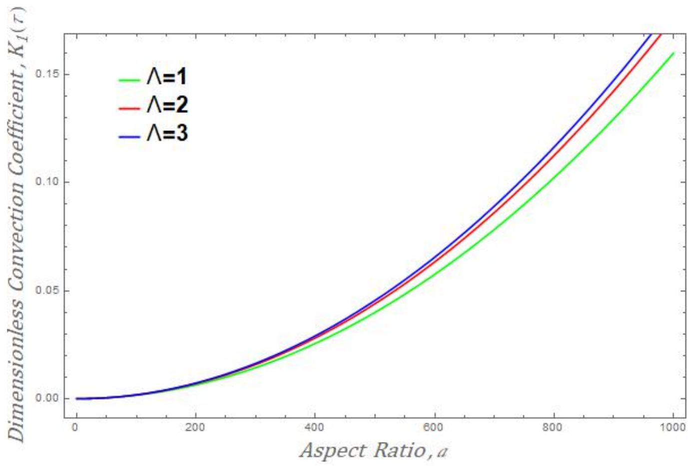

Figure 6: Plots of $K_1(\tau)$ vs a, for various values of λ with $\alpha = 0.8, a = 1$ and $Da = 3$.

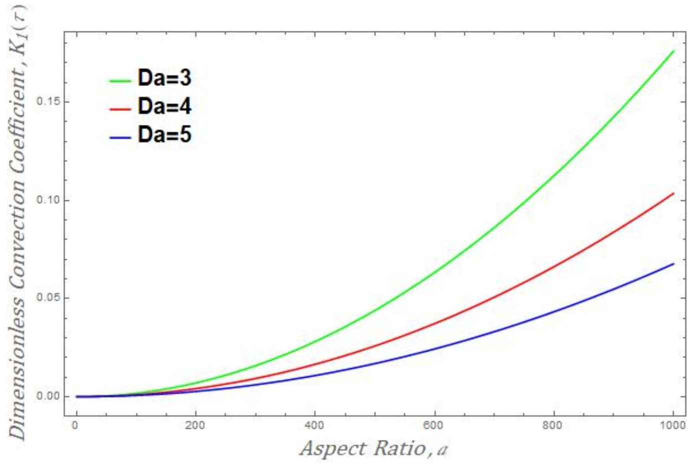

Figure 7: Plots of $K_1(\tau)$ vs a, for various values of Da with $\Lambda = 2, a = 1$ and $\alpha = 0.8$

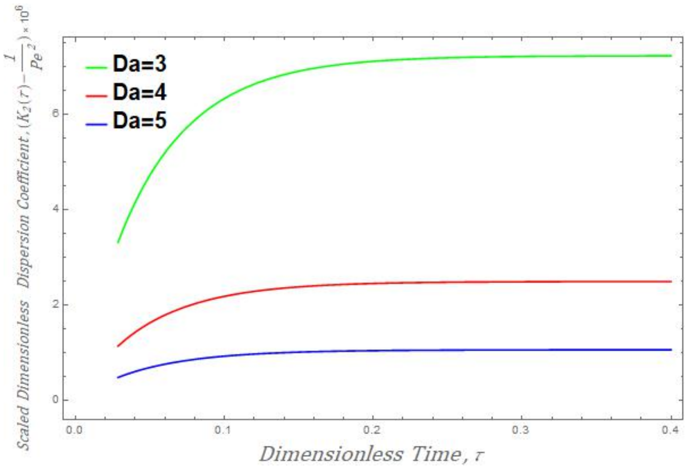

Figure 8: Plots of the scaled dispersion coefficient, $(K_2(\tau) - \frac{1}{Pe^2}) \times 10^6$, vs the dimensionless time, τ, for various values of Da with $\alpha = 0.8$, $\Lambda = 2, a = 1$.

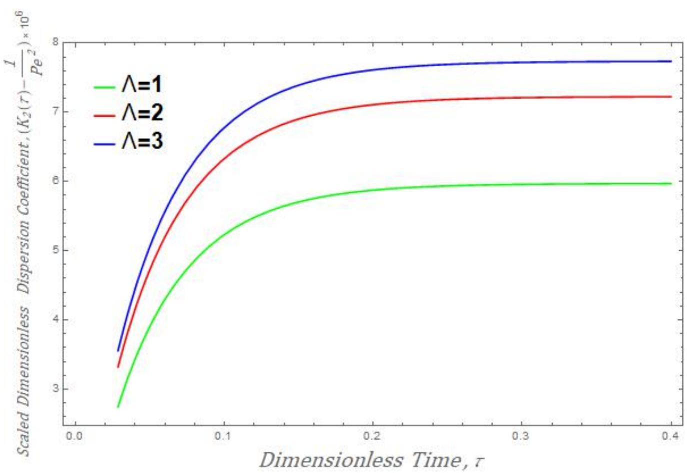

Figure 9: Plots of $(K_2(\tau) - \frac{1}{Pe^2}) \times 10^6$, vs τ for various values of Λ, $Da = 3, a = 1$ and $\alpha = 0.8$.

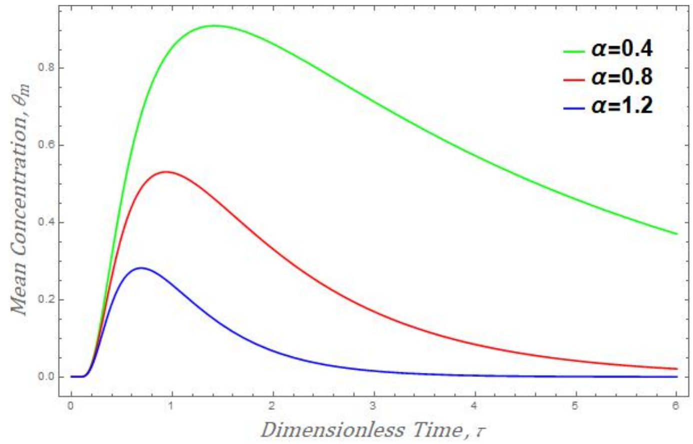

Figure 10: Plots of mean concentration θ_m vs τ at axial distance $Z = 0.2$ for various values of α with $\Lambda = 2, F = 2, a = 1$ and $Da = 3$

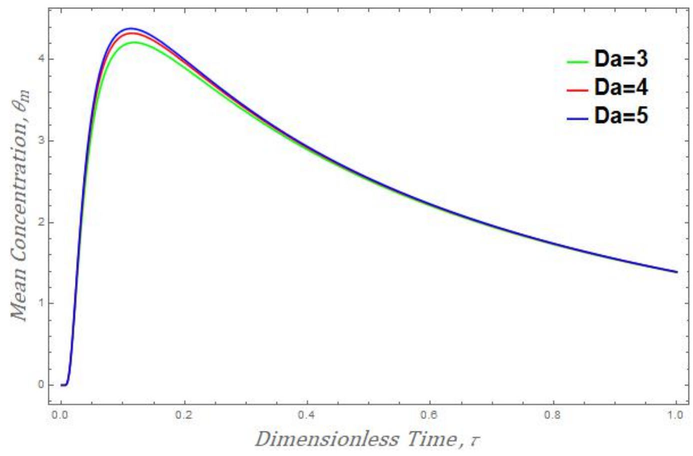

Figure 11: Plots of θ_m vs τ at $Z = 0.05$ for various values of Da with $\Lambda = 2, a = 1$ and $\alpha = 0.8$.

ISSN: 2319-7064
SJIF (2022): 7.942

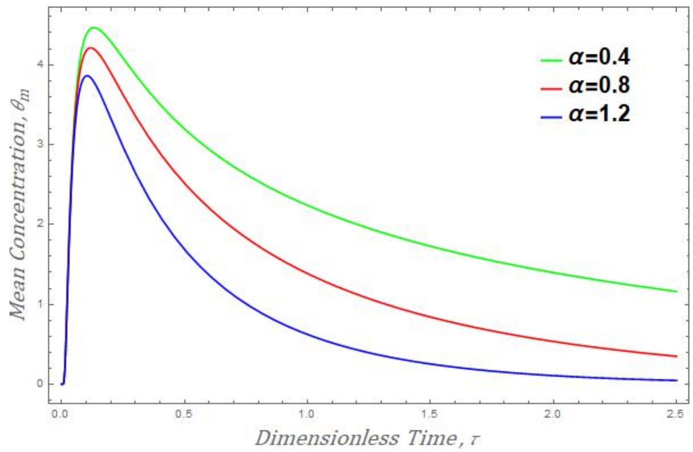

Figure 12: Plots of θ_m vs τ at $Z = 0.05$ for various values of α with $\Lambda = 2, a = 1$ and $Da = 3$.

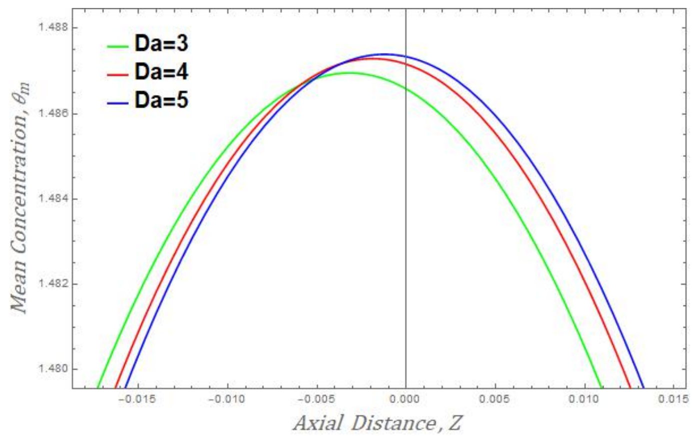

Figure 13: Plots of θ_m vs Z, for various values of Da with $\Lambda = 2, \tau = 1, a = 1$ and $\alpha = 0.8$

Paper ID: SR231007111819 DOI: 10.21275/SR231007111819

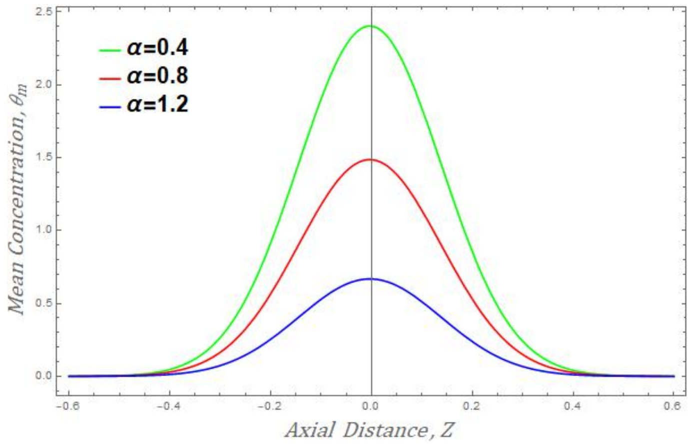

Figure 14: Plots of θ_m vs Z, for various values of α with $\Lambda = 2, \tau = 1 a = 1$ and $Da = 3$

Nomenclature

$w(x, y)$ - Axial Filter Velocity
μ - Dynamic Viscosity
$\mu^{'}$ - Effective Viscosity
$\overline{w}$ - Average Fluid Velocity
$\Lambda = \frac{\mu}{\mu^2}$ - Brinkman number
$Da = \frac{h}{\sqrt{K}}$ - Darcy number
Re - Reynolds number
W - Dimensionless filter velocity
Z - Dimensionless axial distance
τ - Dimensionless time
$\phi(x, y)$ - Strength and location of slug input
C - Local concentration
C_m - Mean Concentration
θ_m - Dimensionless mean concentration
a - Aspect Ratio
α - Chemical reaction rate coefficient
Pe - Peclet number

Acknowledgement

The authors SMS and NPC are grateful for the support and encouragement provided by the Management, The Principal, and The director of B N M Institute of Technology, Bengaluru. The author SMS is grateful to the Management, The Principal, and The Director of R N S Institute of Technology, Bengaluru, for their encouragement to carry out this work. The author SM is grateful to the management, The Vice chancellor and The registrar of RV University, Bengaluru. All the authors are grateful to Prof. P G Siddheshwar, Senior Professor, Christ (Deemed to be University), Bengaluru, for his constant encouragement for their research work.

References

[1] Aris R, 1956. *On dispersion of a solute in a fluid flowing through a tube*, Proc. Roy. Soc. London, Vol A235, pp 67 - 77.

[2] Doshi M R, Daiya P M, Gill W N, 1978. *Three-dimensional laminar dispersion in open and closed rectangular conduits*, Chem Engg Sci. Vol 33, pp 795 - 804.

[3] Gill W N, Sankarasubramanian R, 1970. *Exact analysis of unsteady convective diffusion*, Proc. Roy. London. Series, Vol A316, pp 341 - 350.

[4] Madie C Y, Togue F A, Woafo P, 2022. *Dispersion of pollutants in a porous medium with finite thickness and variable dispersion coefficients*, Helion Vol 8, pp 1 - 8.

[5] Manjunath S, Siddheshwar P G, 2011. *Three-dimensional laminar dispersion of a solute in a couple stress fluid flow through a rectangular channel*, Int. J. Appl. Mech and Engg, Vol. 16(3), pp 775 – 789.

[6] Pal D, 1998. *Effect of chemical reaction on the dispersion of a solute in a porous medium*, Applied mathematical modelling, vol 23(7), pp 557 – 566.

[7] Shivakumar P N, Rudraiah N, Pal D, Siddheshwar P G, 1987. *Closed form solution for unsteady convective diffusion in a fluid saturated sparsely packed porous medium*, Int. Com. Heat mass transfer, Vol. 14(2), pp 137 – 145.

[8] Soundalgekar V M, Gupta A S, 1977. *On the dispersion of a dye with harmonically varying concentration in the hydromagnetic flow in a channel*, J. Appl. Phys. Vol 48(12), pp 5344 – 5346.

[9] Taylor G I, 1953. *Dispersion of soluble matter in solvent slowly flowing through tube*, Proc. Roc. London, Vol A219, pp 186 – 203.