International Journal of Science and Research (IJSR)
ISSN: 23197064
SJIF (2022): 7.942

MINIMUM DEGREE ENERGY FOR SOME CLASSES OF GRAPHS

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Abstract

In this paper we find minimum degree energy for graphs like $C_n \times P_2$, Totalgraph of C_n, Totalgraph of regular graph of degree r and Cocktail party graph

Keywords and Phrases : Minimum degree matrix, minimum degree eigenvalues, minimum degree energy of a graph.

2000 AMS Subject Classification : 05C50.

1 Introduction

Let G be a simple graph and let its vertex set be $V(G) = \{v_1, v_2, ..., v_n\}$. The adjacency matrix $A(G)$ of the graph G is a square matrix of order n whose (i, j)-entry is equal to unity if the vertices v_i and v_j are adjacent, and is equal to zero otherwise. The eigenvalues $\lambda_1, \lambda_2, ..., \lambda_n$ of $A(G)$, assumed in non increasing order, are the eigenvalues of the graph G.

The energy of G was first defined by I.Gutman[4] in 1978 as the sum of the absolute values of its eigenvalues:

$$E(G) = \sum_{i=1}^{n} |\lambda_i|.$$

Ever since the graph energy $E(G)$ of a simple graph G was introduced by I.Gutman [4], there is a constant stream of papers devoted to this topic. Survey of development before 2001 can be found in [5]. For recent developments one can consult [3]. The energy of a graph has close links to chemistry(see for instance [6] . Let G be a simple graph with n vertices $v_1, v_2, \ldots, v_n$ and let d_i be the degree of v_i, $i = 1, 2, 3, \ldots, n$. Define

$$d_{ij} = \begin{cases} min\{d_i, d_j\} & \text{if } v_i \text{ and } v_j \text{ are adjacent,} \\ 0 & \text{otherwise.} \end{cases}$$

Then the $n \times n$ matrix $m(G) = (d_{ij})$ is called the minimum degree matrix of G. This was introduced and studied in[1, 2]. The characteristic polynomial of the minimum degree matrix $m(G)$ is defined by

$$\phi(G; \mu) = det(\mu I - m(G))$$

$$= \mu^{n} + c_1\mu^{n\;\;1} + c_2\mu^{n\;\;2} + \ldots + c_{n\;\;1}\mu + c_n,$$

where I is the unit matrix of order n. The minimum degree eigenvalues $\mu_1, \mu_2, \ldots, \mu_n$ of the graph G, assumed in the non increasing order, are the eigenvalues of its minimum degree matrix $m(G)$. The minimum degree energy of a graph G is defined as

$$E_m(G) = \sum_{i=1}^{n} |\mu_i|.$$

Since $m(G)$ is real symmetric matrix with zero trace, these minimum degree eigenvalues are all real with sum equal to zero. In this paper, we compute the minimum degree Energy $E_m(G)$ of some classes of graphs like $C_n \times P_2$, Totalgraph of C_n, Totalgraph of regular graph of degree r and Cocktail party graph.

2 MIMINIMUM DEGREE ENERGY

In this section we find minimum degree energy of some classes of graphs.

Definition 2.1 If $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are two graphs then the cartesian product of G_1 and G_2 denoted by $G_1 \times G_2 = G(V, E)$ consists a vertex set $V = V_1 \times V_2$ and $(x_1, y_1), (x_2, y_2) \in V_1 \times V_2$ are adjacent if $x_1x_2 \in E_1$ and $y_1 = y_2$ or $y_1y_2 \in E_2$ and $x_1 = x_2$.

Theorem 2.1 *The minimum degree energy of $C_n \times P_2$ is*

$$E_m(C_n \times P_2) = \sum_{i=1}^{n} \left| \;\; 1 + 4\cos\frac{2\pi i}{n} \right| + \sum_{i=1}^{n} \left| 1 + 4\cos\frac{2\pi i}{n} \right|.$$

Proof. The minimum degree characteristic polynomial of C_n is

$$P_{C_n}(\mu) = -2^{n+1} + 2^{n} \sum_{k=0}^{[\frac{n}{2}]} (-1)^k \frac{n}{n-k} \begin{pmatrix} n\;\;\;k \\ k \end{pmatrix} \left(\frac{\mu}{2}\right)^{n\;\;2k}.$$

The minimum degree eigenvalues of cycle C_n are the numbers $4\cos\frac{2\pi i}{n}$, for $i = 1, 2, 3, \ldots, n$. The minimum degree characteristic polynomial of P_2 is

$$P_{P_2}(\mu) = \mu^2 - 1,$$

the minimum degree eigenvalues os P_2 are $\;\;1$ and 1. The eigenvalues of $C_n \times P_2$ are $\;\;1 + 4\cos\frac{2\pi i}{n}$

and $1+4\cos\frac{2\pi i}{n}$ $(i=1,2,3,\ldots,n)$. Thus

$$E_m(C_n\times P_2)=\sum_{i=1}^{n}\left|-1+4\cos\frac{2\pi i}{n}\right|+\sum_{i=1}^{n}\left|1+4\cos\frac{2\pi i}{n}\right|.$$

Also the minimum degree energy of $C_n\times P_3$ is

$$E_m(C_n\times P_3)=\sum_{i=1}^{n}\left|4\cos\frac{2\pi i}{n}\right|+\sum_{i=1}^{n}\left|\ 2+4\cos\frac{2\pi i}{n}\ +\sum_{i=1}^{n}\ 2+4\cos\frac{2\pi i}{n}\right|.$$

Definition 2.2 The total graph $T(G)$ of graph G is that graph whose set of vertices is the union of the set of vertices and of the set of edges of G, with two vertices of $T(G)$ being adjacent if and only if the corresponding elements of G are adjacent or incident.

Let G be regular graph of degree r, having n vertices and m edges, then the minimum degree matrix $m(T(G))$ of total graph $T(G)$ with vertex set $(v_1,v_2,\ldots,v_n,e_{n+1},e_{n+1},\ldots,e_{m+n})$ can be written in the form,

$$m(T(G))=\begin{pmatrix} A & R \\ R^T & B \end{pmatrix},$$

where $A=[a_{ij}]$,

$$a_{ij}=\begin{cases} 2r & \text{if } v_i \text{ and } v_j \text{ are adjacent} \\ 0 & \text{otherwise,} \end{cases}$$

$B=[b_{ij}]$,

$$b_{ij}=\begin{cases} 2r & \text{if } e_{n+1} \text{ and } e_{n+j} \text{ are adjacent} \\ 0 & \text{otherwise,} \end{cases}$$

and $R=[r_{ij}]$,

$$r_{ij}=\begin{cases} 2r & \text{if } v_i \text{ and } e_{n+j} \text{ are adjacent} \\ 0 & \text{otherwise.} \end{cases}$$

Hence the characteristic polynomial $\phi(T(G);\mu)$ of $m(T(G))$ is obtained as follows,

$$P_{T(G)}(\mu)=(2r)^{m+n}\begin{vmatrix} \frac{\mu}{2r}I+rI \quad RR^T & R \\ -R^T & \frac{\mu}{2r}I+2I-R^TR \end{vmatrix}.$$

where $\rho = \frac{\mu}{2r}$.

$$P_{T(G)}(\mu) = (2r)^{m+n} \begin{vmatrix} (\rho+r)I - RR^T + \frac{R}{\rho+2}\left(-(\rho+r+1)R^T + R^TRR^T\right) & 0 \\ -(\rho+r+1)R^T + R^TRR^T & (\rho+2)I \end{vmatrix}.$$

$$\begin{aligned} P_{T(G)}(\mu) &= (2r)^{m+n}(\rho+2)^m \left|\rho I - m(G) + \frac{1}{\rho+2}(m(G)+rI)(m(G)-(\rho+1)I)\right| \\ &= (2r)^{m+n}(\rho+2)^{m-n}\left|(m(G))^2 - (2\rho - r + 3)m(G) + (\rho^2 - (r-2)\rho - r)I\right| \\ &= (2r)^{m+n}(\rho+2)^{m-n}\prod_{i=1}^{n}\left(\rho_i^2 - (2\rho - r + 3)\rho_i + \rho^2 - (r-2)\rho - r\right) \\ &= (2r)^{m+n}(\rho+2)^{m-n}\prod_{i=1}^{n}\left(\rho^2 - (2\rho_i + r - 2)\rho + \rho_i^2 + (r-3)\rho_i - r\right), \end{aligned}$$

where ρ_i, $(i = 1, 2, 3, \ldots, n)$ are minimum degree eigenvalues of A. Now we prove the following theorems.

Theorem 2.2 *If G is a regular graph of degree r, with n vertices and m edges, then*

$$E_m(T(G)) = |4r(m-n)| + n\sum_{i=1}^{n}\left|\left(2r\rho_i + r^2 - 2r + r\sqrt{r^2+4+4\rho_i}\right)\right|$$

$$+n\sum_{i=1}^{n}\left|\left(2r\rho_i + r^2 - 2r - r\sqrt{r^2+4+4\rho_i}\right)\right|.$$

Proof. The minimum degree characteristic polynomial of $T(G)$ is

$$\phi(T(G);\mu) = (2r)^{m+n}\left(\frac{\mu}{2r}+2\right)^{m-n}\prod_{i=1}^{n}\left(\frac{\mu^2}{4r^2} - (2\rho_i + r - 2)\frac{\mu}{2r} + \rho_i^2 + (r-3)\rho_i - r\right).$$

Hence minimum degree eigenvalues of $T(G)$ are $-4r$ [$(m-n)$ times] and $2\rho_i r + r^2 - 2r \pm r\sqrt{r^2+4+4\rho_i}$ [n times]. Thus

$$E_m(T(G)) = 4r(m-n) + n\sum_{i=1}^{n}\left|\left(2r\rho_i + r^2 - 2r + r\sqrt{r^2+4+4\rho_i}\right)\right|$$

$$+n\sum_{i=1}^{n}\left(2r\rho_i + r^2 - 2r - r\sqrt{r^2+4+4\rho_i}\right)\Big|.$$

Theorem 2.3 *For all $n \geq 3$, the minimum degree of $T(C_n)$ is ,*

$$E_m(T(C_n)) = 8(m-n)| + n\sum_{i=1}^{n}\left(16\cos\frac{2\pi i}{n} + 4\sqrt{2 + 4\cos\frac{2\pi i}{n}}\right)$$

$$+n\sum_{i=1}^{n}\left(16\cos\frac{2\pi i}{n}-4\sqrt{2+4\cos\frac{2\pi i}{n}}\right)\Bigg|.$$

Proof. Since C_n is a regular graph of degree 2, the minimum degree characteristic polynomial of $T(C_n)$ is

$$\phi(T(C_n);\mu)=(4)^{m+n}(\frac{\mu}{4}+2)^{m-n}\prod_{i=1}^{n}\left(\frac{\mu^2}{16}-2\cos\frac{2\pi i}{n}\mu+16\cos^2\frac{2\pi i}{n}-4\cos\frac{2\pi i}{n}-2\right).$$

Hence the minimum degree eigenvalues of $T(C_n)$ is 8 [$(m \quad n)$ times] and $16\cos\frac{2\pi i}{n}\pm 4\sqrt{2+4\cos\frac{2\pi i}{n}}$ [n times]. Thus

$$E_m(T(C_n))=|8(m-n)|+n\sum_{i=1}^{n}\left|\left(16\cos\frac{2\pi i}{n}+4\sqrt{2+4\cos\frac{2\pi i}{n}}\right)\right|$$

$$+n\sum_{i=1}^{n}\left|\left(16\cos\frac{2\pi i}{n}-4\sqrt{2+4\cos\frac{2\pi i}{n}}\right)\right|.$$

Theorem 2.4 *If G is a cocktail party graph with $n=2k$ vertices, then*

$$E_m(G)=2(n-2)^2.$$

Proof. Since G is a cocktail party graph with $n=2k$, it is regular graph of degree $n-2$. Then minimum degree characteristic polynomial of G is

$$\phi(G;\mu)=(\mu-2(k-1)(2k-2))\mu^k(\mu+2(2k-2))^{k-1}.$$

Therefore, the minimum degree eigenvalues of G are $2(k-1)(2k-2)$, 0, $-2(2k-2)$ with multiplicity 1, k and $(k-1)$. Hence

$$E_m(G)=|2(k-1)(2k-2)|+|0|k+|-2(2k-2)|(k-1)$$

$$=4(k-1)(2k-2)$$

$$=2(n-2)^2.$$

Theorem 2.5 *For $n\geq 2$, the minimum degree energy of the crown S_n^0 is equal to*

$$E_m(S_n^0)=4(n-1)^2.$$

Proof. The minimum degree characteristic polynomial of S_n^0 is

$$\phi(S_n^0 : \mu) = (\mu^2 \quad (n \quad 1)^2)^{n \quad 1}(\mu^2 \quad (n \quad 1)^2),$$

hence minimum degree eigenvalues of S_n^0 are $(n-1)((n-1)times)$, $-(n-1)((n-1)times)$, $(n-1)^2(1time)$ and $-(n-1)^2(1time)$, thus, minimum degree energy of S_n^0 is

$$E_m(S_n^0) = |n-1|\,(n-1) + |-(n-1)|\,(n-1) + \left|-(n-1)^2\right| + \left|(n-1)^2\right|$$

$$E_m(S_n^0) = 4(n-1)^2.$$

Acknowledgements: The author is thankful to Prof.Chandrashekara Adiga for his encouragement and suggestions.

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