MINIMUM DEGREE ENERGY FOR SOME CLASSES OF GRAPHS

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Abstract

In this paper we find minimum degree energy for graphs like $C_n \times P_2$, Totalgraph of $C_n$, Totalgraph of regular graph of degree $r$ and Cocktail party graph

Keywords and Phrases : Minimum degree matrix, minimum degree eigenvalues, minimum degree energy of a graph.

2000 AMS Subject Classification : 05C50.

1 Introduction

Let $G$ be a simple graph and let its vertex set be $V(G) = \{v_1, v_2, ..., v_n\}$. The adjacency matrix $A(G)$ of the graph $G$ is a square matrix of order $n$ whose $(i, j)$-entry is equal to unity if the vertices $v_i$ and $v_j$ are adjacent, and is equal to zero otherwise. The eigenvalues $\lambda_1, \lambda_2, ..., \lambda_n$ of $A(G)$, assumed in non increasing order, are the eigenvalues of the graph $G$.

The energy of $G$ was first defined by I.Gutman [4] in 1978 as the sum of the absolute values of its eigenvalues:

$$E(G) = \sum_{i=1}^{n} |\lambda_i|.$$ 

Ever since the graph energy $E(G)$ of a simple graph $G$ was introduced by I.Gutman [4], there is a constant stream of papers devoted to this topic. Survey of development before 2001 can be found in [5]. For recent developments one can consult [3]. The energy of a graph has close links to chemistry (see for instance [6]). Let $G$ be a simple graph with $n$ vertices $v_1, v_2, \ldots, v_n$ and let $d_i$ be the degree of $v_i$, $i = 1, 2, 3, \ldots, n$. Define

$$d_{ij} = \begin{cases} 
\min\{d_i, d_j\} & \text{if } v_i \text{ and } v_j \text{ are adjacent,} \\
0 & \text{otherwise.} 
\end{cases}$$

Then the $n \times n$ matrix $m(G) = (d_{ij})$ is called the minimum degree matrix of $G$. This was introduced and studied in [1, 2]. The characteristic polynomial of the minimum degree matrix $m(G)$ is defined by

$$\phi(G; \mu) = \det(\mu I - m(G)).$$
\[
\mu^n + c_1 \mu^{n-1} + c_2 \mu^{n-2} + \ldots + c_n = \mu_n + c_n,
\]

where I is the unit matrix of order n. The minimum degree eigenvalues \(\mu_1, \mu_2, \ldots, \mu_n\) of the graph \(G\), assumed in the non-increasing order, are the eigenvalues of its minimum degree matrix \(m(G)\).

The minimum degree energy of a graph \(G\) is defined as

\[
E_m(G) = \sum_{i=1}^{n} |\mu_i|.
\]

Since \(m(G)\) is a real symmetric matrix with zero trace, these minimum degree eigenvalues are all real with sum equal to zero. In this paper, we compute the minimum degree energy \(E_m(G)\) of some classes of graphs like \(C_n \times P_2\), Totalgraph of \(C_n\), Totalgraph of regular graph of degree \(r\) and Cocktail party graph.

2. MINIMUM DEGREE ENERGY

In this section we find minimum degree energy of some classes of graphs.

**Definition 2.1** If \(G_1 = (V_1, E_1)\) and \(G_2 = (V_2, E_2)\) are two graphs then the cartesian product of \(G_1\) and \(G_2\) denoted by \(G_1 \times G_2 = G(V, E)\) consists a vertex set \(V = V_1 \times V_2\) and \((x_1, y_1), (x_2, y_2) \in V_1 \times V_2\) are adjacent if \(x_1 x_2 \in E_1\) and \(y_1 = y_2\) or \(y_1 y_2 \in E_2\) and \(x_1 = x_2\).

**Theorem 2.1** The minimum degree energy of \(C_n \times P_2\) is

\[
E_m(C_n \times P_2) = \sum_{i=1}^{n} \left| 1 + 4 \cos \frac{2\pi i}{n} \right| + \sum_{i=1}^{n} \left| 1 + 4 \cos \frac{2\pi i}{n} \right|.
\]

**Proof.** The minimum degree characteristic polynomial of \(C_n\) is

\[
P_{C_n}(\mu) = -2^{n+1} + 2^{n} \sum_{k=0}^{[\frac{n}{2}]} (-1)^k \frac{n}{n-k} \left(\begin{array}{c} n \\ k \\ \end{array} \right) \left(\frac{\mu}{2} \right)^{2k}.
\]

The minimum degree eigenvalues of cycle \(C_n\) are the numbers \(4 \cos \frac{2\pi i}{n}\), for \(i = 1, 2, 3, \ldots, n\). The minimum degree characteristic polynomial of \(P_2\) is

\[
P_{P_2}(\mu) = \mu^2 - 1,
\]

the minimum degree eigenvalues os \(P_2\) are 1 and 1. The eigenvalues of \(C_n \times P_2\) are \(1 + 4 \cos \frac{2\pi i}{n}\).
and \(1 + 4 \cos \frac{2\pi i}{n}\) \((i = 1, 2, 3, \ldots, n)\). Thus
\[
E_m(C_n \times P_2) = \sum_{i=1}^{n} \left| -1 + 4 \cos \frac{2\pi i}{n} \right| + \sum_{i=1}^{n} \left| 1 + 4 \cos \frac{2\pi i}{n} \right|.
\]

Also the minimum degree energy of \(C_n \times P_3\) is
\[
E_m(C_n \times P_3) = \sum_{i=1}^{n} \left| 4 \cos \frac{2\pi i}{n} \right| + \sum_{i=1}^{n} \left| 2 + 4 \cos \frac{2\pi i}{n} \right| + \sum_{i=1}^{n} \left| 2 + 4 \cos \frac{2\pi i}{n} \right|.
\]

**Definition 2.2** The total graph \(T(G)\) of graph \(G\) is that graph whose set of vertices is the union of the set of vertices and of the set of edges of \(G\), with two vertices of \(T(G)\) being adjacent if and only if the corresponding elements of \(G\) are adjacent or incident.

Let \(G\) be regular graph of degree \(r\), having \(n\) vertices and \(m\) edges, then the minimum degree matrix \(m(T(G))\) of total graph \(T(G)\) with vertex set \((v_1, v_2, \ldots, v_n, e_{n+1}, e_{n+1}, \ldots, e_{m+n})\) can be written in the form,
\[
m(T(G)) = \begin{pmatrix} A & R \\ R^T & B \end{pmatrix},
\]
where \(A = [a_{ij}]\),
\[
a_{ij} = \begin{cases} 2r & \text{if } v_i \text{ and } v_j \text{ are adjacent} \\ 0 & \text{otherwise}, \end{cases}
\]

\(B = [b_{ij}]\),
\[
b_{ij} = \begin{cases} 2r & \text{if } e_{n+1} \text{ and } e_{n+j} \text{ are adjacent} \\ 0 & \text{otherwise}, \end{cases}
\]
and \(R = [r_{ij}]\),
\[
r_{ij} = \begin{cases} 2r & \text{if } v_i \text{ and } e_{n+j} \text{ are adjacent} \\ 0 & \text{otherwise}. \end{cases}
\]

Hence the characteristic polynomial \(\phi(T(G); \mu)\) of \(m(T(G))\) is obtained as follows,
\[
P_{T(G)}(\mu) = (2r)^{m+n} \begin{vmatrix} \frac{\mu}{2r} I + r I - RR^T & -R \\ -R^T & \frac{\mu}{2r} I + 2I - R^T R \end{vmatrix}.
\]
where $\rho = \frac{\mu}{2r}$.

$$P_{T(G)}(\mu) = (2r)^{m+n}
\begin{vmatrix}
(\rho + r)I - RR^T + \frac{R}{\rho + 2} \left( - (\rho + r + 1)R^T + R^T RR^T \right) & 0 \\
- (\rho + r + 1)R^T + R^T RR^T & (\rho + 2)I
\end{vmatrix} .$$

$$P_{T(G)}(\mu) = (2r)^{m+n}(\rho + 2)^m \left| \rho I - m(G) + \frac{1}{\rho + 2} (m(G) + rI)(m(G) - (\rho + 1)I) \right|
= (2r)^{m+n}(\rho + 2)^{-n} (m(G))^2 - (2\rho - r + 3)m(G) + (\rho^2 - (r - 2)\rho - r)I
= (2r)^{m+n}(\rho + 2)^{-n} \prod_{i=1}^{n} \left( \rho_i^2 - (2\rho_i - r + 2)\rho + \rho_i^2 - (r - 3)\rho_i - r \right),$$

where $\rho_i$, $(i = 1, 2, 3, \ldots, n)$ are minimum degree eigenvalues of $A$. Now we prove the following theorems.

**Theorem 2.2** If $G$ is a regular graph of degree $r$, with $n$ vertices and $m$ edges, then

$$E_m(T(G)) = |4r(m - n)| + n \sum_{i=1}^{n} \left| 2r\rho_i + r^2 - 2r + r\sqrt{r^2 + 4 + 4\rho_i} \right|$$

$$+ n \sum_{i=1}^{n} \left| 2r\rho_i + r^2 - 2r - r\sqrt{r^2 + 4 + 4\rho_i} \right| .$$

**Proof.** The minimum degree characteristic polynomial of $T(G)$ is

$$\phi(T(G); \mu) = (2r)^{m+n}(\frac{\mu}{2r} + 2)^{m-n} \prod_{i=1}^{n} \left( \frac{\mu^2}{4r^2} - (2\rho_i + r - 2)\frac{\mu}{2r} + \rho_i^2 + (r - 3)\rho_i - r \right).$$

Hence minimum degree eigenvalues of $T(G)$ are $-4r$ [(m - n) times] and $2\rho_i r + r^2 - 2r \pm r\sqrt{r^2 + 4 + 4\rho_i}$ [n times]. Thus

$$E_m(T(G)) = 4r(m - n) + n \sum_{i=1}^{n} \left| 2r\rho_i + r^2 - 2r + r\sqrt{r^2 + 4 + 4\rho_i} \right|$$

$$+ n \sum_{i=1}^{n} \left| 2r\rho_i + r^2 - 2r - r\sqrt{r^2 + 4 + 4\rho_i} \right| .$$

**Theorem 2.3** For all $n \geq 3$, the minimum degree of $T(C_n)$ is

$$E_m(T(C_n)) = 8(m - n) + n \sum_{i=1}^{n} \left( 16 \cos \frac{2\pi i}{n} + 4 \sqrt{2 + 4 \cos \frac{2\pi i}{n}} \right).$$
\[ +n \sum_{i=1}^{n} \left( 16 \cos \frac{2\pi i}{n} - 4 \sqrt{2 + 4 \cos \frac{2\pi i}{n}} \right) . \]

**Proof.** Since \( C_n \) is a regular graph of degree 2, the minimum degree characteristic polynomial of \( T(C_n) \) is

\[ \phi(T(C_n); \mu) = (4)^{m+n} \left( \frac{\mu}{4} + 2 \right)^{m-n} \prod_{i=1}^{n} \left( \frac{\mu^2}{16} - 2 \cos \frac{2\pi i}{n} \mu + 16 \cos^2 \frac{2\pi i}{n} - 4 \cos \frac{2\pi i}{n} - 2 \right) . \]

Hence the minimum degree eigenvalues of \( T(C_n) \) is \( 8 \) \([m \ n \ \text{times}]\) and \( 16 \cos \frac{2\pi i}{n} \pm 4 \sqrt{2 + 4 \cos \frac{2\pi i}{n}} \) \([n \ \text{times}]\). Thus

\[ E_m(T(C_n)) = |8(m - n)| + n \sum_{i=1}^{n} \left| \left( 16 \cos \frac{2\pi i}{n} + 4 \sqrt{2 + 4 \cos \frac{2\pi i}{n}} \right) \right| \]

\[ + n \sum_{i=1}^{n} \left| \left( 16 \cos \frac{2\pi i}{n} - 4 \sqrt{2 + 4 \cos \frac{2\pi i}{n}} \right) \right| . \]

**Theorem 2.4** If \( G \) is a cocktail party graph with \( n = 2k \) vertices, then

\[ E_m(G) = 2(n - 2)^2 . \]

**Proof.** Since \( G \) is a cocktail party graph with \( n = 2k \), it is regular graph of degree \( n - 2 \). Then minimum degree characteristic polynomial of \( G \) is

\[ \phi(G; \mu) = (\mu - 2(k - 1)(2k - 2))\mu^k(\mu + 2(2k - 2))^{k-1} . \]

Therefore, the minimum degree eigenvalues of \( G \) are \( 2(k - 1)(2k - 2) \), 0, \(-2(2k - 2)\) with multiplicity 1, \( k \) and \((k - 1)\). Hence

\[ E_m(G) = |2(k - 1)(2k - 2)| + |0|k + | -2(2k - 2)|(k - 1) \]

\[ = 4(k - 1)(2k - 2) \]

\[ = 2(n - 2)^2 . \]

**Theorem 2.5** For \( n \geq 2 \), the minimum degree energy of the crown \( S_n^0 \) is equal to

\[ E_m(S_n^0) = 4(n - 1)^2 . \]
Proof. The minimum degree characteristic polynomial of $S_n^0$ is

$$\phi(S_n^0 : \mu) = (\mu^2 - (n-1)^2)^{n-1} (\mu^2 - (n-1)^2),$$

hence minimum degree eigenvalues of $S_n^0$ are $(n-1)((n-1)\times)_{times}$, $-(n-1)((n-1)\times)_{times}$, $(n-1)^2(1\times)_{time}$ and $-(n-1)^2(1\times)_{time}$, thus, minimum degree energy of $S_n^0$ is

$$E_m(S_n^0) = |n-1|(n-1) + |-(n-1)|(n-1) + |-(n-1)^2| + |(n-1)^2|$$

$$E_m(S_n^0) = 4(n-1)^2.$$

Acknowledgements: The author is thankful to Prof. Chandrashekara Adiga for his encouragement and suggestions.

References


