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A New Modular Relation of Ratio's of Ramanujan Quantity of Degree 9

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1. Introduction

In Chapter 16 of his second notebook [1], Ramanujan develops the theory of theta-function and is defined by

$$f(a,b) := \sum_{n=-\infty}^{\infty} a^{\frac{n(n+1)}{2}} b^{\frac{n(n-1)}{2}}, \mid ab \mid < 1, \tag{1.1}$$

$$= (-a;ab)_\infty(-b;ab)_\infty(ab;ab)_\infty$$

where $(a;q)_0 = 1$ and $(a;q)_\infty = (1-a)(1-aq)(1-aq^2)\cdots$.

Following Ramanujan, we defined

$$\varphi(q) := f(q,q) = \sum_{n=-\infty}^{\infty} q^{n^2} = \frac{(-q;-q)_\infty}{(q;-q)_\infty}, \tag{1.2}$$

$$\psi(q) := f(q,q^3) = \sum_{n=0}^{\infty} q^{\frac{n(n+1)}{2}} = \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty}, \tag{1.3}$$

$$f(-q) := f(-q,-q^2) = \sum_{n=-\infty}^{\infty} (-1)^n q^{\frac{n(3n-1)}{2}} = (q;q)_\infty \tag{1.4}$$

and

$$\chi(q) := (-q;q^2)_\infty. \tag{1.5}$$

Now we define a modular equation in brief. The ordinary hypergeometric series ${}_2F_1(a,b;c;x)$ is defined by

$${}_2F_1(a,b;c;x) := \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n n!} x^n,$$

where $(a)_0 = 1, (a)_n = a(a+1)(a+2)\cdots(a+n-1)$ for any positive integer n, and $|x| < 1$. Let

$$z := z(x) := {}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; x\right) \tag{1.6}$$

and

$$q := q(x) := \exp\left(-\pi \frac{{}_2F_1(\frac{1}{2}, \frac{1}{2}; 1; 1-x)}{{}_2F_1(\frac{1}{2}, \frac{1}{2}; 1; x)}\right), \tag{1.7}$$

where $0 < x < 1$.
Let r denote a fixed natural number and assume that the following relation holds:

$$r\frac{{}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; 1-\alpha\right)}{{}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; \alpha\right)} = \frac{{}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; 1-\beta\right)}{{}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; \beta\right)}. \tag{1.8}$$

Then a modular equation of degree r in the classical theory is a relation between α and β induced by (1.8). We often say that β is of degree r over α and $m := \dfrac{z(\alpha)}{z(\beta)}$ is called the multiplier. We also use the notations $z_1 := z(\alpha)$ and $z_r := z(\beta)$ to indicate that β has degree r over α.

In [6],[7] Nikos Bagis define Ramanujan Quantities $R(a, b, p; q)$ as

$$R(a, b, p; q) = q^{-(a-b)/2+(a^2-b^2)/(2p)} \frac{\prod_{n=0}^{\infty}(1 - q^a q^{np})(1 - q^{p-a} q^{np})}{\prod_{n=0}^{\infty}(1 - q^b q^{np})(1 - q^{p-b} q^{np})}, \tag{1.9}$$

where a, b,and p are positive rationales such that $a + b < p$. General Theorem such

$$\frac{q^{B-A}}{1 - a_1 b_1} + \frac{(a_1 - b_1 q_1)(b_1 - a_1 q_1)}{(1 - a_1 b_1)(q_1^2 + 1)} + \frac{(a_1 - b_1 q_1^3)(b_1 - a_1 q_1^3)}{(1 - a_1 b_1)(q_1^4 + 1)} + \cdots$$
$$= \frac{\prod_{n=0}^{\infty}(1 - q^a q^{np})(1 - q^{p-a} q^{np})}{\prod_{n=0}^{\infty}(1 - q^b q^{np})(1 - q^{p-b} q^{np})} \tag{1.10}$$

where $a_1 = q^A$, $b_1 = q^B$, $q_1 = q^{A+B}$, $a = 2A + 3p/4$, $2B + p/4$, and $p = 4(A + B)$, $|q| < 1$, are proved.

In [3], B. N. Dharmendra, C. S. Shivakumar Swamy, and S. Vasanth Kumar established the several modular equations of Ramanujan Quantities $R(1, 2, 4; q)$ and $R(1, 2, 4; q^n)$ for $n = 4, 6, 8, 9, 10, 11, 13, 14, 15, 17, 19, 23$ and 25.

Recently [4], Dharmendra B. N.* and M. S. Ramesh established a new modular equations of ratios of Ramanujan Quantities $\frac{R(1,2,4;q)}{R(1,2,4;q^5)}$ and $\frac{R(1,2,4;q^n)}{R(1,2,4;q^{5n})}$ degree 5 for $n = 2, 3, 5$ and 7.

Motivated the above work we obtain certain modular relations between $\frac{R(1,2,4;q)}{R(1,2,4;q^9)}$ and $\frac{R(1,2,4;q^n)}{R(1,2,4;q^{9n})}$ for $n = 2, 3, 5$ and 7.

2. Preliminary results

Definition 2.1. [7]

$$[a, p; q] = (q^{p-a}; q^p)_\infty (q^a; q^p)_\infty \tag{2.1}$$

where $q = e^{-\pi\sqrt{r}}$ *and* $a, p, r > 0$.

Definition 2.2. [7]

$$R(a, b, p; q) := q^{-(a-b)/2+(a^2-b^2)/(2p)} \frac{[a, p; q]}{[b, p; q]} \tag{2.2}$$

Lemma 2.1. [1, Ch. 16, Entry 25, p.40]

$$\psi(q)\psi(-q) = \psi(q^2)\varphi(-q^2). \tag{2.3}$$

Lemma 2.2. [1, Ch. 17, Entry 10-11, p.122-123]

$$\varphi(-q^2) = \sqrt{z}(1-\alpha)^{1/8} \tag{2.4}$$

$$\psi(-q) = \sqrt{\frac{1}{2}z}\{\alpha(1-\alpha)q^{-1}\}^{1/8} \tag{2.5}$$

where $q = e^{-y}$

Lemma 2.3. [5] *If* $P = \dfrac{\psi(q)}{q\,\psi(q^9)}$ *and* $Q = \dfrac{\psi(q^2)}{q^2\,\psi(q^{18})}$, *then*

$$\frac{P}{Q} + \frac{Q}{P} + 2 = \frac{3}{P} + P. \tag{2.6}$$

Lemma 2.4. [5] *If* $P = \dfrac{\psi(q)}{q\,\psi(q^9)}$ *and* $Q = \dfrac{\psi(q^3)}{q^3\,\psi(q^{27})}$, *then*

$$\left(3 - P - \frac{3}{P}\right)\left(3 - Q - \frac{3}{Q}\right) = \left(\frac{Q}{P}\right)^2. \tag{2.7}$$

Lemma 2.5. [5] *If* $P = \dfrac{\psi(q)\psi(q^5)}{q^6\psi(q^9)\psi(q^{45})}$ *and* $Q = \dfrac{\psi(q)\psi(q^{45})}{q^{-4}\psi(q^9)\psi(q^5)}$, *then*

$$\begin{aligned} &Q^3 + \frac{1}{Q^3} - 15\left(Q^2 + \frac{1}{Q^2}\right) - 45\left(Q + \frac{1}{Q}\right) - \left(P^2 + \frac{81}{P^2}\right) - 10\left(P + \frac{9}{P}\right) \\ &\times \left[2 + Q + \frac{1}{Q}\right] + 5\left(\sqrt{P^3} + \frac{3^3}{\sqrt{P^3}}\right)\left(\sqrt{Q} + \frac{1}{\sqrt{Q}}\right) + 15\left(\sqrt{P} + \frac{3}{\sqrt{P}}\right) \\ &\times \left[\left(\sqrt{Q^3} + \frac{1}{\sqrt{Q^3}}\right) + 2\left(\sqrt{Q} + \frac{1}{\sqrt{Q}}\right)\right] = 40. \end{aligned} \tag{2.8}$$

Lemma 2.6. [5] *If* $P = \dfrac{\psi(q)\psi(q^7)}{q^8\,\psi(q^9)\psi(q^{63})}$ *and* $Q = \dfrac{\psi(q)\psi(q^{63})}{q^{-6}\,\psi(q^9)\psi(q^7)}$, *then*

$$\begin{aligned} &Q^4 + \frac{1}{Q^4} - 35\left(Q^3 + \frac{1}{Q^3}\right) - 413\left(Q^2 + \frac{1}{Q^2}\right) - 1379\left(Q + \frac{1}{Q}\right) - 1694 \\ &- \left(P^3 + \frac{9^3}{P^3}\right) - 7\left(P^2 + \frac{9^2}{P^2}\right)\left[7 + 3\left(Q + \frac{1}{Q}\right)\right] - 21\left(P + \frac{9}{P}\right) \\ &\left[21 + 14\left(Q + \frac{1}{Q}\right) + 3\left(Q^2 + \frac{1}{Q^2}\right)\right] + 7\left(\sqrt{P^5} + \frac{3^5}{\sqrt{P^5}}\right)\left(\sqrt{Q} + \frac{1}{\sqrt{Q}}\right) \\ &+ 63\left[\sqrt{P} + \frac{3}{\sqrt{P}}\right]\left[7\left(\sqrt{Q^3} + \frac{1}{\sqrt{Q^3}}\right) + 14\left(\sqrt{Q} + \frac{1}{\sqrt{Q}}\right) + \left(\sqrt{Q^5} + \frac{1}{\sqrt{Q^5}}\right)\right] \\ &+ 21\left(\sqrt{P^3} + \frac{3^3}{\sqrt{P^3}}\right)\left[2\left(\sqrt{Q^3} + \frac{1}{\sqrt{Q^3}}\right) + 7\left(\sqrt{Q} + \frac{1}{\sqrt{Q}}\right)\right] = 0. \end{aligned} \tag{2.9}$$

3. Modular relations Ramanujan Quantities of $R(1,2,4;q)$

In this section, we obtain certain modular relations between $\frac{R(1,2,4;q)}{R(1,2,4;q^9)}$ and $\frac{R(1,2,4;q^n)}{R(1,2,4;q^{9n})}$ for $n = 2, 3, 5$ and 7.

Theorem 3.1. *If* $U := \dfrac{R(q)}{R(q^9)}$ *and* $V := \dfrac{R(q^2)}{R(q^{18})}$, *then*

$$\begin{aligned} &(V - 2V^2 + V^3)U^4 + (2V + 2V^3)U^3 + (-1 - 2V^2 - 2V - 2V^3 - V^4)U^2 \\ &+ (2V + 2V^3)U + V - 2V^2 + V^3 = 0. \end{aligned} \tag{3.1}$$

Proof. Employing lemma (2.1) and (2.2) with $a = 1, b = 2$ *and* $p = 4$, we get

$$R(q) := R(1, 2, 4; q) = q^{1/8}\frac{(q; q^4)_\infty(q^3; q^4)_\infty}{(q^2; q^4)_\infty(q^2; q^4)_\infty} \tag{3.2}$$

Using the equations (1.1), (1.2) and (1.3), then the above equation can be written as

$$R(q) = q^{1/8}\frac{f(-q, -q^3)}{f(-q^2, -q^2)} = q^{1/8}\frac{\psi(-q)}{\varphi(-q^2)} \tag{3.3}$$

by using lemma(2.3), the equation (3.3) can be expressed as,

$$R(q) = q^{1/8}\frac{\psi(q^2)}{\psi(q)} \tag{3.4}$$

by substituting the above equation (3.4) in the lemma (2.6), we get

$$\begin{aligned}
&(-U^3V^4 + 2U^2V^4 - 2U^3V^3 + U^4V^2 - UV^4 + 2U^3V^2 - 2UV^3 + 2U^2V^2 \\
&- 2U^3V + 2UV^2 - U^3 + V^2 - 2UV + 2U^2 - U)(U^4V^4 - U^3V^4 - U^4V^3 \\
&+ 3U^3V^3 - U^2V^3 - U^3V^2 - UV^3 + 4U^2V^2 - U^3V - UV^2 - U^2V + 3UV \\
&- V - U + 1)(U^4V^3 - U^2V^4 + 2U^3V^3 - 2U^4V^2 - 2U^2V^3 + U^4V + 2UV^3 \\
&- 2U^2V^2 + 2U^3V + V^3 - 2U^2V - 2V^2 + 2UV - U^2 + V) = 0
\end{aligned} \tag{3.5}$$

By examining the behavior of the above factors near $q = 0$, we can find a neighborhood about the origin, where the third factor is zero; whereas other factors are not equal to zero in this neighborhood. By the Identity Theorem third factor vanishes identically, thus we obtain (3.1).

□

Theorem 3.2. *If* $U := \frac{R(q)}{R(q^9)}$ *and* $V := \frac{R(q^3)}{R(q^{27})}$, *then*

$$\begin{aligned}
&(V^2 - V + 1)U^6 + (4V - 7V^2 - 1 + 5V^3 - V^5 - 2V^4)U^5 \\
&+ (-7V + 13V^2 + 8V^4 + 1 - 2V^5 - 14V^3)U^4 + (22V^3 - 14V^2 \\
&+ 5V - 14V^4 + 5V^5)U^3 + (-2V - 14V^3 + 13V^4 + 8V^2 + V^6 - 7V^5)U^2 \\
&+ (-V - 7V^4 + 4V^5 + 5V^3 - 2V^2 - V^6)U + V^6 - V^5 + V^4 = 0.
\end{aligned} \tag{3.6}$$

Proof. Employing the above equation (3.4) in the lemma (2.7), we get (3.6). □

Theorem 3.3. *If* $U := \frac{R(q)}{R(q^9)}$ *and* $V := \frac{R(q^5)}{R(q^{45})}$, *then*

$$\begin{aligned}
&U^6 + (-5V^4 - 5V - V^5 + 5V^2)U^5 + (20V^4 - 5V^5 + 5V - 5V^2)U^4 \\
&- 20U^3V^3 + (-5V^4 + 5V^5 - 5V + 20V^2)U^2 + (5V^4 - V - 5V^5 - 5V^2)U + V^6 = 0
\end{aligned} \tag{3.7}$$

Proof. Employing the above equation (3.4) in the lemma (2.8), we get (3.7).

Theorem 3.4. *If* $U := \dfrac{R(q)}{R(q^9)}$ *and* $V := \dfrac{R(q^7)}{R(q^{63})}$, *then*

$$\begin{aligned} &U^8 + (-7V^2 + 7V - V^7 - 7V^6 - 7V^5 + 14V^4 - 7V^3)U^7 \\ &+ (-21V^2 + 63V^6 + 63V^3 - 14V^4 - 7V^7 - 49V^5 - 7V)U^6 \\ &+ (63V^2 - 105V^3 - 49V^6 - 7V - 7V^7 - 14V^4 + 63V^5)U^5 \\ &+ (-14V^3 - 14V^2 + 98V^4 + 14V - 14V^6 + 14V^7 - 14V^5)U^4 \\ &+ (-7V^7 - 7V - 49V^2 + 63V^3 - 105V^5 - 14V^4 + 63V^6)U^3 \\ &+ (63V^2 - 7V^7 - 7V - 14V^4 - 21V^6 + 63V^5 - 49V^3)U^2 \\ &+ (-V - 7V^6 - 7V^2 - 7V^5 - 7V^3 + 7V^7 + 14V^4)U + V^8 = 0 \end{aligned} \tag{3.8}$$

Proof. Employing the above equation (3.4) in the lemma (2.9), we get (3.8).

References

[1] B. C. Berndt, Ramanujan's Notebooks, Part III, Springer-Verlag, New York (1991).
[2] Andrews, G.E., Berndt, B.C.: Ramanujan's Lost Notebook, Part I. New York. Springer 2005.
[3] B. N. Dharmendra, C. S. Shivakumar Swamy and S. Vasanth Kumar, On a Ramanujan Quantity, J. Ana. Num. Theor. 2(2) (2014), 83-88.
[4] B. N. Dharmendra and Ramesh M. S, A New Modular Relation of Ratios of Ramanujan Quantity of degree 5, International Journal of Pure and Applied Mathematical Sciences. 15(1) (2022) 1-6.
[5] M. S. Mahadeva Naika, K. Sushan Bairy and S. Chandankumar, Modular Equations for the Ratios of Ramanujans Theta Function *psi* And Evaluations, New Zealand Journal of Mathematics, Volume 40 (2010), 33-48.
[6] Nikos Bagis, and M. L. Glasser, Jacobian Elliptic Function, Continued fractions and Ramanujan Quantities, arXiv:1001.2660v1 [math.GM], 2010.
[7] Nikos Bagis, Generatizations of Ramanujan Continued fractions, arXiv:1107.2393v2 [math.GM], 2012.