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Fixed Point Theorems for Expansion Mappings in Sequentially Complete Quasi-Gauge Function Space

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Abstract: ***In this paper, some fixed point theorems for expansion mappings are proved in sequentially complete quasi-gauge function space generated by the family of pseudo metrics. Our results generalized many known results.***

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1. Introduction

Quasi-gauge space was first developed by Reilly [8,9]. It is one of the space in which Banach contraction principle has been carried over. A quasi-gauge structure for topological spaces (X,T)is a family P of pseudometrics on X such that T has a subbase, i.e., the family β(X, P, ε) is the set {y ∈ X : p(x, y) <ε}. If the topological space (X,T) has a quasi –gauge structure P, it is called a quasi-gauge space and is denoted by (X, P)

To establish our main result, we need the following definitions from [1] [8] and [9]:

2. Preliminaries

Definition 2.1. Let X be a non-empty set and Y^xReilly [8,9] be a quasi-gauge function space. A non-negative real valued function p defined on the function space (X^x× Y^x) having pointwise topology with the properties that:
(i) p(f, g)(x) = 0 if f = g ∈ Y^x and
(ii) p(f, g)(x) ≤ p(f, h)(x) + p(h, g)(x) for all f, g, h ∈ Y^x
Is called a quasi-gauge metric.

Definition 2.2.A sequence {f_n} in a quasi-gauge function space (Y^x, P) is called p-Cauchy, iffor every p ∈ P, there is an integer k, such that p(f_m, f_n)(x) <ε for all m, n ≥ k.

Definition 2.3.A quasi-gauge function space (Y^x, P) is called sequentially complete, if every p-Cauchy sequence in Y^x converges in Y^x .

Definition 2.4. An operator T on a quasi-gauge function space (Y^x, P) into itself is said to be an expansion map, if p(Tf,Tg)(x) ≥ λp(f, g)(x), for all f, g ∈ Y^x , λ > 1.

Throughout in this paper we use the symbol ;
p(f, g)(x). p(f, g)(x) = p^2(f, g)(x)

3. Main Result

Theorem 3.1. Let (Y^x, P) be a sequentially complete quasi-gauge function space generated by the family P of pseudo metrics and let T_1 and T_2 be any two operators on Y^x, such that
(3.1.1) T_1 and T_2 are commutes,
(3.1.2) $\left[p\left(T_1^r(f), T_2^s(g)\right)(x)\right]^2$
≥λmin{$[p(f, g)(x)]^2$, $\left[p\left(f, T_1^r(f)\right)(x)\right]^2$, [p(g,
T2sgx2,
$\left[p\left(f, T_1^r(f)\right)(x)\right].[p(f, g)(x)]$,
$\left[p\left(g, T_2^s(g)\right)(x)\right].[p(f, g)(x)]$,
$\left[p\left(f, T_1^r(f)\right)(x)\right].\left[p\left(g, T_2^s(g)\right)(x)\right]$,
$\left[p\left(T_1^r(f), T_2^s(g)\right)(x)\right].[p(f, g)(x)]$}
Where r and s are positive integer and λ > 1. Then T_1 and T_2 have a fixed point in (Y^x, P).

Proof: Define the sequence {f_n} as follows**,**
$f_0(x) = T_1^r(f_1)(x)$, $f_{2n-2}(x) = T_1^r(f_{2n-1})(x)$
$f_1(x) = T_2^s(f_2)(x)$ and$f_{2n-1}(x) = T_2^s(f_{2n})(x)$
If, $f_m = f_{m-1}$for some m, then f_m has a fixed point of T_1 and T_2 .
Hence, without loss of generality we can assume that $f_n = f_{n-1}$for every n.
From (3.1.1), we have
$[p(f_0, f_1)(x)]^2 = \left[p\left(T_1^r(f_1), T_2^s(f_2)\right)(x)\right]^2$
≥ λmin{$[p(f_1, f_2)(x)]^2$, $\left[p\left(f_1, T_1^r(f_1)\right)(x)\right]^2$, [p($f_2$,
T2sf2x2,
$\left[p\left(f_1, T_1^r(f_1)\right)(x)\right].[p(f_1, f_2)(x)]$,
$\left[p\left(f_2, T_2^s(f_2)\right)(x)\right].[p(f_1, f_2)(x)]$,
$\left[p\left(f_1, T_1^r(f_1)\right)(x)\right].\left[p\left(f_2, T_2^s(f_2)\right)(x)\right]$, [p($T_1^r(f_1)$,
T2sf2(x).pf1, f2(x)}
≥ λmin{$[p(f_1, f_2)(x)]^2$, $[p(f_1, f_0)(x)]^2$, $[p(f_2, f_1)(x)]^2$,
$[p(f_1, f_0)(x)].[p(f_1, f_2)(x)]$, $[p(f_2, f_1)(x)].[p(f_1, f_2)(x)]$,
$[p(f_1, f_0)(x)].[p(f_2, f_1)(x)]$, $[p(f_0, f_1)(x)].[p(f_1, f_2)(x)]$}

Thus,
(3.1.3) $[p(f_0, f_1)(x)]^2 \geq \lambda \min\{[p(f_1, f_2)(x)]^2, [p(f_1, f_0)(x)].[p(f_1, f_2)(x)]\}$
Then from (3.1.3) we have,

Case I: If $[p(f_1, f_2)(x)]^2$ is minimum, then
$[p(f_0, f_1)(x)]^2 \geq \lambda[p(f_1, f_2)(x)]^2$, i.e.
(3.1.4) $[p(f_1, f_2)(x)] \leq \frac{1}{\sqrt{\lambda}}[p(f_0, f_1)(x)]$, as $\lambda > 1$.

Case II: If $[p(f_1, f_0)(x)].[p(f_1, f_2)(x)]$ is minimum, then
$[p(f_0, f_1)(x)]^2 \geq \lambda [p(f_1, f_0)(x)].[p(f_1, f_2)(x)]$
Or, $[p(f_0, f_1)(x)] \geq \lambda [p(f_1, f_2)(x)]$,
(3.1.5) $[p(f_1, f_2)(x)] \leq \frac{1}{\lambda}[p(f_0, f_1)(x)] \leq \frac{1}{\sqrt{\lambda}}[p(f_0, f_1)(x)]$, as $\lambda > 1$.
Therefore from (3.1.3), (3.1./4) and (3.1.5), we have
$[p(f_1, f_2)(x)] \leq \frac{1}{\sqrt{\lambda}}[p(f_0, f_1)(x)]$

Hence in general,
$[p(f_{2n}, f_{2n+1})(x)] \leq \left(\frac{1}{\sqrt{\lambda}}\right)^{2n} [p(f_0, f_1)(x)] \to 0$ as $n \to \infty$.

Hence, $\{f_n\}$ is a Cauchy sequence. Since Y^x is sequentially complete, there exists $u \in Y^x$, such that $\lim_{n\to\infty} f_n = u$, and so we have
$\lim_{n\to\infty} T_1^r(f_{2n-1}) = u$ and $\lim_{n\to\infty} T_2^s(f_{2n}) = u$

Thus, u is a common fixed point of T_1 and T_2.

This completes the proof.

Initially, Maia [5], have proved fixed point theorems in space having two different matrices. On the same line we shall obtain a result having two different quasi-gauge function space.

Theorem 3.2. Let Y^x be a sequentially complete quasi-gauge function space with two quasi-gauge structures P and P_1, such that
(3.2.1) $P_1(f, g)(x) = P(f, g)(x)$,
(3.2.2) T_1 and T_2 are continuous w. r. t. P_1,
(3.2.3) Y^x is sequentially complete w. r. t. P_1 and
(3.2.4) T_1 and T_2 satisfies conditions (3.1.1) and (3.1.2) w. r. t. P.
Then T_1 and T_2 have a fixed point.

Proof: Define the sequence $\{f_n\}$ as follows**,**
$f_0(x) = T_1^r(f_1)(x)$, $f_{2n-2}(x) = T_1^r(f_{2n-1})(x)$
$f_1(x) = T_2^s(f_2)(x)$ and $f_{2n-1}(x) = T_2^s(f_{2n})(x)$
Then proceeding as in the proof of theorem 3.1 with similar arguments, we get
$[P(f_{2n}, f_{2n+1})(x)] \leq \left(\frac{1}{\sqrt{\lambda}}\right)^{2n} [P_1(f_0, f_1)(x)]$
Since, $[P_1(f, g)(x)] \leq P(f, g)(x)$, we have
$[P_1(f_{2n}, f_{2n+1})(x)] \leq [P(f_{2n}, f_{2n+1})(x)] \leq \left(\frac{1}{\sqrt{\lambda}}\right)^{2n} [P_1(f_0, f_1)(x)] \to 0$ as $n \to \infty$.

Hence, $\{f_n\}$ is a Cauchy sequence w.r.t. P_1. Since Y^x is sequentially complete w.r.t. P_1, there exists $u \in Y^x$, such that $\lim_{n\to\infty} f_n = u$.

Also, since T_1 and T_2 are continuous w. r. t. P_1, we have
$u = \lim_{n\to\infty} f_{2n+1}$ implies that,
$\lim_{n\to\infty} T_1(f_{2n+1}) = T_1 \lim_{n\to\infty}(f_{2n+1}) = T_1 u$,

Similarly, $u = \lim_{n\to\infty} f_{2n}$ implies that,
$\lim_{n\to\infty} T_2(f_{2n}) = T_2 \lim_{n\to\infty}(f_{2n}) = T_2 u$,

Thus, u is a common fixed point of T_1 and T_2. This completes the proof.

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