Fixed Point Theorems for Expansion Mappings in Sequentially Complete Quasi-Gauge Function Space

A. S. Saluja
Institute for Excellence in Higher Education, Bhopal, M.P., India.
Email: drassaluja(at)gmail.com

Abstract: In this paper, some fixed point theorems for expansion mappings are proved in sequentially complete quasi-gauge function space generated by the family of pseudo metrics. Our results generalized many known results.

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1. Introduction

Quasi-gauge space was first developed by Reilly [8,9] It is one of the space in which Banach contraction principle has been carried over. A quasi-gauge structure for topological spaces (X,T) is a family P of pseudometrics on X such that T has a subbase, i.e., the family β(X, P, ε) is the set {y ∈ X : p(x, y) < ε}. If the topological space (X,T) has a quasi -gauge structure P, it is called a quasi-gauge space and is denoted by (X, P)

To establish our main result, we need the following definitions from [1] [8] and [9]:

2. Preliminaries

Definition 2.1. Let X be a non-empty set and Y be a quasi-gauge function space. A non-negative real valued function p defined on the function space (X × Y) having pointwise topology with the properties that:
(i) p(f, g)(x) = 0 if f = g ∈ Y
(ii) p(f, g)(x) ≤ p(f, h)(x) + p(h, g)(x) for all f, g, h ∈ Y
Is called a quasi-gauge metric.

Definition 2.2. A sequence {f_n} in a quasi-gauge function space (Y, P) is called p-Cauchy, if for every p ∈ P, there is an integer k, such that p(f_m, f_n)(x) < ε for all m, n ≥ k.

Definition 2.3. A quasi-gauge function space (Y, P) is called sequentially complete, if every p-Cauchy sequence in Y converges in Y.

Definition 2.4. An operator T on a quasi-gauge function space (Y, P) into itself is said to be an expansion map, if p(T_f, T_g)(x) ≥ λp(f, g)(x), for all f, g ∈ Y, λ > 1.

Throughout in this paper we use the symbol ; p(f, g)(x). p(f, g)(x) = p^2(f, g)(x)

3. Main Result

Theorem 3.1. Let (Y, P) be a sequentially complete quasi-gauge function space generated by the family P of pseudo metrics and let T_1 and T_2 be any two operators on Y, such that

(3.1.1) T_1 and T_2 are commutes,
(3.1.2) p(T_1 f(x), T_2 g(x)) ≤ \lambda min(p(f, g)(x)), [p(f, T_1 g(x))(x)]^2, [p(f, T_2 g(x))(x)]^2, [p(g, T_1 f(x))(x)]^2, [p(g, T_2 f(x))(x)]^2

Where r and s are positive integer and \lambda > 1. Then T_1 and T_2 have a fixed point in (Y, P).

Proof: Define the sequence {f_n} as follows, f_0(x) = T_1(f_0(x)), f_{n-1}(x) = T_1 T_2 (f_{n-2}(x)) f_1(x) = T_2 (f_2(x)) and f_{n-1}(x) = T_2 (f_{n-2}(x)) If, f_m = f_{m-1} for some m, then f_m has a fixed point of T_1 and T_2.

Hence, without loss of generality we can assume that f_n = f_{n-1} for every n.

From (3.1.1), we have
p(f_0, f_1)(x) ≤ \lambda min(p(f_1, f_2)(x)), [p(f_1, T_1 g(x))(x)]^2, [p(f_1, T_2 g(x))(x)]^2, [p(f_2, T_1 g(x))(x)]^2, [p(f_2, T_2 g(x))(x)]^2,

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Since, arguments, we get $f_t$. P. (3.1.4) $\lbrack p(f(t), f_2(x)) \rbrack \leq \frac{1}{\lambda} p(f_0, f_1(x))$, as $\lambda > 1$.

**Case I:** If $[p(f(t), f_2(x))]$ is minimum, then $[p(f_0, f_1(x))]^2 \geq \lambda [p(f(t), f_2(x))]^2$, i.e.,

$$\text{(3.1.4)} p(f(t), f_2(x)) \leq \frac{1}{\lambda} p(f_0, f_1(x)) \leq \frac{1}{\lambda} [p(f_0, f_1(x))] > 1.$$

Therefore from (3.1.3), (3.1.4) and (3.1.5), we have

$$\text{(3.1.5)} [p(f(t), f_2(x))] \leq \frac{1}{\lambda} [p(f_0, f_1(x))] \leq 0 \text{ as } n \to \infty.$$

Hence, $(f_n)$ is a Cauchy sequence. Since $Y^x$ is sequentially complete, there exists $u \in Y^x$, such that $\lim_{n \to \infty} f_n = u$, and so we have

$$\lim_{n \to \infty} T_1(f_{2n-1}) = u \text{ and } \lim_{n \to \infty} T_2(f_{2n}) = u.$$

Thus, $u$ is a common fixed point of $T_1$ and $T_2$.

This completes the proof.

**Initially,** Maia [5], have proved fixed point theorems in space having two different matrices. On the same line we shall obtain a result having two different quasi-gauge function space.

**Theorem 3.2:** Let $Y^x$ be a sequentially complete quasi-gauge function space with two quasi-gauge structures $P$ and $P_1$, such that

- (3.2.1) $P(f, g)(x) = P(f, g)(x)$,
- (3.2.2) $T_1$ and $T_2$ are continuous w. r. t. $P_1$,
- (3.2.3) $Y^x$ is sequentially complete w. r. t. $P_1$ and $P_2$.
- (3.2.4) $T_1$ and $T_2$ satisfies conditions (3.1.1) and (3.1.2) w. r. t. $P$. Then $T_1$ and $T_2$ have a fixed point.

**Proof:** Define the sequence $(f_n)$ as follows,

$$f_0(x) = T_1(f_0)(x), \quad f_{2n+2}(x) = T_1(f_{2n})(x), \quad f_1(x) = T_2(f_2)(x) \quad \text{and} \quad f_{2n+1}(x) = T_2(f_{2n})(x).$$

Then proceeding as in the proof of theorem 3.1 with similar arguments, we get

$$\text{(3.2.5)} p(f_{2n}, f_{2n+1})(x) \leq \left(\frac{1}{\lambda^2}\right)^{2n} p(f_0, f_1)(x)$$

Since, $[P(f, g)(x)] \leq P(f, g)(x)$, we have

$$\text{(3.2.5)} p(f_{2n}, f_{2n+1})(x) \leq \frac{1}{\lambda} p(f_0, f_1(x)) \to 0 \text{ as } n \to \infty.$$