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On Computation of Isomorphism Classes of Transversals

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Abstract: ***Let G be a finite group and H a subgroup of G. In this article we have developed a program to compute the number of isomorphism classes of transversals. We also have computed the number of isomorphism classes of transversals of all subgroups for non-abelian groups of order up to 22.***

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1 Introduction

Let G be a finite group and Ha subgroup of G. A groupoid is a pair $(S,\circ)$, where S is a non-empty set and $\circ$is a binary operation on S. A right quasigroup is a groupoid $(S,\circ)$ in which the equation $X \circ a = b$; where X is unknown has a unique solution for all $a, b \in S$. The subset S of a group G is a normalized right transversal (NRT) of H in G, if S is obtained by choosing exactly one element from each right coset of H in G, with identity element 1 (1 is the identity element of the group G) and binary operation defined as, for $x, y \in S$, $\{x \circ y\} := S \cap Hxy$. We denote the set of all normalized right transversals of H in G by $T\ (G, H)$. Let $I\ (G, H)$ be the set of isomorphism classes of normalized right transversals of H in G. Throughout this paper for normalized right transversals, we simply write right transversal.

Two right transversals $S, T \in T\ (G, H)$ are called isomorphic (denoted as $S \cong T$) if their induced right-quasigroup structures are isomorphic. Isomorphism classes of transversals have been studied by several authors. By [7], $|I(G, H)| = 1$ if and only if $H \trianglelefteq G$. Also, $|I(G, H)| \neq 2, 4$ [3, 5]. Also in [4], it is proved that $|I(G, H)| = 3$ if and only if H is not normal in G and $[G : H] = 3$. In [2] it is proved that the number of isomorphism classes of left transversals is equal to the number of isomorphism classes of the right transversals, so it is sufficient to compute the number of isomorphism classes of right transversals only.

Recently, in [8], it has been proved that$|I(G, H)| > 16$ for a finite nilpotent group G non-isomorphic to dihedral group of order 8 and H is its non-normal subgroup.

In [5] it is shown that $|I(D_8, H)| = 6$; where D_8 is the dihedral group of order 8 and H is its non normal subgroup of order 2, $|I(Alt(4), H)| = 5$; where H is a subgroup of $Alt\ (4)$ of order 2.

In Section 2, we develop a code for calculating isomorphism classes of transversals. In Section 3, we use this code to find the isomorphism classes of transversals for subgroups of groups of order less than equal to 22. In Section 4, we propose some problems.

2 GAP Code

The GAP [1] code given below is based on the following facts:

(i) For $r, s \in Sym\ (S)$, symmetric group on S, in GAP $rs(x) := s\big(r(x)\big)$; $x \in S$. So for calculating right cosets we need to take left cosets.

(ii) Identifying S with the set $G/^{l}H$ of all left cosets of H in G; we get a transitive permutation representation$\chi_S : G \rightarrow Sym\ (S)$ defined by$\chi_S(g)(x) = gxH \cap S, g \in G, x \in S$. The kernel ($ker\chi_S$) of this action is $Core_G\ (H)$, the core of H in G.

(iii) By [8, p.163], $I\ (G, H) = I\ (G/Core\chi_S(H), H/Core\chi_S\ (H))$. The GAP command FactorCosetAction return the action of G on the cosets of the subgroup of H of G. We identify G and H with $G/Core\chi_S(H)$ and $H/Core\chi_S\ (H)$ respectively. Clearly G and H are subgroups of $Sym\ (S)$. If S has n elements, then we can identify S with the set $\{1,2, \ldots, n\}$such that identity element of S is identified with 1. So $Sym\ (S)$ means $Sym\ (n)$ and $Sym\ (n-1)$ means $Sym\ (\{2,3, \ldots, n\})$.

(iv) By Lemma 2.2 of [2], any two transversals of H in G are isomorphic if and only if they are conjugate by an element of Symmetric group on the set of left cosets of H in G

2.1 Code for Computing Isomorphism Classes of Transversals

Using above facts we can write the GAP [1] code for calculating number of isomorphism classes of transversals of a subgroup in a group as follows.

```
#g group

#h subgroup

g:=SmallGroup (i,j);

cc:=ConjugacyClassesSubgroups (g);

h:=cc[k][1];

f:=FactorCosetAction (g,h);

g:=Image (f);

h:=Set (h);

h:=List (h,x->x^f);

h:=Group (h);c:=RightCosets (g,h);

c[1]:=[ ()];

d:=Cartesian (c);

d:=Set (List (d, x->Set (x)));

n:=Size (c);

d:=Set (d);

i:=1;

while i<= Length (d) do

for x in SymmetricGroup ([2..n]) do

if Set (d[i])<> Set (x*d[i]*x^-1)then

d:=Difference (d,[Set (x*d[i]*x^-1)]);

fi;

od;

i:=i+1;

od;

Print ("Isomorphism Classes of Transversals is ="); Print (Size (d));
```

3 Computation of Isomorphism Classes of Transversals for groups up to order 22

In each subsection of this section we will calculate the number of non-isomorphic left transversals of a subgroup H of a group G except in the following cases:

- Group G is abelian.
- Subgroup H is normal in G because by Main theorem of [7], $|I(G,H)| = 1$.

3.1 Groups of Order 8

There are two non-abelian groups of order *8*; (i) D_8 dihedral group of order *8*, (ii) Quaternion group. In Quaternion group all subgroups are normal.

GAP ID of Group	Subgroup	Index	NRT	$\|I(G,H)\|$
SmallGroup (8, 3) = D_8	h:=cc[3][1]	4	8	6
	h:=cc[4][1]	4	8	6

3.2 Groups of order 10

Dihedral group of order 10, D_{10}, is the only non-abelian group of order 10.

GAP ID of Group	Subgroup	Index	NRT	$\|I(G,H)\|$
SmallGroup (10, 1) = D_{10}	h:=cc[2][1]	5	16	6

3.3 Groups of order 12

There are three non-abelian groups of order 12. By A_4 we mean alternating group on 4 symbols and D_{12} is the dihedral group of order 12 and $C3 : C4 = \langle x, y \mid x^4 = 1 = y^3, xy = y^2x \rangle$.

GAP ID of Group		Subgroup	Index	NRT	$\|I(G,H)\|$
SmallGroup (12, 1)	=C3:C4	h:=cc[4][1]	3	4	3
SmallGroup (12, 3)	= A_4	h:=cc[2][1]	6	32	5
		h:=cc[3][1]	4	27	7
SmallGroup (12, 4)	= D_{12}	h:=cc[2][1]	6	32	20
		h:=cc[4][1]	6	32	20
		h:=cc[6][1]	3	4	3

3.4 Groups of order 14

Dihedral group of order 14, D_{14}, is the only non-abelian group of order 14.

GAP ID of Group	Subgroup	Index	NRT	$\|I(G,H)\|$
SmallGroup (14, 1) = D_{14}	h:=cc[2][1]	7	64	14

3.5 Groups of order 16

There are nine non-abelian group of order 16. The group $Q_8 \times C_2$ has all its subgroups normal. Except this group the rest groups has been shown in the table. Here D_{16} and QD_{16} denote the dihedral and semidihedral group respectively. Also Q_{16} is the generalized quaternion group.

GAP ID of Group	Subgroup	Index	NRT	$\|I(G,H)\|$
SmallGroup (16, 3) = ($C_4 \times C_2$) : C_2	h:=cc[5][1]	8	128	30
	h:=cc[6][1]	8	128	30
	h:=cc[8][1]	4	64	6
	h:=cc[9][1]	4	64	6
	h:=cc[12][1]	4	8	6
	h:=cc[13][1]	4	64	6
SmallGroup (16, 4) = C_4 : C_4	h:=cc[6][1]	4	8	6
	h:=cc[9][1]	4	8	6
SmallGroup (16, 6) = C_8 : C_2	h:=cc[3][1]	8	128	28

SmallGroup (16, 7) = D_{16}	h:=cc[3][1]	8	128	48
	h:=cc[4][1]	8	128	48
	h:=cc[6][1]	4	8	6
	h:=cc[7][1]	4	8	6
SmallGroup (16, 8) = QD_{16}	h:=cc[3][1]	8	128	48
	h:=cc[5][1]	4	8	6
	h:=cc[6][1]	4	8	6
SmallGroup (16, 9) = Q_{16}	h:=cc[4][1]	4	8	6
	h:=cc[5][1]	4	8	6
SmallGroup (16, 11) = $C_2\times D_8$	h:=cc[3][1]	8	128	38
	h:=cc[4][1]	8	128	38
	h:=cc[7][1]	8	128	38
	h:=cc[8][1]	8	128	38
	h:=cc[16][1]	4	8	6
	h:=cc[17][1]	4	8	6
	h:=cc[18][1]	4	8	6
	h:=cc[19][1]	4	8	6
SmallGroup (16; 13) = $(C_4\times C_2) : C_2$	h:=cc[3][1]	8	128	36
	h:=cc[4][1]	8	128	36
	h:=cc[5][1]	8	128	36

3.6 Groups of order 18

There are three non-abelian groups of order 18. Here SmallGroup (18, 4) is generalized dihedral group and D_{18} is dihedral group.

Groups		Subgroups	Index	NRT	$\lvert I\ (G,H)\rvert$
SmallGroup (18, 1)	= D_{18}	h:=cc[2][1]	9	256	52
		h:=cc[4][1]	3	4	3
SmallGroup (18, 3)	=$C_3\times S_3$	h:=cc[2][1]	9	256	84
		h:=cc[5][1]	6	243	54
		h:=cc[7][1]	3	4	3
SmallGroup (18, 4)	= $(C_3\times C_3) : C_2$	h:=cc[2][1]	9	256	18
		h:=cc[7][1]	3	4	3
		h:=cc[8][1]	3	4	3
		h:=cc[9][1]	3	4	3
		h:=cc[10][1]	3	4	3

3.7 Groups of order 20

Here SmallGroup (20, 1) is Dicyclic group, D_{20} is dihedral group and SmallGroup (20, 3) is general a one group.

Groups			Subgroups	Index	NRT	$\lvert I\ (G;\ H)\rvert$
SmallGroup (20, 1)	= C_5	: C_4	h:=cc[3][1]	5	16	6
SmallGroup (20, 3)	= C_5	: C_4	h:=cc[2][1]	10	1024	140
			h:=cc[3][1]	5	256	70

SmallGroup (20, 4)	= D_{20}	h:=cc[2][1]	10	512	140
		h:=cc[4][1]	10	512	140
		h:=cc[5][1]	5	16	6

3.8 Groups of order 21

There is only one non-abelian group of order 21.

Groups	Subgroups	Index	NRT	$\|I\ (G;\ H)\|$
SmallGroup (21,1) = $C_7 : C_3$	h:=cc[2][1]	7	729	130

3.9 Groups of order 22

Here D_{22} denotes the dihedral group of order 22.

Groups	Subgroups	Index	NRT	$\|I\ (G, H)\|$
SmallGroup (22,1) = D_{22}	h:=cc[2][1]	11	1024	108

4.Problems

As it is clear from the above data that not all natural numbers occurred as number of isomorphism classes of transversals of a subgroup in a group. We would like to propose two problems related to above data.

(i) Prove or disprove that $|I(G,H)| = 5$ implies $[G : H] = 6$.

(ii) Prove or disprove that $|I(G,H)| = 6$ implies $[G : H] = 4$ or $[G:H] = 5$.

(iii) Prove or disprove that $|I(G;\ H)| = 7$ implies $[G : H] = 4$.

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