International Journal of Science and Research (IJSR)
ISSN: 2319-7064
SJIF (2022): 7.942

A Taylor Series Method for the Solution of the Boundary Value Problems for Higher Order Ordinary Differential Equation

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Abstract: ***In this paper a numerical method for solving ordinary differential equation in boundary value condition is presented. In mathematics and other field by the use of Taylor series we solve the problems of linear and nonlinear ordinary differential equations and partial differential equations. Taylor series method is an important analytic-numeric method (algorithm) of ordinary differential equations for approximate solution of initial and boundary value problems due to calculation of higher order derivatives currently this algorithm is not applied frequently. Only explicit version is known in this algorithm. The main idea is based on the approximate calculation of higher derivatives. This paper describes several above-mentioned algorithms and examines its numerical solutions of ODE and It demonstrates some numerical test results for systems of equations The method is computationally attractive and application is demonstrated through illustrative examples.***

Keywords: Taylor's series, n^{th} order linear differential equation, Ordinary differential equation, Initial value condition

Subject Classification Mathematics.

1. Introduction

Differential Equations are essential for a mathematical description in nature so many general and important law of Economics, Chemistry, Physics, Biology, ,Medical and Engineering fields Differential equation occur in connection with numerous problem that are encountered in the various branches of science and engineering .Differential Equation allow us to study all kinds of evolutionary with the property of finite dimensionality and differentiability..Differential equation i.e. existence and uniqueness of solution picards method of successive approximations, which apart from being a mere numerical technique to approximate solutions has far reaching theoretical implications as well, is applied to obtain approximate solution of initial and boundary value problems. The Taylor series algorithm is one of the earliest algorithms (method) for the approximate solution for initial and boundary value problems for ordinary differential equations. Newton used it in his calculation and Euler describe it in his work . Since then one can find many mentions of it such as J. Liouville , G. Peano , E. Picard Many authors have further developed this algorithm, see for example A. Gibbons and R. E. Moore . The Taylor polynomial series approximation method is well known and is used in variety of applications The basic idea of these developments was the recursive calculation of the coefficients of the Taylor series

Modern numerical algorithms for the solution of ordinary differential equations are also depend on the Taylor series. Some algorithms, such as multistep methods or Runge-Kutta are constructed so that they give an expression depending on a parameter (h) called step size as an approximate solution and the first terms of the. It may be possible that an boundary value problem has no solution or it may have exactly one solution or it may have more than one solution our aim in this section is to find under what condition a boundary value problem has a solution Taylor series of this expression must be identical with the terms of the Taylor series of the exact solution.

1.1 Differential Equation

Differential equation first came into existence with the invention of calculus by Newton and Leibniz Isaac Newton listed three kinds of differential equation

$$y' = \frac{dy}{dx} = f(x)$$

$$\frac{dy}{dx} = f(x,y)$$

$$x_1\frac{\partial y}{\partial x_1} + x_2\frac{\partial y}{\partial x_2} = y$$

In all these cases, y is an unknown function of x (or of x_1 and x_2), and
f(x) and f(x,y) is a given functions.

1.2 Ordinary Differential Equation

Definition: A differential equation involving ordinary derivatives of one or more dependent variables with respect to a one independent variable is called ordinary differential equation

$$\frac{dy}{dx} + xy = x \,, \qquad x\left(\frac{dy}{dx}\right)^2 = 4\left(\frac{dy}{dx}\right) - 2y^2 + 2x$$

$\frac{d^2y}{dx^2}+xy\left(\frac{dy}{dx}\right)^2=0$ and $\frac{d^4x}{dt^4}+5\frac{d^2x}{dt^2}+3=\sin t$

An ordinary differential equation (ODEs) is the equation containing an unknown function of one real or complex variable x its derivatives and so me given function of x ,The unknown function is generally represented by a variable Non linear Differential Equation :equation F(x,y,y'y''…….y^n)

1.3 Initial value problem

An initial value problem is a condition of differential equation $y'(t) = f(t, y(t)$ with $f: \Omega \subset R^n \times R^n \to R^n$ where Ω is an open set of $R^n \times R^n$ together with the point in the domain of f $(t_0, y_0) \in \Omega$ A solution of initial value problem is a function yy {\displaystyle y} ythat is a solution for differential equation are satisfiesy$(t_0) = y_0$ is called the initial conditions. The problem of finding a function y of x when we know its derivatives and its value y0 of a particular point x 0is called an initial value problem. differential equation along with subsidiary condition on the unknown function and its derivatives all given at the some value of the independent variable ,constitute an initial value problem (IVP)

i.e $\frac{dy}{dx}+2y = e^y$ y(0)=1,

$y'' + 2y' = e^x$, $y(\pi) = 1, y'(\pi) = 1$

A Boundary value problem is a system of ordinary differential equations with solution and derivative values are specified at more than one point. Most commonly, the solution and derivatives are specified at just two points (the boundaries) defining a two-point boundary value problem. A two-point boundary value problem (BVP) of total order n on a finite interval [a,b] may be written as an explicit first order system of ordinary differential equations (ODEs) with boundary values evaluated at two points as

$y'(x) = f(x, y(x))$, $x \in (a, b)$, $g(y(a), y(b)) = 0$ For second ODE any of the following condition can be used as boundary value problem

$y(x_0)=y_0$

$y(x_1)=y_1$

$y'(x_0)=y_0$

$y'(x_1)=y_1$

$y'(x_0)=y_0$

$y(x_1)=y_1$

$y(x_0)=y_0$

$y'(x_1)=y_1$

Similar combination we can define for higher order differential equations

2. Taylor Series

2.1 Taylor series of single variable

If $f(b + k)$, where b is independent of k be a function of the variable k such that it can be expanded in ascending powers of k and this expansion be differentiable any number of times then

$$f(b+k) = f(b) + kf'(b) + \frac{k^2}{2!}f''(b) + \cdots \ldots \ldots + \frac{k^n}{n!}f^n(b) + \cdots ..$$

If f(x, y) possesses continuous partial derivatives of the n^{th} order in any neighborhood D of a point (a, b) and let (a + h, b + k) be any point of D,

2.2 Taylor's Series method of two variable

If Ω is an open set of ordered pair R^2 which contains the origin $0 = (0,0) \in \Omega$, we define $B(\Omega)$as the set of real analytic function in Ω. Then the neighbourhood of the(0,0) is a function $f = f(x) \in B(\Omega)$,where $x = (x_1, x_2)$, in the form of ordered pair can be expanded in power series as follows

$$f(x_1, x_2) = \sum_{i_1,i_2=0}^{\infty} f_{(i_1 i_2)} x_1^{i_1} x_2^{i_2} \quad (1)$$

The sum of right hand side of equation (1) by grouping the terms in a homogeneous part and then using a lexicographic order.

$$f(x_1, x_2) = f_{(0,0)} + f_{(1,0)}x_1 + f_{(0,1)}x_2 + f_{(2,0)}x_1^2 + f_{(1,1)}x_1x_2 + f_{(0,2)}x_2^2 + f_{(3,0)}x_1^3 + f_{(2,1)}x_1x_2^2 + f_{(0,3)}x_2^3 + \cdots \ldots \ldots \ldots \ldots \quad (2)$$

For our subsequent development, we need to find the position of a term in equation (2) , we write $f(x_1, x_2)$ in equation (2) as

$$f(x_1, x_2) = b_1 + b_2x_1 + b_3x_2 + b_4x_1^2 + b_5x_1x_2 + b_6x_2^2 + bx_1^3 + \cdots \ldots \ldots \ldots \ldots \ldots$$

3. Main Result

Taylor series method in Higher order Ordinary Differential Equation The method of the Taylor series based on numerical derivatives and its numerical approximation values. in this method Taylor algorithm with numerical derivatives is the numerical approximation of derivatives Y(t) (x0) where t=0,1,2,3,4,……….after calculation of Numerical approximation of the second ,third ,fourth derivatives and By summarizing the results explained above we can make some truncation of Taylor's series of the Y solution as an approximate solution at a given sub interval. The basic idea of the this process (construction) is that the expressions, of the derivatives in the truncation of Taylor's expansion are replaced by its approximations which will be constructed by some combination of Matrices. This collection allow us to make a big choice of explicit method of Taylors series in the form of derivatives

Numerical approximation of derivatives in Taylor Series

Differentiating n times with respect to x

$$\frac{d^n}{dx^n}[y''' - f(y, y', y'')] = 0$$

And setting x=a or given value we can obtained y^{n+3}

Taylor series method in initial value condition at $y(x_0) = y_0$

The solution of an initial value problem in ordinary differential equations expanded as a Taylor series has been given as both a classical and numerical methods

$$y_1 = y(x_0) = y_0 + (h)y_0' + \frac{(h)^2 y_0''}{2!} + \frac{(h)^3 y_0'''}{3!} \ldots \ldots \ldots \ldots \ldots \ldots$$

If y'= p and $p(x_0) = p_0$ where p'=f(x,y,p) Taylors algorithm become

$$p_1 = p_0 + (h)p_0' + \frac{(h)^2 p_0'}{2!} + \frac{(h)^3 p_0'}{3!} \ldots \ldots \ldots \ldots \ldots \ldots$$

$$p_2 = p_1 + (h)p_1' + \frac{(h)^2 p_1'}{2!} + \frac{(h)^3 p_1'}{3!} \ldots \ldots \ldots \ldots \ldots \ldots$$

similar (continuous iteration) process can be apply for other higher order derivatives

By summarizing these results explained above we can make the solution of higher order ordinary differential equation in given initial value conditions

Example:-

Solve $\frac{d^2y}{dx^2} + y\frac{dy}{dx} = x$ by Taylor series method for x=1,y=0 , $\frac{dy}{dx} = 2$ use Taylor series method to find a solution to the differential equation in ascending power of (x-1) upto including the term $(x-1)^3$

Solution:-

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \cdots \ldots + \frac{(x-a)^n}{n!}f^n(a) + \cdots$$

At x= 1

$$f(x) = f(1) + (x-a)f'(1) + \frac{(x-a)^2}{2!}f''(1) + \cdots \ldots + \frac{(x-a)^n}{n!}f^n(1) + ..$$

$$\text{let}, y = f(x) \ \text{at} \ x = 1, y = 0, \frac{dy}{dx} = 2$$

f(1)=0 and f'(1)=2

$$\frac{d^2y}{dx^2} + 0.2 = 1 \ , \frac{d^2y}{dx^2} = 1, f''(1)$$

Differentiating equation $\frac{d^2y}{dx^2} + y\frac{dy}{dx} = x$ with respect to x

$$\frac{d^3y}{dx^3} + y\frac{d^2y}{dx^2} + \frac{dy}{dx}\left(\frac{dy}{dx}\right) = 1$$

When x=1,y=0 , $\frac{dy}{dx} = 2, \frac{d^2y}{dx^2} = 1$

$$\frac{d^3y}{x^3} + 0.1 + 4 = 1 \ , \frac{d^3y}{dx^3} = -3 \quad f'''(1) = -3$$

From Taylor series

$$y = 2(x-1) + \frac{(x-1)^2}{2!} - \frac{3(x-1)^3}{3!} \ldots \ldots \ldots + \frac{(x-1)^n}{n!}f^n(1) + \ldots$$

$$y = 2(x-1) + \frac{(x-1)^2}{2} - \frac{(x-1)^3}{2} \ldots\ldots$$

4. Result

In this chapter we introduce the basic properties of Taylors series in two and three variables and using a initial value condition we can easy to solve ordinary differential equation of higher order by expansion of Taylor series method.

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