International Journal of Science and Research (IJSR)
ISSN: 2319-7064
SJIF (2022): 7.942

Certain Subclasses of Convex and Starlike Functions with Negative Coefficients

Yamini J

Department of Mathematics, Government First Grade College, Vijayanagar, Bangalore, Karnataka, India
E-mail: *yaminibalaji[at]gmail.com*

Abstract: ***In this paper, I define a new subclass of uniformly convex functions and corresponding subclass of starlike functions with negative coefficients and obtain coefficient estimates. Further, I study extreme points, growth and distortion bounds, radii of starlikeness and convexity for aforementioned class.***

Keywords: Analytic functions, univalent functions, uniformly convex and uniformly starlike functions.

1. Introduction

Let A(j) denote the class of functions of the form

$$f(z) = z + \sum_{n=j+1}^{\infty} a_n z^n \ , \ \ j \in N \ \ \ldots\ldots\ldots\ldots\ldots\ldots..(1.1)$$

which are analytic in the open unit disc $U = \{z: |z| < 1\}$ note that A$(1) =$ A and A$(j) \subseteq$ A(1). Suppose that S(j) denote the subclass of A(j) consisting of functions that are univalent in U. Also, we note that S$(1) = S$ is subclass of A consisting of functions that are univalent in U. Further, let $S_\alpha^*(j)$ and $C_\alpha(j)$ are the subclasses of S(j) consisting of functions respectively, starlike of order $\alpha(0 \leq \alpha < 1)$ and convex of order $\alpha(0 \leq \alpha<1$.

Let $T(j)$ denote the subclass of S(j) consisting of functions of the form

$$f(z) = z - \sum_{n=j+1}^{\infty} a_n z^n \ , \ \ a_n \geq 0, \forall \ n \geq j+1, j \in N \, , z \in U \ldots (1.2)$$

A function $f \in T(j)$ is called a function with negative coefficients and the class $T(1) = T$ was introduced and studied by Silverman[10]. In 1975, Silverman [10] investigated the subclasses of $T(1)$ denoted by $T^*(\alpha)$ and $K(\alpha)$ for $0 \leq \alpha < 1$. That are, respectively starlike of order α and convex of order α.

Furthermore, denote by $S_\alpha^*(1) \coloneqq S^*$ and $C_0(1) \coloneqq C$ the subclasses of S that are respectively, starlike and convex functions. In 1991, Goodman [3,4] introduced and defined the following subclasses of S^* and C.

A function $f(z)$ is uniformly convex(uniformly starlike) in U, if $f(z)$ is in $C(S^*)$ and has the property that for every circular γ contained in U, with centre ξ also in U and the arc $f(\gamma)$ is convex(starlike) with respect to $f(\xi)$. The class of uniformly convex functions denoted by UCV and the class of uniformly starlike functions denoted by UST (for details see [3]). It is well known from [7,9] that

$$f \in UCV \Leftrightarrow \mathcal{R}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} \geq \left|\frac{zf''(z)}{f'(z)}\right|$$

In [9], Ronning introduced a new class of starlike functions related to UCV and defined as

$$f \in S_p \Leftrightarrow \mathcal{R}\left\{\frac{zf'(z)}{f(z)}\right\} \geq \left|\frac{zf'(z)}{f(z)} - 1\right|$$

Note that $f(z) \in UCV \Leftrightarrow zf'(z) \in S_p$. Further Ronning generalized the class S_p by introducing a parameter α, $-1 \leq \alpha < 1$

$$f \in S_p(\alpha) \Leftrightarrow \mathcal{R}\left\{\frac{zf'(z)}{f(z)} - \alpha\right\} \geq \left|\frac{zf'(z)}{f(z)} - 1\right|$$

In 1997, Bharati et al.[2] introduced the subclasses k-starlike functions of order $(k - ST(\alpha))$ and k-uniformly convex functions of order $\alpha(k - UCV(\alpha))$ of the function class S as follows

$$f \in k - ST(\alpha) \Leftrightarrow \mathcal{R}\left\{\frac{zf'(z)}{f(z)} - \alpha\right\} \geq k\left|\frac{zf'(z)}{f(z)} - 1\right|, \quad k \geq 0\,, \quad 0 \leq \alpha < 1$$

and

$$f \in k - UCV(\alpha) \Leftrightarrow \mathcal{R}\left\{1 + \frac{zf''(z)}{f'(z)} - \alpha\right\} \geq k\left|\frac{zf''(z)}{f'(z)}\right|, k \geq 0, \quad 0 \leq \alpha < 1$$

It follows that $f \in k - UCV(\alpha) \Leftrightarrow zf' \in k - ST(\alpha)$.

Motivated by the above definitions and Murugusundaramoorthy and Magesh[8],we define a new subclass of S(j) as follows.

For $k \geq 0, -1 \leq \alpha < 1$, $0 \leq \mu < 1$ and $\mu \leq \lambda$ we let $S(\lambda, \mu, \alpha, k, j)$ be the subclass of S(j) consisting of functions $f(z)$ of the form (1.1) and satisfying the analytic criterion

$$\mathcal{R}\left\{\frac{zf'(z) + \lambda z^2 f''(z)}{(1-\mu)f(z) + \mu zf'(z) + (\lambda - \mu)z^2 f''(z)} - \alpha\right\} > k\left|\frac{zf'(z) + \lambda z^2 f''(z)}{(1-\mu)f(z) + \mu zf'(z) + (\lambda - \mu)z^2 f''(z)} - 1\right| \ , z \in U \ldots\ldots(1.3)$$

We also let $TS(\lambda, \mu, \alpha, k, j) = S(\lambda, \alpha, k, j) \cap T(j)$.
We note that, by specializing the parameters j, λ, μ, α *and* k, we obtain the following subclasses studied by various authors.

1) $TS(0,0,\alpha,0;1) = S^*(\alpha)$ and $TS(1,1,\alpha,0;1) = K(\alpha)$ (Silverman[10])
2) $TS(0,0,\alpha,0;j) = S^*(\alpha,j)$ and $TS(1,1,\alpha,0;j) = K(\alpha,j)$ (Srivastava et al. [11])
3) $TS(\lambda,\lambda,\alpha,0,1) = S^*(\lambda,\alpha)$ (Altintas[1])
4) $TS(0,0,\alpha,k,1) = k-ST(\alpha)$ and $TS(1,1,\alpha,k,1) = k-UCV(\alpha)$(Bharati et al. [2]
5) $S(0,0,\alpha,1,1) = S_p(\alpha)$ and $S(1,1,\alpha,1,1) = UCV(\alpha)$ (Ronning [9])
6) $S(0,0,1,k,1) = k-ST$ and $S(1,1,1,k,1) = k-UCV$ (Kanas and Wisnowska [5,6] and Subramanian et al. [12]).

The main object of this paper is to obtain a sufficient coefficient condition for functions f of the form (1.1) to be in the class $TS(\lambda,\mu,\alpha,k,j)$. We show that the result is also a necessary condition for functions belong to this class. Further we investigate extreme points, growth and distortion bounds, radii of starlikeness and convexity for the aforementioned class.

2. Coefficient Estimates

In this section we obtain a necessary and sufficient condition for functions $f(z)$ in the classes $TS(\lambda,\mu,\alpha,k,j)$.

Theorem 2.1. A function $f(z)$ of the form (1.1) is in $S(\lambda,\mu,\alpha,k,j)$ if

$$\sum_{n=j+1}^{\infty}[\phi_n(1+k)-(\alpha+k)\psi_n]\,|a_n| \le 1-\alpha \quad \ldots\ldots\ldots(2.1)$$

Where $\phi_n = n[1+\lambda(n-1)]$, $\psi_n = [1+(n-1)(n\lambda - n-1\mu$ and $-1\le\alpha<1$, $0\le\mu<1$, $\mu\le\lambda$ and $k\ge 0$ ……………………………….(2.2)

Proof. It suffices to show that

$$k\left|\frac{zf'(z)+\lambda z^2f''(z)}{(1-\mu)f(z)+\mu zf'(z)+(\lambda-\mu)z^2f''(z)}-1\right| - Re\left\{\frac{zf'(z)+\lambda z^2f''(z)}{(1-\mu)f(z)+\mu zf'(z)+(\lambda-\mu)z^2f''(z)}-1\right\} \le 1-\alpha$$

we have

$$k\left|\frac{zf'(z)+\lambda z^2f''(z)}{(1-\mu)f(z)+\mu zf'(z)+(\lambda-\mu)z^2f''(z)}-1\right| - Re\left\{\frac{zf'(z)+\lambda z^2f''(z)}{(1-\mu)f(z)+\mu zf'(z)+(\lambda-\mu)z^2f''(z)}-1\right\}$$

$$\le (1+k)\left|\frac{zf'(z)+\lambda z^2f''(z)}{(1-\mu)f(z)+\mu zf'(z)+(\lambda-\mu)z^2f''(z)}-1\right|$$

$$\le \frac{(1+k)\sum_{n=j+1}^{\infty}(\phi_n-\psi_n)|a_n|}{1-\sum_{n=j+1}^{\infty}(\psi_n)|a_n|}$$

This last expression is bounded above by $(1-\alpha)$ if

$$\sum_{n=j+1}^{\infty}[\phi_n(1+k)-(\alpha+k)\psi_n]|a_n| \le 1-\alpha$$

hence the proof is complete.

Theorem 2.2: A necessary and sufficient condition for $f(z)$ of the form (1.2) to be in the class $TS(\lambda,\mu,\alpha,k,j)$ is that

$$\sum_{n=j+1}^{\infty}[\phi_n(1+k)-(\alpha+k)\psi_n]|a_n| \le 1-\alpha$$

Proof: In view of Theorem 2.1, we need only to prove the necessity. If $f(z)\in TS(\lambda,\mu,\alpha,k,j)$ and z is real then

$$\frac{1-\sum_{n=j+1}^{\infty}\phi_n a_n z^{n-1}}{1-\sum_{n=j+1}^{\infty}\psi_n a_n z^{n-1}}-\alpha \ge k\left|\frac{\sum_{n=j+1}^{\infty}(\phi_n-\psi_n)a_n z^{n-1}}{1-\sum_{n=j+1}^{\infty}\psi_n a_n z^{n-1}}\right|$$

Letting $z\to 1$ along the real axis, we obtain the desired inequality.

$$\sum_{n=j+1}^{\infty}[\phi_n(1+k)-(\alpha+k)\psi_n]a_n \le 1-\alpha\,, -1\le\alpha<1, k \ge 0$$

Finally the function $f(z)$ given by

$$f(z) = z-\frac{1-\alpha}{[\phi_{j+1}(1+k)-(\alpha+k)\psi_{j+1}]}z^{j+1} \quad \ldots\ldots\ldots\ldots(2.4)$$

where ϕ_{j+1} and ψ_{j+1} as written in (2.2)is extremal for the function.

Corollary 2.3: Let the function $f(z)$ defined by (1.2) be in the class $TS(\lambda,\mu,\alpha,k,j)$. Then

$$a_n \le \frac{1-\alpha}{[\phi_n(1+k)-(\alpha+k)\psi_n]}, n\ge j+1 \quad \ldots\ldots\ldots(2.5)$$

this equality in (2.5) is attained for the function $f(z)$ given by (2.4).

3. Growth and Distortion Theorem

Theorem 3.1: Let the function $f(z)$ defined by (1.2) be in the class $TS(\lambda,\mu,\alpha,k,j)$. Then for $|z|<r=1$

$$r-\frac{1-\alpha}{[\phi_{j+1}(1+k)-(\alpha+k)\psi_{j+1}]}r^{j+1} \le |f(z)| \le r+\frac{1-\alpha}{[\phi_{j+1}(1+k)-(\alpha+k)\psi_{j+1}]}r^{j+1} \quad \ldots\ldots\ldots(3.1)$$

The result (3.1) is attained for the function $f(z)$ given by (2.4) for $z=\pm r$.

Proof: Note that

$$[\phi_{j+1}(1+k)-(\alpha+k)\psi_{j+1}]\sum_{n=j+1}^{\infty}a_n \le \sum_{n=j+1}^{\infty}[\phi_n(1+k)-(\alpha+k)\psi_n]a_n \le 1-\alpha,$$

this last inequality follows from Theorem 2.2. Thus

$$|f(z)| \le |z| - \sum_{n=j+1}^{\infty} a_n |z|^n \ge r - r^{j+1} \sum_{n=j+1}^{\infty} a_n$$
$$\ge r - \frac{1-\alpha}{[\,\phi_{j+1\_}(1+k) - (\alpha+k)\psi_{j+1}]} r^{j+1}$$

Similarly

$$|f(z)| \le |z| + \sum_{n=j+1}^{\infty} a_n |z|^n \le r + r^{j+1} \sum_{n=j+1}^{\infty} a_n$$
$$\le r + \frac{1-\alpha}{[\,\phi_{j+1\_}(1+k) - (\alpha+k)\psi_{j+1}]} r^{j+1}$$

This completes the proof.

Theorem 3.2: Let the function $f(z)$ defined by (1.2) be in the class $TS(\lambda,\mu,\alpha,k,j)$. Then for $|z| < r = 1$

$$r - \frac{(j+1)(1-\alpha)}{[\,\phi_{j+1\_}(1+k) - (\alpha+k)\psi_{j+1}]} r^j \le |f'(z)|$$
$$\le r + \frac{(j+1)(1-\alpha)}{[\,\phi_{j+1\_}(1+k) - (\alpha+k)\psi_{j+1}]} r^j \quad \ldots\ldots\ldots..(3.2)$$

Proof: We have

$$|f'(z)| \ge 1 - \sum_{n=j+1}^{\infty} n a_n |z|^{n-1}$$
$$\ge 1 - r^j \sum_{n=j+1}^{\infty} n a_n \quad \ldots\ldots\ldots(3.3)$$
$$|f'(z)| \le 1 + \sum_{n=j+1}^{\infty} n a_n |z|^{n-1}$$
$$\le 1 + r^j \sum_{n=j+1}^{\infty} n a_n \quad \ldots\ldots\ldots(3.4)$$

In view of theorem 2.2,

$$\frac{[\,\phi_{j+1\_}(1+k) - (\alpha+k)\psi_{j+1}]}{j+1} \sum_{n=j+1}^{\infty} n a_n$$
$$\le \sum_{n=j+1}^{\infty} [\,\phi_n(1+k) - (\alpha+k)\psi_n] a_n$$
$$\le 1-\alpha\,.(3.5)$$

or equivalently

$$\sum_{n=j+1}^{\infty} n a_n \le \frac{(j+1)(1-\alpha)}{[\,\phi_{j+1}\ \ (1+k) - (\alpha+k)\psi_{j+1}]} \ldots\ldots\ldots..(3.6)$$

A substitution of (3.6) into (3.3) and (3.4) yields the inequality (3.2). This completes the proof.

Theorem 3.3: Let $f_j(z) = z$, and

$$f_n(z) = z - \frac{1-\alpha}{[\,\phi_n(1+k) - (\alpha+k)\psi_n]} z^n,$$
$$n \ge j+1 \ \ldots\ldots(3.7)$$

for $0 \le \lambda \le 1, k \ge 0, -1 \le \alpha < 1$. Then $f(z)$ is in the class $TS(\lambda,\mu,\alpha,k,j)$ if and only if it can be expressed in the form

$$f(z) = \sum_{n=j}^{\infty} \mu_n f_n(z) \ldots\ldots.(3.8)$$

where $\mu_n \ge 0\ \ (n \ge j)$ *and* $\sum_{n=j}^{\infty} \mu_n = 1$.

Proof: Assume that

$$f(z) = \mu_j f_j(z) + \sum_{n=j+1}^{\infty} \mu_n \left[z - \frac{1-\alpha}{[\,\phi_n(1+k) - (\alpha+k)\psi_n]} z^n\right]$$
$$= \sum_{n=j}^{\infty} \mu_n z - \sum_{n=j+1}^{\infty} \frac{1-\alpha}{[\,\phi_n(1+k) - (\alpha+k)\psi_n]} \mu_n z^n$$

Then it follows that

$$\sum_{n=j+1}^{\infty} \frac{1-\alpha}{[\,\phi_n(1+k) - (\alpha+k)\psi_n]} \mu_n \frac{[\,\phi_n(1+k) - (\alpha+k)\psi_n]}{1-\alpha}$$
$$= \sum_{n=j+1}^{\infty} \mu_n \le 1,$$

so by Theorem 2.2 , $f(z) \in TS(\lambda,\mu,\alpha,k,j)$.

Conversely, assume that the function $f(z)$ defined by (1.2) belongs to the class $TS(\lambda,\mu,\alpha,k,j)$. Then

$$a_n \le \frac{1-\alpha}{[\,\phi_n(1+k) - (\alpha+k)\psi_n]}, \qquad n \ge j+1.$$

Setting

$$\mu_n = \frac{[\,\phi_n(1+k) - (\alpha+k)\psi_n]}{1-\alpha} a_n,$$
$$(n \ge j+1) \ and \ \ \mu_j = 1 - \sum_{n=j+1}^{\infty} \mu_n\,,$$

We have

$$f(z) = z - \sum_{n=j+1}^{\infty} a_n z^n$$

$$f(z) = z - \sum_{n=j+1}^{\infty} \frac{1-\alpha}{[\,\phi_n(1+k) - (\alpha+k)\psi_n]} \mu_n z^n \quad \ldots\ldots\ldots(3.9)$$

Then (3.8) gives

$$f(z) = z + \sum_{n=j+1}^{\infty} (f_n(z) - z)\,\mu_n = f_j(z)\mu_j + \sum_{n=j+1}^{\infty} f_n(z)\,\mu_n$$
$$= \sum_{n=j}^{\infty} f_n(z)\,\mu_n$$

and hence the proof is complete.

4. Radii of close-to-convexity, Starlikeness and Convexity

In this subsection, we obtain the radii of close-to-convexity, starlikeness and convexity for the class $TS(\lambda,\mu,\alpha,k,j)$.

Theorem 4.1: Let $f \in TS(\lambda,\mu,\alpha,k,j)$. Then $f(z)$ is close-to-convex of order σ $(0 \le \sigma < 1)$ in the disc $|z| < r_1$ where

$$r_1 \coloneqq inf \left[\frac{(1-\sigma)[\,\phi_n(1+k) - (\alpha+k)\psi_n]}{n(1-\alpha)}\right]^{\frac{1}{n-1}}, \quad n \ge j+1 \ldots\ldots..(4.1)$$

The result is sharp with extremal function $f(z)$ given by (2.4).

Proof: Given $f \in T$, and f is close-to-convex of order σ, we have

$$|f'(z) - 1| < 1 - \sigma \quad \dots\dots\dots (4.2)$$

For the left hand side of (4.2) we have

$$|f'(z) - 1| \leq \sum_{n=j+1}^{\infty} n a_n |z|^{n-1}$$

The last expression is less than $1 - \sigma$ if

$$\sum_{n=j+1}^{\infty} \frac{n}{1-\sigma} a_n |z|^{n-1} < 1.$$

Using the fact that $f \in TS(\lambda, \mu, \alpha, k, j)$ if and only if

$$\sum_{n=j+1}^{\infty} \frac{[\phi_n(1+k) - (\alpha+k)\psi_n]}{(1-\alpha)} a_n \leq 1.$$

We can say (4.2) is true if

$$\frac{n}{1-\sigma}|z|^{n-1} \leq \frac{[\phi_n(1+k) - (\alpha+k)\psi_n]}{(1-\alpha)}$$

Or, equivalently,

$$|z|^{n-1} = \left[\frac{(1-\sigma)[\phi_n(1+k) - (\alpha+k)\psi_n]}{n(1-\alpha)}\right]$$

which completes the proof.

Theorem 4.2: Let $f \in TS(\lambda, \mu, \alpha, k, j)$. Then

i) f *is starlike of order* σ $(0 \leq \sigma < 1)$ *in the disc* $|z| < r_2$, *where*

$$r_2 = \inf\left[\left(\frac{1-\sigma}{n-\sigma}\right)\frac{[\phi_n(1+k) - (\alpha+k)\psi_n]}{(1-\alpha)}\right]^{\frac{1}{n-1}}, n \geq j+1 \dots\dots\dots\dots (4.3)$$

ii) f *is convex of order* σ $(0 \leq \sigma < 1)$ *in the unit disc* $|z| < r_3$, *where*

$$r_3 = \inf\left[\left(\frac{1-\sigma}{n(n-\sigma)}\right)\frac{[\phi_n(1+k) - (\alpha+k)\psi_n]}{(1-\alpha)}\right]^{\frac{1}{n-1}}, n \geq j+1 \dots\dots\dots\dots (4.4)$$

Each of these results are sharp for the extremal function $f(z)$ given by (2.4).

Proof: i) Given $f \in T$, and f is starlike of order σ, we have

$$\left|\frac{z f'(z)}{f(z)} - 1\right| < 1 - \sigma \quad \dots\dots\dots\dots (4.5)$$

For the left hand side of (4.5) we have

$$\left|\frac{z f'(z)}{f(z)} - 1\right| \leq \frac{\sum_{n=j+1}^{\infty}(n-1)a_n|z|^{n-1}}{1 - \sum_{n=j+1}^{\infty} a_n |z|^{n-1}}$$

The last expression is less than $1 - \sigma$ if

$$\sum_{n=j+1}^{\infty} \frac{n-\sigma}{1-\sigma} a_n |z|^{n-1} < 1$$

Using the fact that $f \in TS(\lambda, \mu, \alpha, k, j)$ if and only if

$$\sum_{n=j+1}^{\infty} \frac{[\phi_n(1+k) - (\alpha+k)\psi_n]}{(1-\alpha)} a_n \leq 1$$

We can say (4.5) is true if

$$\frac{n-\sigma}{1-\sigma}|z|^{n-1} \leq \frac{[\phi_n(1+k) - (\alpha+k)\psi_n]}{(1-\alpha)}$$

Or, equivalently,

$$|z|^{n-1} = \left[\left(\frac{1-\sigma}{n-\sigma}\right)\frac{[\phi_n(1+k) - (\alpha+k)\psi_n]}{(1-\alpha)}\right]$$

which yields the starlikeness of the family.

ii) Using the fact that f is convex if and only if zf' is starlike, we can prove (ii), on lines similar to the proof of (i).

References

[1] O.Altinta,s, On a subclass of certain starlike functions with negative coefficients, Math. Japon.36 (1991), no. 3, 489-495.

[2] R. Bharati, R. Parvatham and A. Swaminathan, On subclasses of uniformly convex functions and corresponding class of starlike functions, Tamkang J. Math.28(1997). no.1, 17-32.

[3] A.W. Goodman, On uniformly convex functions, Ann.Polon.Math.56(1991), no.1, 87-92.

[4] A.W. Goodman, On uniformly convex functions, J.Math.Anal.Appl.155(1991), no.2, 364- 370.

[5] S.Kanas and A.Wisniowska, Conic regions and k-uniform convexity, J.Comput.Appl. Math. 105 (1999), no.1-2, 327-336.

[6] S.Kanas and A.Wisniowska, Conic domains and starlike functions, Rev. Roumaine Math. Pures Appl. 45(2000), no.4, 647-657 (2001).

[7] W.C. Ma and D. Minda, Uniformly convex functions, Ann. Polon. Math.57(1992), no.2, 165-175.

[8] G. Murugusundaramoorthy and N. Magesh, On certain subclasses of analytic functions associated with hypergeometric functions, Appl.Math.Lett.24 (2011), no. 4,494-500.

[9] F. Ronning, Uniformly convex functions and a corresponding class of starlike functions, Proc.Amer.Math.Soc.118(1993), no.1,189-196.

[10] H. Silverman, Univalent functions with negative coefficients, Proc.Amer.Math.Soc.51 (1975).

[11] H. M. Srivastava, S. Owa and S. K. Chatterjea, A note on certain classes of starlike functions, Rend. Sem. Mat.Univ. Padova77 (1987), 115-124.

[12] K. G. Subramanian et al.,Classes of uniformly starlike functions, Publ. Math. Debrecen 53 (1998), no. 3-4, 309-315.