International Journal of Science and Research (IJSR)
ISSN: 2319-7064
SJIF (2022): 7.942

# Positive Integer Solutions of Some Pell Equations via Generalized Bi-Periodic Fibonacci and Generalized Bi-Periodic Lucas Sequences

**S. Sriram[1], P. Veeramallan[2]**

[1]Assistant Professor, National College (Affiliated to Bharathidasan University), Tiruchirappalli-620 001, Tamilnadu, India
*Corresponding author: *sriram.priya02[at]yahoo.com*

[2]P.G Assistant in Mathematics, GHSS, Yethapur, Salem-636 117, Tamilnadu, India
(Part-time Research Scholar, National College (Affiliated to Bharathidasan University), Tiruchirappalli), veeramallan1986[at]gmail.com

**Abstract:** ***Let C be a non-perfect square positive-integer and $C = m^2 \pm 1, m^2 \pm 2, m^2 \pm m$. The basic solution of the Pell equation is found in the present article $x^2 - Cy^2 = \pm 1$ by using Continued fraction expansion of $\sqrt{C}$. Also, in terms of Generalized Bi-Periodic Fibonacci and Lucas sequences, we obtain all positive-integer solutions of the Pell equation $x^2 - Cy^2 = \pm 1$.***

**Keywords:** Continued fraction, Pell equations, Generalized Bi-Periodic Fibonacci and Lucas sequences

**2010 Mathematics Subject Classifications:** 11A55, 11B39, 11D55, 11D09, 11J70

## 1. Introduction

It is generally recognized that the Pell equation $x^2 - Cy^2 = 1$ always have positive-integer solutions, where C is a positive-integer which is not a perfect square. When N is not equal to 1, there may be no positive-integer solution for $x^2 - Cy^2 = N$. The positive-integer solution for $x^2 - Cy^2 = -1$ equation depends on the period length of $\sqrt{C}$ continued fraction expansion. When $m$ is a positive integer as well as $C = m^2 \pm 1, m^2 \pm 2, m^2 \pm m$, particularly if a solution is available, all positive integer solutions are provided in terms of Generalized Bi-Periodic Fibonacci and Lucas sequences. In the present article, we will utilize $\sqrt{C}$ continued fraction expansion to obtain all positive integer solutions of the equations for different values of C, if a solution exists.

## 2. Preliminaries

Some writers have generalized the sequences, Fibonacci and Lucas, by altering their initial conditions and recurring relations. Yayenie and Edson [11] generalize the Fibonacci sequence to the new set of sequences denoted as $\{p_n\}$ and is defined by

$$p_0 = 0, p_1 = 1,\ p_n = \begin{cases} ap_{n-1} + p_{n-2}, & if\ n\ is\ even \\ bp_{n-1} + p_{n-2}, & if\ n\ is\ odd, \end{cases} \quad n \geq 2$$

Bilgici [1], generalized the Lucas sequence by presenting a bi-periodic Lucas sequence denoted as $\{l_n\}$ and is expressed as:

$$l_0 = 2, l_1 = a,\ l_n = \begin{cases} bl_{n-1} + l_{n-2}, & if\ n\ is\ even \\ al_{n-1} + l_{n-2}, & if\ n\ is\ odd, \end{cases} \quad n \geq 2$$

as well as several interesting associations between $\{p_n\}$ and $\{l_n\}$ have been proven.

We now consider a generalized bi-periodic Fibonacci sequence $\{f_n\}$ and Lucas sequence $\{q_n\}$ which are the generalization of $\{p_n\}$ and $\{l_n\}$, defined by:

$$f_0 = 0, f_1 = 1,\ f_n = \begin{cases} af_{n-1} + cf_{n-2}, & if\ n\ is\ even \\ bf_{n-1} + cf_{n-2}, & if\ n\ is\ odd, \end{cases} \quad n \geq 2$$

and

$$q_0 = 2d, q_1 = ad,\ q_n = \begin{cases} bq_{n-1} + cq_{n-2}, & if\ n\ is\ even \\ aq_{n-1} + cq_{n-2},, & if\ n\ is\ odd \end{cases} \quad n \geq 2$$

where $a, b, c, d$ are nonzero real numbers.

Yayenie and Choo [11] and [3] gave Binet's formulas for $\{f_n\}$ and $\{q_n\}$ are given by

$$f_n(a,b,c) = \frac{a^{\zeta(n+1)}}{(ab)^{\left\lfloor \frac{n}{2} \right\rfloor}} \left( \frac{\alpha^n - \beta^n}{\alpha - \beta} \right) \quad (1)$$

$$q_n(a,b,c,d) = \frac{d}{(ab)^{\left\lfloor \frac{n}{2} \right\rfloor} b^{\zeta(n)}} (\alpha^n + \beta^n) \quad (2)$$

where $\alpha = \frac{ab + \sqrt{a^2b^2 + 4abc}}{2}$ and $\beta = \frac{ab - \sqrt{a^2b^2 + 4abc}}{2}$, i.e., $\alpha$ and $\beta$ are equation roots $x^2 - abx - abc = 0$, and $\zeta(n) = n - 2\left\lfloor \frac{n}{2} \right\rfloor$ is the parity function such that

$$\zeta(n) = \begin{cases} 0\ if\ n\ is\ even \\ 1\ if\ n\ is\ odd. \end{cases}$$

We now provide the fundamental solution to an equation $x^2 - Cy^2 = \pm 1$ utilizing the length ofa period of $\sqrt{C}$ continued fraction expansion.

**Lemma 2.1:** Suppose $l$ be period length of $\sqrt{C}$ continued fraction expansion. When $l$ is even, then the fundamental solution for $x^2 - Cy^2 = 1$ equation is given by

$$x_1 + y_1\sqrt{C} = p_{l-1} + q_{l-1}\sqrt{C}$$

and $x^2 - Cy^2 = -1$ equation has no integer solution. In case of $l$ is odd, then the fundamental solution for $x^2 - Cy^2 = 1$ equation is given by

$$x_1 + y_1\sqrt{C} = p_{2l-1} + q_{2l-1}\sqrt{C}$$


Licensed Under Creative Commons Attribution CC BY

and fundamental solution for $x^2 - Cy^2 = -1$ equation is given by

$$x_1 + y_1\sqrt{C} = p_{l-1} + q_{l-1}\sqrt{C}$$

**Cognition 2.1** Let $x_1 + y_1\sqrt{C}$ be the fundamental solution of $x^2 - Cy^2 = 1$ equation. Then all positive-integer solutions of $x^2 - Cy^2 = 1$ equation is given by

$$x_n + y_n\sqrt{C} = \left(x_1 + y_1\sqrt{C}\right)^n$$

with $n \geq 1$.

**Cognition 2.2** Let $x_1 + y_1\sqrt{C}$ be the fundamental solution of $x^2 - Cy^2 = -1$ equation. Then all positive-integer solutions for $x^2 - Cy^2 = -1$ equation are given by

$$x_n + y_n\sqrt{C} = \left(x_1 + y_1\sqrt{C}\right)^{2n-1}$$

with $n \geq 1$.

**Cognition 2.3** Let $C = m^2 \pm 1, m^2 \pm 2, m^2 \pm m$. Then $\sqrt{C}$ continued fraction expansionis given by

$$\sqrt{C} = \begin{cases} [m; \overline{2m}], & \text{if } C = m^2 + 1 \ with\ m \geq 1 \\ [m-1; \overline{1, 2m-2}], & \text{if } C = m^2 - 1 \ withm > 1 \\ [m; \overline{m, 2m}], & \text{if } C = m^2 + 2 \ with\ m > 1 \\ [m-1; \overline{1, m-2, 1, 2m-2}], & \text{if } C = m^2 - 2 \ with\ m > 1 \\ [m; \overline{2, 2m}], & \text{if } C = m^2 + m \ with\ m > 1 \\ [m-1; \overline{2, 2m-2}], & \text{if } C = m^2 - m \ with\ m > 1 \end{cases}$$

**Corollary 2.1** Let $C = m^2 \pm 1, m^2 \pm 2, m^2 \pm m$. The basic solution of $x^2 - Cy^2 = 1$ equation is given by

$$x_1 + y_1\sqrt{C} = \begin{cases} (2m^2+1) + 2m\sqrt{C}, & if\ C = m^2 + 1 \\ (2m^2-1) + 2m\sqrt{C}, & if\ C = m^2 - 1 \\ (m^2+1) + m\sqrt{C}, & if\ C = m^2 + 2 \\ (m^2-1) + m\sqrt{C}, & if\ C = m^2 - 2 \\ (2m+1) + 2\sqrt{C}, & if\ C = m^2 + m \\ (2m-1) + m\sqrt{C}, & if\ C = m^2 - m \end{cases}$$

**Corollary 2.2** Let $m > 0$ and $C = m^2 + 1$. The basic solution of $x^2 - Cy^2 = -1$ equation is $x_1 + y_1\sqrt{C} = m + \sqrt{C}$.

## 3. Main Theorems

**Theorem 3.1:** Suppose $m > 0$ and $C = m^2 + 1$. Then all positive solution of the equation
$x^2 - Cy^2 = 1$ are given by

$$(x, y) = \left(2^{2n-1} m^{\lfloor n \rfloor} q_{2n}\left(1, m, \frac{1}{4m}, 1\right), 2^{2n-1} m^{\lfloor n \rfloor} f_{2n}\left(1, m, \frac{1}{4m}\right)\right)$$

(or)

$$(x, y) = \left(2^{2n-1} m^{\lfloor n \rfloor} q_{2n}\left(m, 1, \frac{1}{4m}, 1\right), 2^{2n-1} \frac{m^{\lfloor n \rfloor}}{m} f_{2n}\left(m, 1, \frac{1}{4m}\right)\right)$$

with $n \geq 1$.

**Proof**

By Corollary 2.1, Cognition 2.1, and 2.3, Then all positive solution of the equation $x^2 - Cy^2 = 1$ are given by

$$x_n + y_n\sqrt{C} = \left((2m^2+1) + 2m\sqrt{C}\right)^n$$

with $n \geq 1$. Let $\alpha_1 = (2m^2+1) + 2m\sqrt{C}$ and $\beta_1 = (2m^2+1) - 2m\sqrt{C}$. Then,

$$x_n + y_n\sqrt{C} = {\alpha_1}^n \text{ and } x_n - y_n\sqrt{C} = {\beta_1}^n$$

Thus, it follows that

$$x_n = \frac{{\alpha_1}^n + {\beta_1}^n}{2} \text{ and } y_n = \frac{{\alpha_1}^n - {\beta_1}^n}{2\sqrt{C}}$$

Let

$$\alpha = \frac{ab + \sqrt{a^2b^2 + 4abc}}{2} \text{ and } \beta = \frac{ab - \sqrt{a^2b^2 + 4abc}}{2}$$

**Case (i)**

Take $a = 1, b = m, c = \frac{1}{4m}$, we get

$$\alpha = \frac{m + \sqrt{m^2+1}}{2} \text{ and } \beta = \frac{m - \sqrt{m^2+1}}{2}$$

Thus, $4\alpha^2 = (2m^2+1) + 2m\sqrt{m^2+1} = \alpha_1$ and $4\beta^2 = (2m^2+1) - 2m\sqrt{m^2+1} = \beta_1$

Therefore, we get,

$$x_n = \frac{(4\alpha^2)^n + (4\beta^2)^n}{2} = 2^{2n-1}(\alpha^{2n} + \beta^{2n})$$
$$= 2^{2n-1} m^{\lfloor n \rfloor} q_{2n}\left(1, m, \frac{1}{4m}, 1\right) \text{ by (2)}$$

and

$$y_n = \frac{(4\alpha^2)^n - (4\beta^2)^n}{2\sqrt{m^2+1}} = 2^{2n-1}\frac{\alpha^{2n} - \beta^{2n}}{\alpha - \beta}$$
$$= 2^{2n-1} m^{\lfloor n \rfloor} f_{2n}\left(1, m, \frac{1}{4m}\right) \text{ by (1)}$$

Thus, $(x, y) =$
$$\left(2^{2n-1} m^{\lfloor n \rfloor} q_{2n}\left(1, m, \tfrac{1}{4m}, 1\right), 2^{2n-1} m^{\lfloor n \rfloor} f_{2n}\left(1, m, \tfrac{1}{4m}\right)\right)$$


Licensed Under Creative Commons Attribution CC BY

**Case (ii)**

Take, $a = m, b = 1, c = \frac{1}{4m}$, we get

$$\alpha = \frac{m+\sqrt{m^2+1}}{2} \text{ and } \beta = \frac{m-\sqrt{m^2+1}}{2}$$

Thus, $4\alpha^2 = (2m^2+1) + 2m\sqrt{m^2+1} = \alpha_1$ and $4\beta^2 = (2m^2+1) - 2m\sqrt{m^2+1} = \beta_1$

Therefore, we get,

$$x_n = \frac{(4\alpha^2)^n + (4\beta^2)^n}{2} = 2^{2n-1}(\alpha^{2n}+\beta^{2n})$$
$$= 2^{2n-1} m^{\lfloor n \rfloor} q_{2n}\left(m, 1, \frac{1}{4m}, 1\right) \text{ by (2)}$$

and

$$y_n = \frac{(4\alpha^2)^n - (4\beta^2)^n}{2\sqrt{m^2+1}} = 2^{2n-1}\frac{\alpha^{2n}-\beta^{2n}}{\alpha-\beta}$$
$$= 2^{2n-1}\frac{m^{\lfloor n \rfloor}}{m} f_{2n}\left(m, 1, \frac{1}{4m}\right) \text{ by (1)}$$

Thus,

$$(x,y) = \left(2^{2n-1} m^{\lfloor n \rfloor} q_{2n}\left(m,1,\frac{1}{4m},1\right), 2^{2n-1}\frac{m^{\lfloor n \rfloor}}{m} f_{2n}\left(m,1,\frac{1}{4m}\right)\right)$$

From cases (i) and (ii) we get the required solution.

Now we examine the remaining instances of C without providing evidence since they can be proven to be identical to those of Theorem 3.1.

**Theorem 3.2** Let $m > 0$ and $C = m^2 + 1$. Then all positive solution of the equation $x^2 - Cy^2 = -1$ are given by

$$(x,y) = \left(2^{2n-2}.m.m^{\left\lfloor\frac{2n-1}{2}\right\rfloor}.q_{2n-1}\left(1,m,\frac{1}{4m},1\right), 2^{2n-2}.m^{\left\lfloor\frac{2n-1}{2}\right\rfloor}.f_{2n-1}\left(1,m,\frac{1}{4m}\right)\right)$$

(or)

$$(x,y) = \left(2^{2n-2}.m^{\left\lfloor\frac{2n-1}{2}\right\rfloor}.q_{2n-1}\left(m,1,\frac{1}{4m},1\right), 2^{2n-2}.m^{\left\lfloor\frac{2n-1}{2}\right\rfloor}.f_{2n-1}\left(m,1,\frac{1}{4m}\right)\right)$$

with $n \geq 1$.

**Theorem 3.3** Let $m > 0$ and $C = m^2 - 1$.

Then all positive solution of the equation $x^2 - Cy^2 = 1$ are given by

$$(x,y) = \left(2^{n-1} m^{\left\lfloor\frac{n}{2}\right\rfloor} m^{\zeta(n)} q_n\left(1,m,\frac{-1}{4m},1\right), 2^{n-1} m^{\left\lfloor\frac{n}{2}\right\rfloor} f_n\left(1,m,\frac{-1}{4m}\right)\right)$$

(or)

$$(x,y) = \left(2^{n-1} m^{\left\lfloor\frac{n}{2}\right\rfloor} q_n\left(m,1,\frac{-1}{4m},1\right), 2^{2n-1}\frac{m^{\left\lfloor\frac{n}{2}\right\rfloor}}{m^{\zeta(n+1)}} f_n\left(m,1,\frac{-1}{4m}\right)\right)$$

with $n \geq 1$.

**Theorem 3.4:** Suppose $m > 0$ and $C = m^2 + 2$. Then all positive solution of the equation $x^2 - Cy^2 = 1$ are given by

$$(x,y) = \left(2^{n-1} m^{\lfloor n \rfloor} q_{2n}\left(1,m,\frac{1}{2m},1\right), 2^{n-1} m^{\lfloor n \rfloor} f_{2n}\left(1,m,\frac{1}{2m}\right)\right)$$

(or)

$$(x,y) = \left(2^{n-1} m^{\lfloor n \rfloor} q_{2n}\left(m,1,\frac{1}{2m},1\right), 2^{n-1}\frac{m^{\lfloor n \rfloor}}{m} f_{2n}\left(m,1,\frac{1}{2m}\right)\right)$$

with $n \geq 1$.

**Theorem 3.5:** Suppose $m > 0$ and $C = m^2 - 2$. Then all positive solution of the equation $x^2 - Cy^2 = 1$ are given by

$$(x,y) = \left(2^{n-1} m^{\lfloor n \rfloor} q_{2n}\left(1,m,\frac{-1}{2m},1\right), 2^{n-1} m^{\lfloor n \rfloor} f_{2n}\left(1,m,\frac{-1}{2m}\right)\right)$$

(or)

$$(x,y) = \left(2^{n-1} m^{\lfloor n \rfloor} q_{2n}\left(m,1,\frac{-1}{2m},1\right), 2^{n-1}\frac{m^{\lfloor n \rfloor}}{m} f_{2n}\left(m,1,\frac{-1}{2m}\right)\right)$$

with $n \geq 1$.

**Theorem 3.6** Let $m > 0$ and $C = m^2 + m$. Then all positive solution of the equation $x^2 - Cy^2 = 1$ are given by

$$(x,y) = \left(2^{2n-1}\frac{m^{\lfloor n \rfloor}}{m^n} q_{2n}\left(1,m,\frac{1}{4},1\right), 2^{2n-1}\frac{m^{\lfloor n \rfloor}}{m^n} f_{2n}\left(1,m,\frac{1}{4}\right)\right)$$

(or)

$$(x,y) = \left(2^{2n-1}\frac{m^{\lfloor n \rfloor}}{m^n} q_{2n}\left(m,1,\frac{1}{4},1\right), 2^{2n-1}\frac{m^{\lfloor n \rfloor}}{m^n} f_{2n}\left(m,1,\frac{1}{4}\right)\right)$$

with $n \geq 1$.

**Theorem 3.7** Let $m > 0$ and $C = m^2 - m$. Then all positive solution of the equation $x^2 - Cy^2 = 1$ are given by

$$(x,y) = \left(2^{2n-1}\frac{m^{\lfloor n \rfloor}}{m^n} q_{2n}\left(1,m,\frac{-1}{4},1\right), 2^{2n-1}\frac{m^{\lfloor n \rfloor}}{m} f_{2n}\left(1,m,\frac{-1}{4}\right)\right)$$

(or)

$$(x,y) = \left(2^{2n-1}\frac{m^{\lfloor n \rfloor}}{m^n} q_{2n}\left(m,1,\frac{-1}{4},1\right), 2^{2n-1}\frac{m^{\lfloor n \rfloor}}{m^{n+1}} f_{2n}\left(m,1,\frac{-1}{4}\right)\right)$$

with $n \geq 1$.

**Theorem 3.8** Let $C = \begin{cases} m^2 - 1, m > 1 \\ m^2 + 2, m \geq 0 \\ m^2 - 2, m \geq 2 \\ m^2 + m, m \geq 1 \\ m^2 - m, m \geq 2 \end{cases}$ then the equation $x^2 - Cy^2 = -1$ has no solution in positive integers.


Licensed Under Creative Commons Attribution CC BY

**Proof**

Since by Cognition 2.3, the period length of $\sqrt{C}$continued fraction expansion is even always. It follows from Lemma 2.1 that equation does not have a positive-integer value

$$x^2 - Cy^2 = -1.$$

## 4. Conclusion

In this paper, we investigate the Pell equation $x^2 - Cy^2 = \pm 1$, $C = m^2 \pm 1, m^2 \pm 2, m^2 \pm m$ and in $x$and$y$, we are seeking positive-integer values. We have all positive integer values in the Pell equations $x^2 - Cy^2 = \pm 1$for generalized Bi-Periodic Fibonacci and Lucas sequences when $C = m^2 \pm 1, m^2 \pm 2, m^2 \pm m$.

## References

[1] Bilgici, G. Two generalizations of Lucas sequence. *Appl. Math. Comput*. 2014, 245, 526-538.

[2] Burton, D.M. Elementary Number Theory, Seventh Edition, *The McGraw Hill Companies*, New York (2011).

[3] Choo, Y. On the generalized bi-periodic Lucas quaternions. *Int. J. Math. Anal*.2020, 14, 137-145.

[4] Choo, Y. Relations between Generalized Bi-Periodic Fibonacci and Lucas Sequences, *MDPI*, 2020, 8, 1527.

[5] Duman, M.G. Positive Integer Solutions of Some Pell equations, *MATHEMATIKA*, 2014, Vol. 30(1), 97-108.

[6] Keskin, R. and Duman, M.G. Positive integer solutions of some Pell equations, *Palestine Journal of Mathematics*, Vol. 8(2) (2019), 213-226.

[7] LeVeque, W. J. Topics in Number Theory. Volume 1 and 2. *Dover Publications*. 2002.

[8] Nagell, T. Introduction to Number Theory. *New York: Chelsea Publishing Company*. 1981.

[9] Tekcan, A. "Continued Fractions Expansions of $\sqrt{D}$ and Pell Equation $x^2 - Dy^2 = 1$", *Mathematica Moravica*, Vol. 15-2 (2011), 19-27.

[10] TituAndreescu, DorinAndrica and Ion Cucurezeanu., An introduction to Diophantine Equations, *Birhauser*, New York, 2010.

[11] Yayenie, O. A note on generalized Fibonacci sequences. *Appl. Math. Comput*. 2011, 217, 5603-5611.


Licensed Under Creative Commons Attribution CC BY