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On Certain Classes of Generalized Rational Functions

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Abstract: ***A normalized function f analytic in the open unit disc around the origin and nonvanishing outside the origin can be expressed in the form z/g(z) where g (z) has Taylor coefficients b_n's. Necessary and sufficient conditions in terms of b_n's are derived for some classes of analytic functions.***

1. Introduction

Let A_1 be the class of functions f analytic in $U = \{z \in C; |z| < 1\}$, and normalized by $f(0)=0$, $f'(0)=1$ where C is the set of complex numbers. An f in A_1 with $f(z) \neq 0$ in the punctured disc $U/\{0\}$, may be expressed as $f(z) = \psi(g) = z/g(z)$ in U,
where $g(z) = 1 + \sum_{n=1}^{\infty} b_n z^n$ in U.

Mitrinovic [2], Reade et.al [5], Silverman and Silvia [6] and Srinivas [7, 8] studied these coefficients.

Mitrinovic [3] obtained estimates for the radii on univalence of certain generalized rational functions $z/g(z)$. In particular, he found sufficient conditions for functions of the form
(1) $\frac{z}{1+b_1+b_2z^2+\dots+b_nz^n}$

$b_n \neq 0$, to be univalent in the unit disk U.

A function
(2) $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$
in A_1 is said to be starlike with respect to the origin in U, if it satisfies $Re\frac{zf'(z)}{f(z)} > \alpha$ in U. A function $f(z)$ in A_1 is said to be convex, if $Re\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > 0$ in the unit disc U.

Mac Gregor [1] showed the following.

Theorem A: If $f \in A$ satisfies
$$\left|\frac{f(z)}{z} - 1\right| < 1 (z \in U),$$
then
$$\left|\frac{zf'(z)}{f(z)} - 1\right| < 1 \left(|z| < \frac{1}{2}\right)$$
so that
$$Re\left(\frac{zf'(z)}{f(z)}\right) > 0 \left(|z| < \frac{1}{2}\right)$$
Therefore, $f(z)$ is univalent and starlike for $|z| < \frac{1}{2}$.

Also, Mac Gregor [2] had given the following result.

Theorem B. If $f \in A$ satisfies
$$|f'(z) - 1| < 1 (z \in U)$$
then
$$Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > 0 \left(|z| < \frac{1}{2}\right).$$
Therefore $f(z)$ is convex for $|z| < \frac{1}{2}$

The condition domains to Theorem A and Theorem Bare some circular domains whose centre is the point z = 1.

In the research paper Nunokawa et.al [4], some sufficient conditions for starlikeness and convexity under the hypotheses whose condition domains were centered at the origin were obtained as follows.

A result for starlikeness of $f(z)$ is
Theorem C. Let for $f \in A_1$ suppose that
$0.10583 \dots = \exp\left(-\frac{\pi^2}{4 \log 3}\right) < \left|\frac{zf'(z)}{f(z)}\right| < exp\left(\frac{\pi^2}{4 \log 3}\right) = 9.44915\dots (z \in U)$.

Then $f(z)$ is starlike for $|z| < \frac{1}{2}$.

Theorem D. Let for $f \in A_1$ suppose that
$0.472367 \dots = \exp\left(-\frac{3}{4}\right) < \left|\frac{f(z)}{z}\right| < exp\left(\frac{3}{4}\right) = 2.1777\dots (z \in U)$.

Then we have
$$\left|\frac{zf'(z)}{f(z)} - 1\right| < 1 \left(|z| < \frac{1}{2}\right)$$
and $f(z)$ is starlike for $|z| < \frac{1}{2}$.

For convexity of functions $f(z)$, the following result was derived.

Corollary E. Let $f \in A_1$ and suppose that

$0.472367 \dots = \exp\left(-\frac{3}{4}\right) < |f'(z)| < exp\left(\frac{3}{4}\right) = 2.1777\dots (z \in U)$.

Then $f(z)$ is convex for $|z| < \frac{1}{2}$.

A result for convexity of functions $f(z)$ was derived in

Theorem F. Let $f \in A_1$ and suppose that
$0.778801 \dots = \exp\left(-\frac{1}{4}\right) < \left|\frac{zf'(z)}{f(z)}\right| < exp\left(\frac{1}{4}\right) = 1.28403\dots (z \in U)$.
Then $f(z)$ is convex for $|z| < \frac{1}{2}$.

Theorem G Let $f \in A_1$ and suppose that

$0.10583\ \ldots = \exp\left(-\frac{\pi^2}{4\log 3}\right) < \left|\frac{zf'(z)}{f(z)}\right| < exp\left(\frac{\pi^2}{4\log 3}\right) = 9.44915\ldots (z \in U)$.

Then $f(z)$ is convex for $|z| < r_0$ where r_0 is the root of the equation
$(4\log 3)r^2 - 2(4\log 3 + \pi^2 + r + 4\log 3) = 0,$

$$r_0 = \frac{\pi^2 - 4\log 3 - \pi\sqrt{\pi^2 + 8\log 3}}{4\log 3} = 0.15787\ldots.$$

In this paper we derive sufficient conditions on $b_n's$ for f to be starlike or convex in $|z| < \frac{1}{2}$.

SECTION – 1

First we determine some sufficient conditions on f in terms of $b_n's$ for f to be starlike in $|z| < \frac{1}{2}$, in the following Theorems 1 to 3.

Theorem 1: Let $f(z) = z/(1 + \sum_{n=1}^{\infty} b_n z^n) \in A_1$ with $b_n's$ satisfying.

$$\sum_{n=1}^{\infty} |b_n| < \frac{1}{2} \qquad \ldots..(1)$$

Then $f(z)$is univalent and starlike for $|z| < \frac{1}{2}$

Theorem 2: Let $f(z) = z/(1 + \sum_{n=1}^{\infty} b_n z^n) \in A_1$. If $b_n's$ satisfy

(i) $1 - e\left(\frac{-\pi^2}{4\log 3}\right) > e\left(\frac{-\pi^2}{4\log 3}\right)|b_1| + \sum_2^{\infty}\left[(n-1) + \right.$
$e-\pi24log3bn$ ….. (2)

or

(ii)$\sum_2^{\infty}\left[(n-1) + e\left(\frac{-\pi^2}{4\log 3}\right)\right]|b_n| < e\left(\frac{\pi^2}{4\log 3}\right) - 1$

then $f(z)$ is starlike for $|z| < \frac{1}{2}$

Theorem 3: Let $f(z) = z/(1 + \sum_{n=1}^{\infty} b_n z^n) \in A_1$. If $b_n's$ satisfy

(i) $\sum_1^{\infty}|b_n| < \frac{1}{2}$

or

(ii) $\sum_1^{\infty}|b_n| < 1 - e^{3/4}$

or

(iii) $\sum_1^{\infty}|b_n| < e^{3/4} - 1$

then $f(z)$ is starlike for $|z| < \frac{1}{2}$.

Finally we obtain some sufficient conditions on f in terms of $b_n's$ for f to be convex in $|z| < \frac{1}{2}$, in the following result.

Theorem 4. Let $f(z) = z/(1 + \sum_{n=1}^{\infty} b_n z^n) \in A$. If $b_n's$ satisfy

(i) $1 - e^{-1/4} > \sum_1^{\infty}\{(n-1) + e^{1/4}\}|b_n|$

or

(ii) $\sum_{n=1}^{\infty}\left\{(n-1) + e\left(\frac{1}{4}\right)\right\}|b_n| < e\left(\frac{1}{4}\right) - 1$

then $f(z)$ is convex for $|z| < \frac{1}{2}$.

Proof of Theorem 1 : For $f(z) = z/g(z)$ where $g(z) = (1 + \sum_{n=1}^{\infty} b_n z^n), z \in U$, we have

$$\left|\frac{f(z)}{z} - 1\right| = \left|\frac{1}{g(z)} - 1\right|$$

$$= \left|\frac{1 - g(z)}{|g(z)|}\right|$$

$$= \left|\frac{-\sum_1^{\infty} b_n z^n}{1 + \sum_1^{\infty} b_n z^n}\right|$$

$$\le \frac{\sum_1^{\infty}|b_n|}{1 - |\sum_1^{\infty} b_n z^n|}$$

$$\le \frac{\sum_1^{\infty}|b_n|}{1 - \sum_1^{\infty}|b_n|} < 1$$

for z in U, since (1) implies that

$$2\sum_1^{\infty}|b_n| < 1 \Rightarrow \sum_1^{\infty}|b_n| < 1 - \sum_1^{\infty}|b_n|.$$

Therefore $f(z)$ is univalent and starlike for $|z| < \frac{1}{2}$ by Theorem A of Mac Gregor [1]

Proof of Theorem 2 : Let $f(z) = z/g(z)$ where $g(z) = (1 + \sum_{n=1}^{\infty} b_n z^n), z \in U$

Part (i) : We have

$$\left|\frac{zf'(z)}{f(z)}\right| = \left|1 - \frac{zg'(z)}{g(z)}\right| = \left|\frac{1 + \sum_1^{\infty}(1-n)b_n z^n}{\sum_0^{\infty} b_n z^n}\right|$$

$$\ge \frac{1 - |\sum_1^{\infty}(1-n)b_n z^n|}{1 + |\sum_0^{\infty} b_n z^n|}$$

$$\ge \frac{1 - \sum_2^{\infty}(n-1)|b_n|}{1 + \sum_1^{\infty}|b_n|}$$

$$> e\left(\frac{-\pi^2}{4\log 3}\right) \qquad \ldots..(4)$$

For z in U, since (2) implies

$1 - e\left(\frac{-\pi^2}{4\log\ \ 3}\right) > e\left(\frac{-\pi^2}{4\log\ \ 3}\right)|b_1| + \sum_2^{\infty}\left[(n-1) + \right.$
$e-\pi24log3bn$,

$$1 - \sum_2^{\infty}(n-1)|b_n| > e\left(\frac{-\pi^2}{4\log\ \ 3}\right)(1 + \sum_1^{\infty}\ \ |b_n|)$$

Therefore $f(z)$ is starlike for $|z| < \frac{1}{2}$ by Theorem C of Nunokawa et.al. [4] and the inequality (4).

Part (iii) : We have

$$\left|\frac{zf'(z)}{f(z)}\right| = \left|1 - \frac{zg'(z)}{g(z)}\right| = \left|1 - \frac{\sum_1^{\infty} nb_n z^n}{1 + \sum_0^{\infty} b_n z^n}\right|$$

$$= \frac{|\sum_1^{\infty}(1-n)b_n z^n|}{\sum_0^{\infty} b_n z^n}$$

$$\le \frac{1 - \sum_1^{\infty}(n-1)|b_n|}{1 + \sum_1^{\infty}|b_n|} < e\left(\frac{\pi^2}{4\log 3}\right) \qquad \ldots..(5)$$

for z in U, since (3) implies

$$1 - \sum_1^{\infty}(n-1)|b_n| < e\left(\frac{\pi^2}{4\log\ \ 3}\right)(1 - \sum_1^{\infty}\ \ |b_n|)$$

$\sum_1^{\infty}\ \ \left\{(n-1) + e\left(\frac{\pi^2}{4\log\ \ 3}\right)\right\}|b_n| < e\left(\frac{\pi^2}{4\log\ \ 3}\right) -$
1

Therefore $f(z)$ is starlike for $|z| < \frac{1}{2}$ by Theorem C of Nunokawa et.al [4] and the inequality (5).

Proof of Theorem 3 :Consider $f(z) = z/g(z)$ where $g(z) = (1 + \sum_{n=1}^{\infty} b_n z^n), z \in U$.

Part (i): Follows from Theorem 1.

Part (ii): $1 - e^{-3/4} > \sum_1^{\infty}\ \ \ |b_n|$

$$\Rightarrow e^{-3/4}(1 - \sum_1^\infty |b_n|) > 1$$
$$\Rightarrow \left|\frac{f(z)}{z}\right| = \left|\frac{1}{g(z)}\right| = \frac{1}{|1+\sum_1^\infty b_n z^n|} \le \frac{1}{1-\sum_1^\infty |b_n|} < e^{3/4}$$

Now Theorem D of Nunokawa et al. [4] gives the Part (ii).

Part (iii): We have
$$\left|\frac{f(z)}{z}\right| = \left|\frac{1}{g(z)}\right| = \frac{1}{|1+\sum_1^\infty b_n z^n|} \ge \frac{1}{1+\sum_1^\infty |b_n|} > e^{-3/4}$$

Now Theorem D of Nunokawa et al … [4] gives the Part (iii).

Proof of Theorem 4 : Follows from that of Theorem 2 via Theorem F.

References

[1] Macgregor, T.H. The radius of univalence of certain analytic functions. II, Proc. Amer Math Soc., 14(3) (1963), 521-524

[2] Macgregor T.H. A class of univalent functions, Proc. Amer Math Soc., 15 (1964), 311-317.

[3] MitrinovicDragoslavS. : On the univalence of rational functions, Univ. Beograd Publ. Elektrotech, Fak., Ser, Mat. Fiz., No.634-No.637 (1979), pp.221-227

[4] Nunokawa, M., Owa S., Nakamura Y. and Hayami T. : Sufficient conditions for starlikeness and convexity in $|z| < 1/2$., JIPAM, Vol.9 (2008), Issue 2, Article 32, 6pp.

[5] Reade, M.O., Silverman : H and Todorov, P.G. : On the starlikenes and convexity of a class analytic functions, RendicontidelCircoloMatematico di Palemo, serie II, Tomo XXXIII (1984), 265-272

[6] Silverman, H and Silvia, E.M. : Univalence preserving operators, Complex Variable 5 (1986), 313-312

[7] Srinivas, V. On Inverse coefficients, Journal of Orissa math Soc, Vol.25 (1/2), (2006) 67-73.

[8] Srinivas, V. : On inverse coefficients, Journal of Analysis, Vol. 15 (1/2), (2007), 255-262

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