International Journal of Science and Research (IJSR)
ISSN: 2319-7064
SJIF (2022): 7.942

Supersymmetric Approach to Solve Interpolated Position - Dependent Mass Hamiltonians

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Abstract: ***Quantum dots, liquid crystals, compositionally graded crystals, and other condensed matter systems all use position dependent effective mass (PDEM) Hamiltonians to describe the dynamics of electrons. Because the PDEM quantum Hamiltonians are not Hermitian, we employ the effective mass kinetic energy operator in Von Ross's two-parameter form, which is Hermitian by default and contains additional reasonable forms as special instances. It is shown that Hamiltonians of the form $H(s) = (1-s)H_{-} + sH_{+}$, $0 \leq s \leq 1$ where $H_{\pm}$ are supersymmetric partner Hamiltonians corresponding to position dependent mass Schrödinger equations are exactly solvable for a number of deformed shape invariant potentials.***

Keywords: Interpolation, Position-Dependent mass, Shape Invariance

1. Introduction

During the past few years PDMSE has attracted a lot of attention due to it's possible applications in a variety of fields like describing the dynamics of electrons in many condensed matter systems, such as compositionally graded crystals [2], quantum dots [3] and quantum liquids [4], in the determination of the electronic properties of semiconductors [5], ^{3} He cluster [6] etc. and also due to intrinsic interest in such systems. For this reason there have been a growing interest in obtaining the exact solutions of PDMSE in different contexts e.g, solutions of the nonrelativistic wave equation with position-dependent effective mass has been obtained [7], N-fold supersymmetry is studied in [8], exact solvability of complexified Von Ross Hamiltonian is discussed [9], singular position dependent mass is cosidered [10] etc. Exact solutions of PDMSE's can be obtained using different methods e.g, via supersymmetric quantum mechanics [11, 12], point canonical transformation approach [7, 13], series solutions [14] and etc. in [15, 16, 17].

Recently using the shape invariance property Odake et al [18] obtained a new class of exactly solvable Hamiltonians of the form

$$H(s) = (1-s)A^{\dagger}A + sAA^{\dagger}, \quad 0 \leq s \leq 1 \tag{1}$$

where the operators A and $A^{\dagger}$ are given by

$$A = \frac{d}{dx} + W(x), \quad A^{\dagger} = -\frac{d}{dx} + W(x), \tag{2}$$

and $W(x)$ is the superpotential corresponding to a shape invariant potential. The formalism was also found to be useful in the context of discrete quantum mechanical systems [18]. Here our another objective is to examine whether or not this formalism can be extended to shape invariant Hamiltonians with a position dependent mass [19, 20, 21, 22]. In particular we shall obtain the spectrum of a number of interpolating Hamiltonians constructed using shape invariant Hamiltonians with a position dependent mass.

2. Constant Mass Hamiltonian

In conventional quantum mechanics, the Hamiltonian in constant mass Schrödinger equation is

$$H_{cm} = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + V(x) \tag{3}$$

which is in general Hermitian. Let us assume $(for\ m = \frac{1}{2}\ and\ \hbar = 1)$

$$H_{cm} = H_{1,cm} = A^{\dagger}(a)A(a) = -\frac{d^2}{dx^2} + V_1(x,a) \tag{4}$$

Where "a" represents the set of parameters and A(a) and $A^{\dagger}(a)$ are defined as

$$A(a) = \frac{d}{dx} + W(x,a)$$
$$A^{\dagger}(a) = -\frac{d}{dx} + W(x,a) \tag{5}$$

Now let us define the Hamiltonian $H_{2,cm}$ as follows

$$H_{2,cm} = A(a)A^{\dagger}(a) = -\frac{d^2}{dx^2} + V_2(x,a) \tag{6}$$

The Hamiltonian $H_{2,cm}$ is called the supersymmetric partner of $H_{1,cm}$. It can be easily shown that $H_{1,cm}$ and $H_{2,cm}$ are isospectral except for the ground state. We have $V_1(x,a) = W^2(x,a) - W'(x,a)$ and $V_2(x,a) = W^2(x,a) + W'(x,a)$, which are the partner potentials for constant mass and $W(x,a)$ is known as superpotential.

3. Position dependent effective mass (PDEM) Hamiltonian

For $\hbar = 2m_0 = 1$, we may therefore write the PDEM SE as

$$H_{pdm}\,\psi(x) = \left[-\frac{1}{2}\begin{pmatrix} M^{\xi'}(\alpha,x)\frac{d}{dx}M^{\eta'}(\alpha,x)\frac{d}{dx}M^{\varsigma'}(\alpha,x) \\ +M^{\varsigma'}(\alpha,x)\frac{d}{dx}M^{\eta'}(\alpha,x)\frac{d}{dx}M^{\xi'}(\alpha,x)\end{pmatrix} + V(a,x)\right]\psi(x) = E\psi(x) \quad (7)$$

where α and a denotes the two sets of parameters and ξ', η', ς' are the Von Ross ambiguity parameters, constrained by the condition $\xi' + \eta' + \varsigma' = -1$.

On setting

$$M(\alpha,x) = \frac{1}{f^2(\alpha,x)} \qquad f(\alpha,x) = 1 + g(\alpha,x) \quad (8)$$

(note that $g(\alpha,x) = 0$ corresponds to the constant mass case), Eqn.(7) becomes

$$\left[-\frac{1}{2}\left(f^{\xi}(\alpha,x)\frac{d}{dx}f^{\eta}(\alpha,x)\frac{d}{dx}f^{\varsigma}(\alpha,x) + \right.\right.$$
$$f\varsigma\alpha,xddxf\eta\alpha,xddxf\xi\alpha,x+Va,x\psi x=E\psi x \quad (9)$$

with $\xi + \eta + \varsigma = 2$. In simplified form Eqn. (9) can be written as

$$\left[\Pi^2 + V_{eff}(b,x)\right]\psi(x) = E\psi(x) \quad (10)$$

where

$$\Pi = \sqrt{f(\alpha,x)}p\sqrt{f(\alpha,x)} = -i\sqrt{f(\alpha,x)}\frac{d}{dx}\sqrt{f(\alpha,x)},\ p = -i\frac{d}{dx}$$
$$V_{eff}(b,x) = V(a,x) + \tilde{V}(\alpha,\xi,x)$$
$$\tilde{V}(\alpha,\xi,x) = \frac{1}{2}(1-\xi-\varsigma)f(\alpha,x)f''(\alpha,x) + (\tfrac{1}{2}-\xi)(\tfrac{1}{2}-\varsigma)f'^2(\alpha,x) \quad (11)$$

and b is a set of parameters depends on a, α, ξ.

Let us now assume

$$H_{pdm} = A^{\dagger}(\lambda)A(\lambda) = H_{1,pdm} = -\Pi^2 + V_{1,eff}(\lambda,x) \quad (12)$$

where λ is a parameter determined entirely by the potential parameter b and the first order differential operators A and $A^{\dagger}$ are given by

$$A(\lambda) = \sqrt{f(x,\alpha)}\frac{d}{dx}\sqrt{f(x,\alpha)} + W(x,\lambda)$$
$$A^{\dagger}(\lambda) = -\sqrt{f(x,\alpha)}\frac{d}{dx}\sqrt{f(x,\alpha)} + W(x,\lambda) \quad (13)$$

Therefore the isospectral partner of $H_{1,pdm}$ is defined as

$$H_{2,pdm} = A(\lambda)A^{\dagger}(\lambda) = -\Pi^2 + V_{2,eff}(\lambda,x) \quad (14)$$

Thus we have from above

$$V_{1,eff}(\lambda,x) = W^2(x,\lambda) - f(\alpha,x)W'(x,\lambda)$$
$$V_{2,eff}(\lambda,x) = W^2(x,\lambda) + f(\alpha,x)W'(x,\lambda) \quad (15)$$

3.1 Special csae:

As a special case let us choose $\xi = \varsigma = 0$, then PDEM SE Eqn.(9) reduces to

$$\left[-\frac{d}{dx}f^2(x,\alpha)\frac{d}{dx} + V(x,a)\right]\psi(x) = E\psi(x) \quad (16)$$

In this case partner Hamiltonians are

$$H_{1,pdm} = A^{\dagger}(a)A(a) = -\frac{d}{dx}f^2(x,\alpha)\frac{d}{dx} + V_1(x,a)$$
$$H_{2,pdm} = A(a)A^{\dagger}(a) = -\frac{d}{dx}f^2(x,\alpha)\frac{d}{dx} + V_2(x,a) \quad (17)$$

Where the first order differential operators A(a) and $A^{\dagger}(a)$ are of the form

$$A(a) = f(x)\frac{d}{dx} + W(x,a)$$
$$A^{\dagger}(a) = -f(x)\frac{d}{dx} - U'(x) + W(x,a) \quad (18)$$

and

$$V_1(x,a) = W^2(x,a) - (f(x)W(x,a))'$$
$$V_2(x,a) = W^2(x,a) + (f(x)W(x,a))' - 2f'(x)W(x,a) - f(x)f''(x) \quad (19)$$

4. Interpolation of two supersymmetric partner Hamiltonians when mass is constant

Recently, the interpolation of the isospectral partner Hamiltonian have been considered for constant mass case by *S. Odake et al*. They took the interpolating Hamiltonian as a covex combination of the partner Hamiltonian, which reads

$$H_{s,cm} = (1-s)H_{1,cm} + sH_{2,cm} = -\frac{d^2}{dx^2} + W^2(x,a) + (2s-1)W'(x,a),\quad 0 \le s \le 1 \quad (20)$$

They had shown that for a wide class of shape invariant Hamiltonians, the interpolating Hamiltonian also retain the shape invariance property. That is the interpolating Hamiltonian $H_{s,cm}$ has the same form as the original Hamiltonian $H_{1,cm}$ with shifted coupling constant(s) and shifted ground state energy.

4.1 General case

We have from Eqn.(12) and (14) the isospectral partner Hamiltonians of position dependent mass are $H_{1,pdm}$ and $H_{2,pdm}$. The interpolation Hamiltonian of the partner Hamiltonians, is given by

$$H_{s,pdm} = (1-s)H_{1,pdm} + sH_{2,pdm}$$
$$= \Pi^2 + W^2(x,\lambda) + (2s-1)f(x,\alpha)W'(x,\lambda) \quad (21)$$
$$= \Pi^2 + V_{s,eff}(x,\lambda')$$

We note that λ' depends on s.

Here we assume that the original potential $V_{1,eff}$ belongs to shape invariance class. Next our plan is to check weather the interpolation Hamiltonian $H_{s,pdm}$ has the same form as the $H_{1,pdm}$ with coupling constant λ is replaced by λ'.

4.2 Shape invariance property and solvability of the original Hamiltonian $H_{1,pdm}$

Now to solve

$$H_{1,pdm}\,\psi(x,\alpha) = \left[-\Pi^2 + V_{1,eff}(\lambda,x)\right]\psi(x,\alpha) = E\psi(x,\alpha) \quad (22)$$

means that we are to find a superpotential $W(x,\lambda)$, the function $f(x,\alpha)$ and the parameter λ, such that the following conditions are satisfied.

$$V_{1,eff} = W^2(x,\lambda) - f(x,\alpha)W'(x,\lambda), \text{ and the Shape Invariance condition } A^\dagger(\lambda_1)A(\lambda_1) = A(\lambda_2)A^\dagger(\lambda_2) + R(\lambda_1) \quad (23)$$

Let us consider the superpotential be of the form

$$W(x,\lambda) = \lambda\phi(x) + \mu \quad (24)$$

where $\phi(x)$ is a parameter independent function.

Then from Eqn.(23), the SI condition gives

$$(\lambda_2^2 - \lambda_1^2)\phi^2(x) + 2(\lambda_2\mu_2 - \lambda_1\mu_1)\phi(x) - (\lambda_2 + \lambda_1)(1 + g(x,\alpha))\phi'(x) + R(\lambda_1,\mu_1) + \mu_2^2 - \mu_1^2 = 0$$

Assuming

$$\lambda_2 - \lambda_1 = \alpha \;\; and \qquad \mu_2 = \frac{\alpha\beta + \lambda_1\mu_1 + 2\lambda_1\beta}{\lambda_1 + \alpha} \quad (26)$$

we have

$$\alpha\phi^2(x) + 2\beta\phi(x) - (1 + g(x))\phi'(x) + \frac{R(\lambda_1,\mu_1) + \mu_2^2 - \mu_1^2}{2\lambda_1 + \alpha} = 0 \quad (27)$$

Since $\phi(x)$ is independent of parameters, Eqn. (27) implies that

$$\alpha\phi^2(x) + 2\beta\phi(x) - (1 + g(x))\phi'(x) = -c_1 \qquad \frac{R(\lambda_1,\mu_1) + \mu_2^2 - \mu_1^2}{2\lambda_1 + \alpha} = c_1 \quad (28)$$

where c_1 is a constant.

Therefore for a given $\phi(x)$ we can calculate $g(x,\alpha)$ and consequently $V_{1,eff}$. The corresponding energy eigenvalues can be obtained by the following relation

$$E_n = \sum_{i=1}^{n} R(\lambda_i,\mu_i) = c_1(2n\lambda + n^2\alpha) + \mu^2 - \left[\frac{n^2\alpha\beta + \lambda\mu + 2n\beta\lambda}{\lambda + n\alpha}\right] \quad (29)$$

where

$$\lambda_{i+1} = \lambda_1 + i\alpha \qquad \mu_i = \frac{\lambda_1\mu_1 + 2i\beta\lambda_1 + i^2\alpha\beta}{\lambda_1 + i\alpha} \qquad \lambda_1 = \lambda, \;\; \mu_1 = \mu \quad (30)$$

4.3 Form of Interpolation Hamiltonian for given $\phi(x)$

We have for the chosen $W(x,\lambda)$ in Eq.(24), from Eq. (28), $f(x)$ can be calculated as

$$f(x) = \frac{\alpha\phi^2(x) + 2\beta\phi(x) + c_1}{\phi'(x)} \quad (25) \quad (31)$$

Therefore the interpolation Hamiltonian has the form

$$H_{s,pdm} = \Pi^2 + \lambda(\lambda + (2s-1)\alpha)\phi^2(x) + 2\lambda(\mu + (2s-1)\beta)\phi(x) + \mu^2 + (2s-1)\lambda c_1 \quad (32)$$

5. Example

Let us take $\phi(x) = x$ and $c_1 = 1$, then we have

$$\begin{aligned} W(x,\lambda) &= \lambda x + \mu \\ g(x) &= \alpha x^2 + 2\beta x \\ V_{1,eff}(x,\lambda) &= (\lambda^2 - \alpha\lambda)x^2 + 2\lambda(\mu - \beta)x + \mu^2 - \lambda \\ V_{s,eff}(x,\lambda) &= (\lambda^2 + 2s\alpha\lambda - \alpha\lambda)x^2 + 2\lambda(\mu + 2s\beta - \beta)x + \mu^2 + (2s-1)\lambda \end{aligned} \quad (33)$$

Therefore it is obvious that the interpolating Hamiltonian $H_{s,pdm}$ has the same form as the original Hamiltonian $H_{1,pdm}$ with the coupling constants λ and μ are replaced by λ' and μ', given by :

$$\lambda' = \frac{\alpha + \sqrt{(2\lambda-\alpha)^2 + 8s\alpha\lambda}}{2} \qquad \mu' = \beta + \frac{\lambda}{\lambda'}(\mu + 2s\beta - \beta) \quad (34)$$

together with the shift of the ground state energy $\Delta E = \mu^2 - \mu'^2 + (2s-1)\lambda + \lambda'$

6. Conclusion

In conclusion, using the idea of deformed shape invariance, we were able to develop certain precisely solved position dependent mass interpolating Schrödinger equations. The similar method can be used with a few other shape invariant potentials, such as [23]. It would be intriguing to investigate whether this formalism may be used to self-similar shape invariant potentials, as mentioned by [24, 25]. If the original system is precisely solvable but not necessarily shape invariant, it would be interesting to investigate whether perfectly solvable interpolating Hamiltonians can be found.

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