International Journal of Science and Research (IJSR)
ISSN: 2319-7064
SJIF (2022): 7.942

Fixed Point for Contraction Mapping in Complete Parametric B Metric Space

Arun Kumar Garg[1], Bharti Mishra[2]

[1, 2]Department of Mathematics, Madhyanchal University, Bhopal, Madhya Pradesh, India
[1]*gargarun1956 [at]gmail.com*
[2]*bmrani123 [at]gmail.com*

Abstract: ***There are many generalization of metric space. Complete parametric bmetric space is the generalization of metric space too. Which was introduced and studied by Hussian [8] (a new approach to metric space) in 2014. In present paper we prove two new fixed point theorems based on injective mapping using contraction conditions.***

Mathematics Subject Classification (MSC): 47H10, 54H25

Keyboards: unique fixed point, contraction, parametric metric space, injective mapping

1. General Introduction

Let X be a non empty set and $T: X \to X$ be a self mapping. A point $p \in X$ such that$= p$; then p is called a fixed point of mapping T. If $T: X \to X$ is a multi valued map (i.e. from X to all non empty subsets of X) then point$p \in X$ is called a fixed point of mapping Tif$p \in Tp$. Most of the physical problems may be transferred to fixed point problems as $Tx = x$ (to find a point x in a domain of an appropriate mapping T).Fixed point theory has fascinated lots of researchers since 1922 with the celebration of Banach (Polish mathematician) contraction principle [1] which provided a constructive method to find a fixed point of a map**.** Banach's contraction principal stated as a mapping $T: X \to X$ defined on metric space $(X, d)$is called contraction mapping if

$$d(Tx, Ty) \le kd(x, y) \quad \forall\, x, y \in X \,\&\, 0 < k < 1$$

However, only drawback of Banach contraction principle was the mapping T must be continuous throughout space X. Kannan [2] rectified this problem and proved a fixed point theorem for operators that need not be continuous. Further, Chatterjea [3], in 1972, also proved a fixed point theorem for discontinuous mapping, which is actually a kind of dual of Kannan mapping. A lucid survey shows that there exist a vast literature available on fixed point theory. Fixed point theorems are important tools for proving the existence and uniqueness to the solution inmetric equation, convex minimization and split feasibility, Iteration methods, differential equations, partial differential equations, integral differential equations, variational inequalities as well as for finding zeros of contractive mappings.

The concept of metric space is generalized in many direction. Wang et.al. [4] introduced and defined expansive mapping and proved some fixed point theorems in complete metric space for expansive mapping wherein Daffer and Kaneko [5] proved some fixed point theorems for a pair of mappings in complete metric space for expansive mapping. The notion of a b-metric space was studied by Czerwik [6, 7] and many fixed point theorems were proved for single mapping and multi-valued mappings by different authors have been obtained on b-metric spaces and in ordered b-metric spaces. Alghamdi, et al. defined [8] b-metric-like spaces and obtained some fixed point theorems for single mapping and two mappings. The concept of fuzzy sets was initiated by Zadeh [9] in 1965. The fuzzy metric space was introduced by Kramosil and Michalek [10]. Also, Grabeic [11] proved the contraction principle in the setting of fuzzymetric space. Also, George and Veeramani [12] modified the notion of fuzzy metric space with the help of continuous t-norm, by generalizing the concept of probabilistic metric space to fuzzy situation. Later several authors, for example, Bariet. al. [13], Vetro et. al. [14] etc. proved fixed and common fixed point theorems in fuzzy metric spaces. In 2004, Park [14] introduced the notion of intuitionistic fuzzy metric space. In 2014Hussain et al. [15] studied and introduced New Approach to Fixed Point Results in Triangular Intuitionistic Fuzzy Metric Space. Hussain et. al. [16] introduced fixed point results for various contractions in parametric and fuzzy b-metric spaces. The notion of parametric metric space. Later on gave generalized parametric metric space and introduced parametric b-metric space which is combination of both metric space and b-metric spaces. In the present paper we prove fixed point theorems on complete parametric b metric space.

2. Preliminaries

Proceeding to our main result, let we furnish some definitions, proposition, properties &lemmas needed in sequel.

2.1 Definition: Let X be a non empty set and$\mathrm{T_p}: \mathrm{X} \times \mathrm{X} \times (0, \infty) \to (0, \infty)$ be a map on X such that $\forall\, x, y, z \in X$ and $t > 0$

(a)$\mathrm{T_p}(x, y, t) = 0$ if$x = y$

(b). $\mathrm{T_p}(x, y, t) = \mathrm{T_p}(y, x, t)$

(c)$\mathrm{T_p}(x, y, t) \le \mathrm{T_p}(x, z, t) + \mathrm{T_p}(z, y, t)$

Then$\mathrm{T_p}$ is called parametric metric and pair (X, d) is called parametric metric space.

2.2 Properties:

a) If $\lim_{n\to\infty}(x_n, x, t) = 0 \Rightarrow \lim_{n\to\infty} x_n = x$, for all $t > 0$ then sequence $\{x_n\}_{n=1}^{\infty}$ converses $x \in X$
b) If $\lim_{n\to\infty}(x_n, x_m, t) = 0$ for all $t > 0$; then sequence $\{x_n\}_{n=1}^{\infty}$ is called Cauchy sequence.
c) If every Cauchy sequence is convergent, then parametric metric space (X, d) is a complete parametric metric space.
d) Let(X, d)be a parametric metric space and $T: X \to X$ be a mapping, then we say T is a continuous mapping at p in X, if for any sequence $\{x_n\}_{n=1}^{\infty} \in X$such that $\lim_{n\to\infty} x_n = x \Rightarrow \lim_{n\to\infty} Tx_n = Tx$.

2.3 Example

Let X denote the set of all functions$f: (0;\infty) \to R$.Define $p \in X \times X \times (0;\infty) \to (0;\infty)$ by $p(f;g;t) = |f(t) - g(t)|\ \forall f;g \in X$and $t>0$. Then p is a parametric metric on X and the pair (X, p) is a parametric metric spaces.

Hussain et al. [15, 16] introduced the concept of parametric b-metric space as follows.

2.4 Definition

Let X be a non empty set; $s \geq 1$ be a real number and $T_{pb}: \mathrm{X} \times \mathrm{X} \times (0, \infty) \to (0, \infty)$ be a map on X such that $\forall\, x, y, z \in X$ and $t > 0$
(a) $T_{pb}(x, y, t) = 0$ if$x = y$
(b). $T_{pb}(x, y, t) = T_{pb}(y, x, t)$
(c)$T_{pb}(x, y, t) \leq s\,[T_{pb}(x, z, t) + T_{pb}(z, y, t)]$

ThenT_{pb} is called parametric b- metric and pair (X, d) is called parametric b- metric space.

2.5 Properties

Let $\{x_n\}_{n=1}^{\infty}$be a sequence in parametric b- metric space; then
(a). If$\lim_{n\to\infty}(x_n, x, t) = 0 \Rightarrow \lim_{n\to\infty} x_n = x$, for all $t > 0$ then sequence $\{x_n\}_{n=1}^{\infty}$ converses $x \in X$
(b).If $\lim_{n\to\infty}(x_n, x_m, t) = 0$ for all $t > 0$; then sequence $\{x_n\}_{n=1}^{\infty}$ is called Cauchy sequence.
(c)If every Cauchy sequence is convergent, then parametric b- metric space (X, d) is a complete parametric b- metric space.
(d)Let(X, d)be a parametric b- metric space and $T_{pb}: X \to X$ be a mapping, then We sayT_{pb} is a continuous mapping at x in X, if for any sequence $\{x_n\}_{n=1}^{\infty} \in X$ such that $\lim_{n\to\infty} x_n = x \Rightarrow \lim_{n\to\infty} Tx_n = Tx$

2.6 Example

Let $\mathrm{X} = (0, \infty)$ and define $T_{pb}: \mathrm{X} \times \mathrm{X} \times (0, \infty) \to (0, \infty)$as $p(x; y; t) = t(x - y)^p$. Then P is a parametric b-metric with constant 2^p.

Proof: Axioms 1 and 2 are obvious. Now for Third
$p(x; y; t) = t(x - y)^p = t(s\,[x - z + z - y])^p \leq t\ \ s^p\,[(x - z)^p + (z - y)^p]fors = 2$

3. Main Result

The objective of this paper is to prove new fixed point theorems in complete b metric space of. In present paper we prove two fixed point theorems using contraction conditions. The authors are unknown from this fact that others authors had already proved these theorems.

3.1 Theorem

Let(X, d) be a complete parametric b metric space and $T_{pb}: X \to X$ be an injective mapping satisfying the condition

$$(3.1)\quad d(T_{pb}x, T_{pb}y, t) \leq \alpha . d(x, y, t) + \beta\left(\frac{d(x, T_{pb}x, t)\,.\,d(y, T_{pb}y, t)}{d(x, y, t)}\right) + \gamma\left(\frac{d(x, T_{pb}x, t).\,d(x, T_{pb}y, t)}{d(x, y, t) + d(y, T_{pb}y, t)}\right)$$

$\forall\, t \in [0,1)$; $\alpha, \beta, \gamma > 0; x, y \in X\ \&\ x \neq y$ have a fixed point if $s\alpha + \beta + \gamma < 1$and moreover a unique fixed point if $\alpha < 1$.

Proof: Let $x_0 \in X$, Define iterative sequence$\{x_n\}_{n=1}^{\infty}$follows:$T_p x_n = x_{n+1}$ for $n = 1, 2, 3, \ldots$. If for some n, $T_p x_n = x_n$, then x_n is the fixed point. Otherwise$T_p x_n \neq x_n$, using inequality (3.1)

$$d(x_{n+1}, x_{n+2}, t) = d(T_p x_n, T_p x_{n+1}, t) \leq \alpha d(x_n, x_{n+1}, t) + \beta\left(\frac{d(x_n, T_{pb}x_n, t).\,d(x_{n+1}, T_{pb}x_{n+1}, t)}{d(x_n, x_{n+1}, t)}\right) + \gamma\left(\frac{d(x_n, T_{pb}x_n, t).\,d(x_n, T_{pb}x_{n+1}, t)}{d(x_n, x_{n+1}, t) + d(x_{n+1}, T_{pb}x_{n+1}, t)}\right)$$

$$d(x_{n+1}, x_{n+2}, t) \leq \alpha d(x_n, x_{n+1}, t) + \beta\left(\frac{d(x_n, x_{n+1}, t).\,d(x_{n+1}, x_{n+2}, t)}{d(x_n, x_{n+1}, t)}\right) + \gamma\left(\frac{d(x_n, x_{n+1}, t).\,d(x_n, x_{n+2}, t)}{d(x_n, x_{n+1}, t) + d(x_{n+1}, x_{n+2}, t)}\right)$$

$$d(x_{n+1}, x_{n+2}, t) \leq \alpha d(x_n, x_{n+1}, t) + \beta\left(\frac{d(x_n, x_{n+1}, t).\,d(x_{n+1}, x_{n+2}, t)}{d(x_n, x_{n+1}, t)}\right) + \gamma\left(\frac{d(x_n, x_{n+1}, t).\,d(x_n, x_{n+2}, t)}{d(x_n, x_{n+1}, t) + d(x_{n+1}, x_{n+2}, t)}\right)$$

$$d(x_{n+1}, x_{n+2}, t) \leq \alpha d(x_n, x_{n+1}, t) + \beta d(x_{n+1}, x_{n+2}, t) + \gamma\left(\frac{d(x_n, x_{n+1}, t).\,s\,[d(x_n, x_{n+1}, t) + d(x_{n+1}, x_{n+2}, t)]}{d(x_n, x_{n+1}, t) + d(x_{n+1}, x_{n+2}, t)}\right)$$

$$d(x_{n+1}, x_{n+2}, t) \leq \alpha d(x_n, x_{n+1}, t) + \beta d(x_{n+1}, x_{n+2}, t) + \gamma . s\; d(x_n, x_{n+1}, t)$$

$$(1 - \beta) d(x_{n+1}, x_{n+2}, t) \leq (\alpha + \gamma . s) d(x_n, x_{n+1}, t)$$

$$d(x_{n+1}, x_{n+2}, t) \leq \frac{(\alpha + \gamma . s)}{(1 - \beta)} d(x_n, x_{n+1}, t) s$$

$$d(x_{n+1}, x_{n+2}, t) \leq k\; d(x_n, x_{n+1}, t)$$

$\forall\, t \in [0,1)$ and $k = \frac{(\alpha+\gamma.s)}{(1-\beta)} < \frac{1}{s} \Rightarrow s\alpha + \beta + \gamma < 1$.

Therefore by successive iterations

$$d(x_{n+1}, x_{n+2}, t) \leq k^n\; d(x_0, x_1, t)$$

As we know if$\{x_n\}_{n\to\infty}$be a sequence in parametric b metric space(X, d)such that $d(x_{n+1}, x_{n+2}, t) \leq k\, d(x_n, x_{n+1}, t) \forall\, t \in [0,1) \&\ n = 1,2,3,\ldots$ then $\{x_n\}_{n\to\infty}$ is a Cauchy sequence inparametric b metric space (X,d). Since (X,d) is a complete parametric b metric space; therefore $\{x_n\}_{n\to\infty}$ converses. Let $x^* \in X$, then $lim_{n\to\infty} x_n \to x^*$. Again T_{pb} is continuous, therefore

$$T_{pb}x^* = T_{pb}(lim_{n\to\infty} x_n) = lim_{n\to\infty} T_{pb}x_n = x^* \Rightarrow T_{pb}x^* = x^*$$

Implies T_{pb} has a fixed point $T_{pb}x^* = x^*$ in X.

Now we will show that x^* is unique. For that; suppose y^* is another fixed point therefore $T_{pb}y^* = y^*$. Therefore by inequality (3.1) we have

$$d(T_{pb}x^*, T_{pb}y^*, t) \leq \alpha\, d(x^*, y^*, t)$$

$$+\beta\left(\frac{d(x^*T_{pb}, x^*, t).d(y^*, T_{pb}y^*, t)}{d(x^*, y^*, t)}\right) + \gamma\left(\frac{d(x^*, T_{pb}x^*, t).d(x^*, T_{pb}y^*, t)}{d(x^*, y^*, t) + d(y^*, T_{pb}y^*, t)}\right)$$

$$d(x^*, y^*, t) \leq \alpha\, d(x^*, y^*, t) + \beta\left(\frac{d(x^*, x^*, t).d(y^*, y^*, t)}{d(x^*, y^*, t)}\right) + \gamma\left(\frac{d(x^*, x^*, t).d(x^*, y^*, t)}{d(x^*, y^*, t) + d(y^*, y^*, t)}\right)$$

$$d(x^*, y^*, t) \leq \alpha\, d(x^*, y^*, t)$$
$$d(x^*, y^*, t) \leq \alpha\, d(x^*, y^*, t)$$
$$\Rightarrow (1-\alpha)d(x^*, y^*, t) \leq 0$$

$\Rightarrow d(x^*, y^*, t) = 0$ Since $\alpha < 1 \Rightarrow x^* = y^*$. Hence T_p has a unique point.

4. Theorems

Let (X, T_{pb}) be a complete parametric b metric space and $T_{pb}: X \to X$ be a orbitally continuous self-map satisfying the condition

(4.1) $$d(T_{pb}x, T_{pb}y, t) \leq \alpha\, Max\{d(x,y,t), \frac{d(x, T_{pb}x, t)d(y, T_{pb}y, t)}{d(x,y,t)}, \frac{d(x, T_{pb}y, t)d(y, T_{pb}x, t)}{d(x,y,t)}, \frac{d(x, T_{pb}x, t)d(x, T_{pb}y, t)}{2.s.d(x,y,t)}\}$$

$\forall\, t \in [0,1)$; $\alpha > 0$; $x, y \in X \ \&\ x \neq y$ and $\alpha \in [0,1]$, then T_{pb} has a unique fixed point.

Proof: Let $x_0 \in X$, Define iterative sequence $\{x_n\}_{n=1}^{\infty}$ follows: $T_p x_n = x_{n+1}$ for $n = 1,2,3,\ldots$. If for some n, $T_p x_n = x_n$, then x_n is the fixed point. Asserting $T_p x_n \neq x_n$, using in equality (4.1)

$$d(T_{pb}x_n, T_{pb}x_{n+1}, t) = d(x_{n+1}, x_{n+2}, t)$$

$$\leq \alpha\, Max\{d(x_n, x_{n+1}, t), \frac{d(x_n, T_{pb}\,x_n, t).d(x_{n+1}, T_{pb}x_{n+1}, t)}{d(x_n, x_{n+1}, t)}, \frac{d(x_n, T_{pb}x_{n+1}, t).d(x_{n+1}, T_{pb}x_n, t)}{d(x_n, x_{n+1}, t)}, \frac{d(x_n, T_{pb}x_n, t).d(x_n, T_{pb}x_{n+1}, t)}{2.s.d(x_n, x_{n+1}, t)}\}$$

$$\leq \alpha\, Max\{d(x_n, x_{n+1}, t), \frac{d(x_n, x_{n+1}, t).d(x_{n+1}, x_{n+2}, t)}{d(x_n, x_{n+1}, t)}, \frac{d(x_n, x_{n+2}, t).d(x_{n+1}, x_{n+1}, t)}{d(x_n, x_{n+1}, t)}, \frac{d(x_n, x_{n+1}, t).d(x_n, x_{n+2}, t)}{2d(x_n, x_{n+1}, t)}\}$$

$$\leq \alpha\, Max\{d(x_n, x_{n+1}, t), d(x_{n+1}, x_{n+2}, t).0, \frac{s\{d(x_n, x_{n+1}, t) + d(x_{n+1}, x_{n+2}, t)\}}{2.s.d(x_n, x_{n+1}, t)}\}$$

$$\leq \alpha\, Max\{d(x_n, x_{n+1}, t), d(x_{n+1}, x_{n+2}, t).0, \frac{\{d(x_n, x_{n+1}, t) + d(x_{n+1}, x_{n+2}, t)\}}{2.d(x_n, x_{n+1}, t)}\}$$

$$d(x_{n+1}, x_{n+2}, t) \leq \alpha\, Max\{d(x_n, x_{n+1}, t), d(x_{n+1}, x_{n+2}, t)\}$$

$$\Rightarrow d(x_{n+1}, x_{n+2}, t) \leq \alpha\, d(x_n, x_{n+1}, t)$$

Therefore by successive iteration

$$d(x_{n+1}, x_{n+2}, t) \leq \alpha^n\, d(x_0, x_1, t)$$
$$d(x_{n+1}, x_{n+2}, t) \leq \alpha^n\, d(x_0, x_1, t)$$

As we know if $\{x_n\}_{n\to\infty}$ be a sequence in parametric space (X,d) such that $d(x_{n+1}, x_{n+2}, t) \leq \alpha^n\, d(x_0, x_1, t) \forall\, t \in [0,1)\ \&\ n = 1,2,3,\ldots$ then $\{x_n\}_{n\to\infty}$ is a Cauchy sequence in (X,d). Since (X,d) is a complete parametric space; $\{x_n\}_{n\to\infty}$ converses. Let $x^* \in X$, then $lim_{n\to\infty} x_n \to x^*$. Again T_{pb} is continuous, therefore

$$T_p x^* = T_p(lim_{n\to\infty} x_n) = lim_{n\to\infty} T_p x_n = x^* \Rightarrow T_p x^* = x^*$$

Implies T_p has a fixed point $T_p x^* = x^*$ in X.

Now we will show that x^* is unique. for that suppose y^* is another fixed point therefore $T_p y^* = y^*$. Therefore by inequality (4.1) we have

$$d(T_p x^*, T_p y^*, t) \leq \alpha Max\{d(x^*, y^*, t), \frac{d(x^*, T_p x^*, t) d(y^*, T_p y^*, t)}{d(x^*, y^*, t)}, \frac{d(x^*, T_p y^*, t) d(y^*, T_p x^*, t)}{d(x^*, y^*, t)}, \frac{d(x^*, T_p x^*, t) d(x^*, T_p y^*, t)}{2d(x^*, y^*, t)}\}$$

$$d(T_p x^*, T_p y^*, t) \leq \alpha Max\{d(x^*, y^*, t), \frac{d(x^*, x^*, t) d(y^*, y^*, t)}{d(x^*, y^*, t)}, \frac{d(x^*, y^*, t) d(y^*, x^*, t)}{d(x^*, y^*, t)}, \frac{d(x^*, x^*, t) d(x^*, y^*, t)}{2d(x^*, y^*, t)}\}$$

$$d(T_p x^*, T_p y^*, t) \leq \alpha Max\{d(x^*, y^*, t), 0, d(x^*, y^*, t), 0\}$$

$$d(x^*, y^*, t) \leq \alpha d(x^*, y^*, t)$$

$$\Rightarrow (1 - \alpha) d(x^*, y^*, t) \leq 0$$

$\Rightarrow d(x^*, y^*, t) = 0$since $\alpha > 1 \Rightarrow x^* = y^*$. Hence T_{pb} has a unique point.

References

[1] Banach, S. (1922). Sur les operations dans les ensembles abstraitsetleur application aux equations intégrales. Fund. math, 3(1), 133-181.

[2] Kannan, R. (1968). Some results on fixed points. Bull. Cal. Math. Soc., 60, 71-76.

[3] Chatterjea, S.K. (1972). Fixed-point theorems. Dokladina Bolgarskata Akademiyana Naukite, 25(6), 727-730.

[4] S. Z. Wang, B. Y. Li, Z. M. Gao and K. Iseki, Some fixed point theorems for expansion mappings, Math. Japonica., 29(1984), 631-636.

[5] P. Z. Daffer and H.Kaneko, On expansive mappings, Math. Japonica., 37(1992), 733-735.

[6] S. Czerwik, Contraction mappings in b-metric spaces, Acta Math. Inf. Univ. Ostravensis, 1(1993), 5-11.

[7] S .Czerwik, Nonlinear set-valued contraction mappings in b-metric spaces, Atti Sem. Mat. Fis. Univ. Modena, 46(1998), 263-276.

[8] M. A. Alghamdi, N.Hussain and P. Salimi, Fixed point and coupled fixed point theorems on b-metric-like spaces, J. Inequal. Appl., 2013(2013).

[9] L. A. Zadeh, Fuzzy sets, Information and Computation, 8(1965), 338353.

[10] I. Kramosil and J. Michalek, Fuzzy metrics and statistical metric spaces, Kybernetika, 11(5) (1975), 336344.

[11] M. Grabiec, Fixed points in fuzzy metric spaces, Fuzzy Sets and Systems, 27(3) (1988), 385389. A. George and P. Veeramani, On some results in fuzzy metric spaces, Fuzzy Sets and Systems, 64(3), (1994), 395399.

[12] C. Di Bari and C. Vetro, Fixed points, attractors and weak fuzzy contractive mappings in a fuzzy metric space, Journal of Fuzzy Mathematics, 13(4) (2005), 973982.

[13] J. H. Park, Intuitionistic fuzzy metric spaces, Chaos, Solitons and Fractals, 22(5), (2004), 10391046

[14] N. Hussain, S. Khaleghizadeh, P. Salimi and Afrah A. N. Abdou, A New Approach to Fixed Point Results in Triangular Intuitionistic Fuzzy Metric Spaces, Abstract and Applied Analysis, 2014(2014), Article ID 690139.

[15] N. Hussain, P. Salimi and V. Parvaneh, Fixed point results for various contractions in parametric and fuzzy b-metric spaces, J. Nonlinear Sci. Appl., 8(2015), 719-739