Some More Results of Generalized Parametric R-Norm Entropy in Quantum Logic

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Abstract: ***In the present communication, I deal with the modelling of Generalized Parametric R-Norm Information measure in quantum logic. This information measure is extended to Joint and Conditional measure in quantum logic.***

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1. Introduction

Birkhoffand Neumann introduce the approach of quantum logic. With the help of information measure, the uncertainty in random events can be measured. The concept of information measure has been applied in many areas such as in physics, information theory, computer science, general systems theory, sociology, statistics, biology, chemistry and many other fields. There are several information measure presented in history such as Shannon's information measure (C. E. Shannon, 1948) and the measure of order α and of type β, introduced by Renyi (Alfred Renyi, 1961), Havrda-Charvat (Havrda & Charvát, 1967) and Daroczy (Daróczy, 1970) respectively. There are many applications of them in Statistics, Pattern recognition and Coding theory.

We define $\triangle_p$ as the set of all n-ary probability distributions

$$\Delta_p = \left\{P = (t_1, t_2, t_3, \dots, t_p) | t_i \geq 0, i = 1,2, \dots p; \sum_{i=1}^{p} t_i = 1\right\}$$

Information measure given by Shannon (C. E. Shannon, 1948) is

$$H(P) = -\sum_{i=1}^{p} t_i \log t_i \tag{1}$$

where $D > 0$.

The RIM of the distribution P is given by Boekke and Lubbe (Boekee & Van der Lubbe, 1980) and is defined for $R \in \Re$ by

$$H_R(P) = \frac{R}{R-1}\left[1 - \left[\sum_{i=1}^{p} t_i^R\right]^{\frac{1}{R}}\right] \tag{2}$$

Where $\Re = \{R: R > 0, R \neq 1\}$ and the RIM is a real function from Δ_n to $\Re$ defined on Δ_n, ($n \geq 2$) R is real numbers set (see (Boekee & Van der Lubbe, 1980)). When $R \to 1$, RIM approaches to Shannon's information measure and when $R \to \infty, H_R(P) \to (1 - \max t_i); i = 1,2, \dots, n.$ Satish Kumar (Kumar, 2009) studied Parametric R-Norm Information Measure which is given by

$$H_R^\delta = \frac{R}{R-\delta}\left[1 - \left(\sum_{i=1}^{p} t_i^{\frac{R}{\delta}}\right)^{\frac{\delta}{R}}\right]; \delta \neq R, R \in \Re. \tag{3}$$

Information measure given in eq. (2) can be generalized parametrically in many other ways. Satish Kumar (Kumar & Choudhary, 2012) consider generalized parametric R-Norm Information measure:

$$H_R^m = \frac{R-m+1}{R-m}\left[1 - \left(\sum_{i=1}^{p} t_i^{R-m+1}\right)^{\frac{1}{R-m+1}}\right]; m \neq R, R, m \in \Re. \tag{4}$$

In (Zadeh (*Zadeh L.a.-Fuzzy Sets (1965) | PDF*, n.d.)), (Khare(Khare & Roy, 2007)) and (Zhao (Zhao & Ma, 2007)) the concepts of information measure, joint information measure and conditional information measure of partitions and dynamical systems have been studied in quantum logic. We can study quantum information theory in an easy way by using the concepts of information measure in quantum logic (see (*2005 — the Zhang Lab*, n.d.)). In this paper, we study the notion of Generalized Parametric R-Norm Information Measure, joint Generalized Parametric R-Norm Information Measure and conditional Generalized Parametric R-Norm Information Measure in quantum logic. Some of the results of the information measure are generalized to quantum logic.

2. Some Definitions

Definition 1. (Khare & Roy, 2007). *Consider a quantum logic* QL. *This is a σ-orthomodular lattice, i.e., a lattice $\Omega(\Omega, R, \vee, \wedge, 0, 1)$ where 0 is the minimal element and 1 is the maximal element. We can define an operation ′: $\Omega \to \Omega$ such that the following properties hold $\forall u, v \in \Omega$:*

(i) $(u')' = u.$

(ii) $uRv \Rightarrow v'Ru'.$

(iii) Given any finite sequence $< u_i >, u_i R u_j'$, then $V_{i \in \mathbf{N}} u_i$ lies in Ω.

(iv) Ω is orthomodular: $uRv \Rightarrow v = u \vee (v \wedge u').$

In quantum logic we have $u \wedge u' = 0$ and $u \vee u' = 1$.

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Definition 2.(Khare & Roy, 2007). *Consider a quantum logic Ω. A state ζ is a mapping $\Omega \to [0,1]$ such that:*
(i) $\zeta(1) = 1$.
(ii) $\zeta(u \vee v) = \zeta(u) + \zeta(v)$ *if* $u \perp v$.
It may be observed that $\zeta(0) = 0, \zeta$ *is monotone and* $\zeta(u') = 1 - \zeta(u), u \in \Omega$.

Definition 3.(Khare & Roy, 2007). *Consider* $u_1, \dots, u_n$ *are the finite elements of a quantum logic.* $P = \{u_1, \dots, u_n\}$ *is* $\vee$-*orthogonal iff* $\vee_{i=1}^{k} u_i \perp u_{k+1}, \forall k = 1,2, \dots, n-1$.

Definition 4.(Khare & Roy, 2007). *Consider Ω is a quantum logic.* $P = \{u_1, \dots, u_n\} \subset \Omega$ *is called a partition of Ω w.r.t. a state ζ on Ω if:*
(i) P *is* $\vee$-*orthogonal.*
(ii) $\zeta(\vee_{i=1}^{n} u_i) = 1$.

Remarks 1. *From the above definitions, we can find that* $\sum_{i=1}^{p} \zeta(u_i) = 1$.

Definition 5.(Zarenezhad & Ebrahimzadeh, 2021). *Let* (Ω, ζ) *be any couple in quantum logic and* $P = \{u_1, u_2, \dots, u_p\}$ *and* $Q = \{v_1, v_2, \dots, v_q\}$ *be any two partitions of it. Q is said to be a ζ-refinement of P if $\exists$ a partition $I(1), \dots, I(p)$ of the set $\{1, \dots, q\}$ such that* $u_i = \vee_{j \in I(i)} v_j, \forall i = 1, \dots, p$.

Definition 6.(Khare & Roy, 2007). *Consider a partition* $Q = \{v_1, \dots, v_q\}$ *of a couple* (Ω, ζ) *and* $u \in \Omega$. *The state ζ satisfy Bayes' property if*

$$\zeta\left(\bigvee_{j=1}^{q} (u \wedge v_j)\right) = \zeta(u).$$

In this case, we get

$$\sum_{j=1}^{q} \zeta(u \wedge v_j) = \zeta(u).$$

Let us consider $P = \{u_1, u_2, \dots, u_p\}$ *and* $Q = \{v_1, v_2 \dots, v_q\}$ *are the two partitions of* (Ω, ζ). *Now,* $P \vee Q$ *is a partition of* (Ω, ζ) *if ζ has Bayes's property. Then the refinement of this partition is given by*

$$P \vee Q = \{u_i \wedge v_j \mid u_i \in P, v_j \in Q\}.$$

Definition 7. *Two partitions P and Q are said to be ζ-independent if*

$$\zeta(u \wedge v) = \zeta(u)\zeta(v) \text{ for } u \in P \text{ and } v \in Q.$$

Remarks 2.(Beckenbach & Bellman, 2012). *The well-known Minkowski inequality is given by*

Case1. *If* $\tau > 1$,

$$\left(\sum_{i=1}^{k} x_i^{\tau}\right)^{\frac{1}{\tau}} + \left(\sum_{i=1}^{k} y_i^{\tau}\right)^{\frac{1}{\tau}} \geq \left(\sum_{i=1}^{k} (x_i + y_i)^{\tau}\right)^{\frac{1}{\tau}}$$

Case2. *If* $0 < \tau < 1$,

$$\left(\sum_{i=1}^{k} x_i^{\tau}\right)^{\frac{1}{\tau}} + \left(\sum_{i=1}^{k} y_i^{\tau}\right)^{\frac{1}{\tau}} \leq \left(\sum_{i=1}^{k} (x_i + y_i)^{\tau}\right)^{\frac{1}{\tau}}$$

where x_i, y_j *are non negative numbers.*

Remarks 3.(Beckenbach & Bellman, 2012). *Also, the well-known Jensen's inequality is given by*

Case1. *If ϕ is a real convex function,*

$$\phi\left(\sum_{i=1}^{p} u_i x_i\right) \leq \sum_{i=1}^{p} u_i \phi(x_i)$$

Case2. *If ϕ is a real concave function,*

$$\phi\left(\sum_{i=1}^{p} u_i x_i\right) \geq \sum_{i=1}^{p} u_i \phi(x_i)$$

where x_i's are real numbers and u_i's are non negative real numbers with condition $\sum_{i=1}^{p} u_i = 1$.

Generalized Parametric R-Norm Information Measure in Quantum Logic

In this section, we define Generalized Parametric R-Norm Information Measure for a finite partition on a quantum logic. Some of the properties of suggested measure are proved on Quantum logic.

Definition 8. *Let us consider a partition* $P = \{u_1, u_2, \dots, u_p\}$ *of a couple* (Ω, ζ). *We can define the Generalized Parametric R-Norm Information Measure w.r.t. state ζ as:*

$$^{\zeta}H_R^m(P) = \frac{R - m + 1}{R - m}\left(1 - \left(\sum_{i=1}^{p} \left(\zeta(u_i)\right)^{R-m+1}\right)^{\frac{1}{R-m+1}}\right) \tag{5}$$

for $R \neq m$ *and* $R, m > 0 (\neq 1)$.

Theorem 1. *For a partition* $P = \{u_1, u_2, \dots, u_p\}$ *of a couple* (Ω, ζ), $^{\zeta}H_R^m(P) \geq 0$.

Proof. Case1. If $0 < R < m$ or $R - m + 1 < 1$, then we have

$$\left(\zeta(u_i)\right)^{R-m+1} \geq \zeta(u_i), i = 1,2, \dots, p.$$

therefore,

$$\sum_{i=1}^{p} \left(\zeta(u_i)\right)^{R-m+1} \geq \sum_{i=1}^{p} \zeta(u_i) = 1$$

it follows that

$$\left(\sum_{i=1}^{p} \left(\zeta(u_i)\right)^{R-m+1}\right)^{\frac{1}{R-m+1}} \geq 1$$

Also, for $0 < R < m$, we have $\frac{R-m+1}{R-m} < 0$. So, from eq. (4) we have $^{\zeta}H_R^m(P) \geq 0$.

Case2. if $R > m$ or $R - m + 1 > 1$, then we have

$$\left(\zeta(u_i)\right)^{R-m+1} \leq \zeta(u_i), i = 1,2, \dots, p.$$

therefore,

$$\sum_{i=1}^{p} \left(\zeta(u_i)\right)^{R-m+1} \leq \sum_{i=1}^{p} \zeta(u_i) = 1$$

it follows that

$$\left(\sum_{i=1}^{p} \left(\zeta(u_i)\right)^{R-m+1}\right)^{\frac{1}{R-m+1}} \leq 1$$

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Also, for $R > m$, we have $\frac{R-m+1}{R-m} > 0$. So, from eq. (4), we have ${}^{\zeta}H_R^m(P) \geq 0$.

Theorem 2. *Let us consider a family of all the states denoted by K, on quantum logic. Let* $\zeta,\eta \in K$ *and P is a partition of* (Ω,ζ), (Ω,η). *Then for* $\lambda \in [0,1]$, *we have*

$$\lambda^{\zeta}H_R^m(P) + (1-\lambda)^{\eta}H_R^m(P) \leq {}^{\lambda\zeta+(1-\lambda)\eta}H_R^m(P).$$

Proof. Let $P = \{u_1, u_2, \dots, u_p\}$ be a partition. In the Minkowski inequality put $x_i = \lambda\zeta(u_i)$ and $y_i = (1-\lambda)\eta(u_i)$, we have

For $R > m$,

$$\lambda\left(\sum_{i=1}^{p}(\zeta(u_i))^{R-m+1}\right)^{\frac{1}{R-m+1}} + (1-\lambda)\left(\sum_{i=1}^{p}(\eta(u_i))^{R-m+1}\right)^{\frac{1}{R-m+1}} \geq \left(\sum_{i=1}^{p}(\lambda\zeta(u_i)+(1-\lambda)\eta(u_i))^{R-m+1}\right)^{\frac{1}{R-m+1}},$$

and for $0 < R < m$,

$$\lambda\left(\sum_{i=1}^{p}(\zeta(u_i))^{R-m+1}\right)^{\frac{1}{R-m+1}} + (1-\lambda)\left(\sum_{i=1}^{p}(\eta(u_i))^{R-m+1}\right)^{\frac{1}{R-m+1}} \leq \left(\sum_{i=1}^{p}(\lambda\zeta(u_i)+(1-\lambda)\eta(u_i))^{R-m+1}\right)^{\frac{1}{R-m+1}}.$$

Since, $\frac{R-m+1}{R-m} > 0$ for $R > m$ and $\frac{R-m+1}{R-m} < 0$ for $0 < R < m$, therefore we get

$$\lambda^{\zeta}H_R^m(P) + (1-\lambda)^{\eta}H_R^m(P) \leq {}^{\lambda\zeta+(1-\lambda)\eta}H_R^m(P).$$

Theorem 3. *Let us consider a couple* (Ω,ζ). *Let* $P = \{u_1, u_2, \dots, u_p\}$ *and* $Q = \{v_1, v_2, \dots, v_q\}$ *are two partitions of it and Q is a* ζ-*refinment of P. Then,*

$${}^{\zeta}H_R^m(P) \leq {}^{\zeta}H_R^m(Q).$$

Proof. Since, Q is a ζ-refinment of P, therefore ∃ a partition $I_1, I_2, \dots, I_p$ of the set $\{1,2,\dots,q\}$ s.t. $u_i = \vee_{j\in I_i} v_j$ for every $i \in \{1,2,\dots,m\}$. From Definition **2**, we have $\zeta(u_i) = \sum_{j\in I_i}\zeta(v_j)$.

Case1. Now, for $R > m$,

$$(\zeta(u_i))^{R-m+1} = \left(\sum_{j\in I_i}\zeta(v_j)\right)^{R-m+1} \geq \sum_{j\in I_i}(\zeta(v_j))^{R-m+1} \tag{6}$$

Since, the partition of the set $\{1,2,\dots,q\}$ is $I_1, I_2, \dots, I_p$, therefore, we have

$$\cup_{i=1}^{p} I_i = \cup_{j=1}^{q}\{j\},$$

and

$$I_u \cap I_v = \phi; \forall\, u,v \in \{1,2,\dots,p\}.$$

So,

$$\sum_{i=1}^{p}\sum_{j\in I_i}(\zeta(v_j))^{R-m+1} = \sum_{j=1}^{q}(\zeta(v_j))^{R-m+1} \tag{7}$$

Taking summation on both sides of eq. (6) w.r.t. i varies from 1 to p and using eq. (7), we get

$$\sum_{i=1}^{p}(\zeta(u_i))^{R-m+1} \geq \sum_{i=1}^{p}\sum_{j\in I_i}(\zeta(v_j))^{R-m+1} = \sum_{j=1}^{q}(\zeta(v_j))^{R-m+1} \tag{8}$$

Thus, we have

$$\sum_{i=1}^{p}(\zeta(u_i))^{R-m+1} \geq \sum_{j=1}^{q}(\zeta(v_j))^{R-m+1} \tag{9}$$

Raising power to $\frac{1}{R-m+1}$ on both sides, we have

$$\left(\sum_{i=1}^{p}(\zeta(u_i))^{R} - m + 1\right)^{\frac{1}{R-m+1}} \geq \left(\sum_{j=1}^{q}(\zeta(v_j))^{R} - m + 1\right)^{\frac{1}{R-m+1}} \tag{10}$$

Now, $\frac{R-m+1}{R-m} > 0$ (for $R > m$), so we have

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$$\frac{R-m+1}{R-m}\left(1-\left(\sum_{i=1}^{p}(\zeta(u_i))^{R-m+1}\right)^{\frac{1}{R-m+1}}\right) \leq \frac{R-m+1}{R-m}\left(1-\left(\sum_{j=1}^{q}\left(\zeta(v_j)\right)^{R-m+1}\right)^{\frac{1}{R-m+1}}\right)$$

i.e.

$$^{\zeta}H_R^m(P) \leq {}^{\zeta}H_R^m(Q). \tag{11}$$

Case2. Now, for $0 < R < m$,

$$(\zeta(u_i))^{R-m+1} = \left(\sum_{j\in I_i}\zeta(v_j)\right)^{R-m+1} \leq \sum_{j\in I_i}\left(\zeta(v_j)\right)^{R-m+1} \tag{12}$$

Continuing as in above case, we get

$$\sum_{i=1}^{p}(\zeta(u_i))^{R}-m+1 \leq \sum_{j=1}^{q}\left(\zeta(v_j)\right)^{R}-m+1$$

Using $\frac{R-m+1}{R-m} < 0$ (for $0 < R < m$) and eq.(4), we get the same result as in eq. (11).

Theorem 4. *Let $P = \{u_1, u_2, \dots, u_p\}$ and $Q = \{v_1, v_2, \dots, v_q\}$ are two partitions of a couple (Ω, ζ) satisfying Bayes' property, then*

$$\max\{^{\zeta}H_R^m(P), {}^{\zeta}H_R^m(Q)\} \leq {}^{\zeta}H_R^m(P \vee Q). \tag{13}$$

Proof. As we know that (P ∨ Q) is ζ-refinement of both P and Q Therefore, by using result of Theorem **3**, we have

$$^{\zeta}H_R^m(P) \leq {}^{\zeta}H_R^m(P \vee Q),$$

and

$$^{\zeta}H_R^m(Q) \leq {}^{\zeta}H_R^m(P \vee Q).$$

Hence, the result.

Theorem 5. *Let $P = \{u_1, u_2, \dots, u_p\}$ and $Q = \{v_1, v_2, \dots, v_q\}$ be partitions of a couple (Ω, ζ) having Bayes' property, then the Generalized Parametric R-Norm Information Measure with respect to state ζ is sub-additive.*
i.e.

$$^{\zeta}H_R^m(P \vee Q) = {}^{\zeta}H_R^m(P) + {}^{\zeta}H_R^m(Q) - \frac{R-m}{R-m+1}{}^{\zeta}H_R^m(P)^{\zeta}H_R^m(Q). \tag{14}$$

where P and Q are ζ-independent partitions.

Proof. Consider

$$^{\zeta}H_R^m(P) + {}^{\zeta}H_R^m(Q) - \frac{R-m}{R-m+1}{}^{\zeta}H_R^m(P)^{\zeta}H_R^m(Q)$$

$$= \frac{R-m+1}{R-m}\left(1-\left(\sum_{i=1}^{p}(\zeta(u_i))^{R-m+1}\right)^{\frac{1}{R-m+1}}\right) + \frac{R-m+1}{R-m}\left(1-\left(\sum_{j=1}^{q}\left(\zeta(v_j)\right)^{R-m+1}\right)^{\frac{1}{R-m+1}}\right)$$

$$-\frac{R-m}{R-m+1}\left(\frac{R-m+1}{R-m}\right)^2\left(1-\left(\sum_{i=1}^{p}(\zeta(u_i))^{R-m+1}\right)^{\frac{1}{R-m+1}}\right)\left(1-\left(\sum_{j=1}^{q}\left(\zeta(v_j)\right)^{R-m+1}\right)^{\frac{1}{R-m+1}}\right)$$

$$= \frac{R-m+1}{R-m}\left(1-\left(\sum_{i=1}^{p}(\zeta(u_i))^{R-m+1}\right)^{\frac{1}{R-m+1}}\left(\sum_{j=1}^{q}\left(\zeta(v_j)\right)^{R-m+1}\right)^{\frac{1}{R-m+1}}\right)$$

$$= \frac{R-m+1}{R-m}\left(1-\left(\sum_{i=1}^{p}\sum_{j=1}^{q}\left(\zeta(u_i)\zeta(v_j)\right)^{R-m+1}\right)^{\frac{1}{R-m+1}}\right)$$

$$= \frac{R-m+1}{R-m}\left(1-\left(\sum_{i=1}^{p}\sum_{j=1}^{q}\left(\zeta(u_i \wedge v_j)\right)^{R-m+1}\right)^{\frac{1}{R-m+1}}\right)$$

$$= {}^{\zeta}H_R^m(P \vee Q)$$

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Hence, the result.

Joint and Conditional Generalized Parametric R-Norm Information Measure in Quantum Logic

Let us assume that Ω is a quantum logic. Let $P = \{u_1, u_2, \dots, u_p\}$ and $Q = \{v_1, v_2, \dots, v_q\}$ are two partitions of Ω corresponding to a state ζ on Ω. Let us consider a set π given by

$$\pi = \{\pi_{11}, \pi_{12}, \dots, \pi_{pq}\},$$

where $\pi = x_{ij} y_j = y_{ji} x_i$; $\forall\, i = 1,2, \dots, p; j = 1,2, \dots, q.$

$$x_{ij} = \frac{\zeta(u_i \wedge v_j)}{\zeta(v_j)}, \quad y_{ji} = \frac{\zeta(v_j \wedge u_i)}{\zeta(u_i)}, \qquad \forall\, i = 1,2, \dots, p; j = 1,2, \dots, q. \tag{15}$$

and

$$x_i = \sum_{j=1}^{m} \pi_{ij}\ , \quad y_i = \sum_{i=1}^{n} \pi_{ij}\ , \quad \forall\, i = 1,2, \dots, p; j = 1,2, \dots, q. \tag{16}$$

Definition 9. *The Joint Generalized Parametric R-Norm Information Measure of partition P and Q w.r.t. state ζ is defined by:*

$$^{\zeta}H_R^m(P,Q) = \frac{R-m+1}{R-m}\left(1 - \left(\sum_{i=1}^{p}\sum_{j=1}^{q}(\pi_{ij})^{R-m+1}\right)^{\frac{1}{R-m+1}}\right) \tag{17}$$

Definition 10. *The Conditional Generalized Parametric R-Norm Information Measure of P given Q w.r.t. state ζ is defined by:*

$$'^{\zeta}H_R^m(Q|P) = \frac{R-m+1}{R-m}\left[1 - \sum_{i=1}^{p}\zeta(u_i)\left(\sum_{j=1}^{q}\left(\frac{\zeta(v_j \wedge u_i)}{\zeta(u_i)}\right)^{R-m+1}\right)^{\frac{1}{R-m+1}}\right] \tag{18}$$

and

$$''^{\zeta}H_R^m(Q|P) = \frac{R-m+1}{R-m}\left[1 - \left(\sum_{i=1}^{p}\zeta(u_i)\sum_{j=1}^{q}\left(\frac{\zeta(v_j \wedge u_i)}{\zeta(u_i)}\right)^{R-m+1}\right)^{\frac{1}{R-m+1}}\right] \tag{19}$$

for $R > 0 (R \neq m)$.

Theorem 6. *Let Ω is a quantum logic w.r.t. a state ζ. If P and Q are two partitions of it satisfying Bayes' property. Then the following results hold:*

(i). $'^{\zeta}H_R^m(Q|P) \leq {}^{\zeta}H_R^m(Q)$,

(ii). $''^{\zeta}H_R^m(Q|P) \leq {}^{\zeta}H_R^m(Q)$,

(iii). $''^{\zeta}H_R^m(Q|P) \leq {}'^{\zeta}H_R^m(Q|P)$.

Proof. (i) **Case1.** For $R > m$,

Using Jensen's inequality (Beckenbach & Bellman, 2012) for $R - m + 1 > 1$, we get

$$\left[\sum_{j=1}^{q}\left(\sum_{i=1}^{p}\pi_{ij}\right)^{R-m+1}\right]^{\frac{1}{R-m+1}} \leq \left[\sum_{i=1}^{p}\left(\sum_{j=1}^{q}(\pi_{ij})^{R-m+1}\right)^{\frac{1}{R-m+1}}\right]$$

Using eq. (16) on LHS and $\pi_{ij} = y_{ji} x_i$ on RHS, we have

$$\left[\sum_{j=1}^{q}(y_j)^{R-m+1}\right]^{\frac{1}{R-m+1}} \leq \left[\sum_{i=1}^{p}\left(\sum_{j=1}^{q}(y_{ji} x_i)^{R-m+1}\right)^{\frac{1}{R-m+1}}\right]$$

i.e.

$$1 - \left[\sum_{i=1}^{p}\left(\sum_{j=1}^{q}(y_{ji} x_i)^{R-m+1}\right)^{\frac{1}{R-m+1}}\right] \leq 1 - \left[\sum_{j=1}^{q}(y_j)^{R-m+1}\right]^{\frac{1}{R-m+1}}$$

i.e.

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$$1-\left[\sum_{i=1}^{p} x_i\left(\sum_{j=1}^{q}(y_{ji})^{R-m+1}\right)^{\frac{1}{R-m+1}}\right] \leq 1-\left[\sum_{j=1}^{q}(y_j)^{R-m+1}\right]^{\frac{1}{R-m+1}}$$

Putting the values from eq.(15), we get

$$1-\sum_{i=1}^{p}\zeta(u_i)\left(\sum_{j=1}^{q}\left(\frac{\zeta(v_j\wedge u_i)}{\zeta(u_i)}\right)^{R-m+1}\right)^{\frac{1}{R-m+1}} \leq 1-\left(\sum_{j=1}^{q}\left(\zeta(v_j)\right)^{R-m+1}\right)^{\frac{1}{R-m+1}}$$

Using $\frac{R-m+1}{R-m} > 0$ (for $R > m$), we have

$$'^{\zeta}H_R^m(Q|P) \leq {}^{\zeta}H_R^m(Q)$$

Case2. On the same line we can prove the same result for $0 < R < m$.
Also, the equality sign holds iff $\pi_{ij} = x_i y_j$.

Proof. (i) **Case1.** For $R > m$,
Using Jensen's inequality (Beckenbach & Bellman, 2012), we have

$$\sum_{i=1}^{p} x_i\, y_{ji}^{R-m+1} \geq \left[\sum_{i=1}^{p} x_i\, y_{ji}\right]^{R-m+1}$$

use eq. (15) and(16), we get

$$\sum_{i=1}^{p}\zeta(u_i)\left(\frac{\zeta(v_j\wedge u_i)}{\zeta(u_i)}\right)^{R-m+1} \geq \left(\sum_{i=1}^{p}\zeta(u_i)\left(\frac{\zeta(v_j\wedge u_i)}{\zeta(u_i)}\right)\right)^{R-m+1} = \zeta(v_j)^{R-m+1}$$

i.e.

$$\sum_{i=1}^{p}\zeta(u_i)\left(\frac{\zeta(v_j\wedge u_i)}{\zeta(u_i)}\right)^{R-m+1} \geq \zeta(v_j)^{R-m+1}$$

Take summation w.r.t j varying from 1 to q and then raising power to $\frac{1}{R-m+1}$ on both sides, we have

$$\left(\sum_{i=1}^{p}\zeta(u_i)\sum_{j=1}^{q}\left(\frac{\zeta(v_j\wedge u_i)}{\zeta(u_i)}\right)^{R-m+1}\right)^{\frac{1}{R-m+1}} \geq \left(\sum_{j=1}^{q}\zeta(v_j)^{R-m+1}\right)^{\frac{1}{R-m+1}}$$

i.e.

$$1-\left(\sum_{i=1}^{p}\zeta(u_i)\sum_{j=1}^{q}\left(\frac{\zeta(v_j\wedge u_i)}{\zeta(u_i)}\right)^{R-m+1}\right)^{\frac{1}{R-m+1}} \leq 1-\left(\sum_{j=1}^{q}\zeta(v_j)^{R-m+1}\right)^{\frac{1}{R-m+1}}$$

Using $\frac{R-m+1}{R-m} > 0$ (for $R > m$), we have

$$''^{\zeta}H_R^m(Q|P) \leq {}^{\zeta}H_R^m(Q)$$

Case2. On the same line we can prove the same result for $0 < R < m$.

Proof. (i) **Case1.** For $R > m$,

Using Jensen's inequality (Beckenbach & Bellman, 2012), we have

$$\left[\sum_{i=1}^{p} x_i\left(\sum_{j=1}^{q} y_{ji}^{R-m+1}\right)^{\frac{1}{R-m+1}}\right] \leq \left[\sum_{i=1}^{p} x_i\sum_{j=1}^{q} y_{ji}^{R-m+1}\right]^{\frac{1}{R-m+1}}$$

use eq. (15) and(16) in above eq., we get

$$\left[\sum_{i=1}^{p}\zeta(u_i)\left(\sum_{j=1}^{q}\left\{\frac{\zeta(v_j\wedge u_i)}{\zeta(u_i)}\right\}^{R-m+1}\right)^{\frac{1}{R-m+1}}\right] \leq \left[\sum_{i=1}^{p}\zeta(u_i)\sum_{j=1}^{q}\left\{\frac{\zeta(v_j\wedge u_i)}{\zeta(u_i)}\right\}^{R-m+1}\right]^{\frac{1}{R-m+1}}$$

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i.e.

$$1-\left[\sum_{i=1}^{p}\zeta\left(u_i\right)\left(\sum_{j=1}^{q}\left\{\frac{\zeta\left(v_j\wedge u_i\right)}{\zeta(u_i)}\right\}^{R-m+1}\right)^{\frac{1}{R-m+1}}\right]\geq 1-\left[\sum_{i=1}^{p}\zeta\left(u_i\right)\sum_{j=1}^{q}\left\{\frac{\zeta\left(v_j\wedge u_i\right)}{\zeta(u_i)}\right\}^{R-m+1}\right]^{\frac{1}{R-m+1}}$$

Using $\frac{R-m+1}{R-m}>0$ (for $R>m$), we have

$$''^{\zeta}H_R^m(Q|P)\leq {}'^{\zeta}H_R^m(Q|P)$$

Case 2. On the same line we can prove the same result for $0<R<m$. Hence, Theorem is proved.

3. Conclusion

In this paper, I have defined the Generalized Parametric R-Norm Information Measure, joint and conditional Generalized Parametric R-Norm Information Measure in quantum logic. I generalize R-norm information measure introduced by (Boekee and Van der Lubbe 1980) and studied a new function in quantum logic.

References

[1] *2005 — the Zhang Lab*. (n.d.). Retrieved July 29, 2022, from https://www.zhanglab.tch.harvard.edu/2005

[2] Alfred Renyi. (1961). On Measures of Entropy and Infromation. *Fourth Berkeley Symposium on Mathematical Statistics and Probability*, *1*, 547–561.

[3] Beckenbach, E., & Bellman, R. (2012). an Introduction To Inequalities. *An Introduction to Inequalities*, 1–2. https://doi.org/10.5948/upo9780883859216.002

[4] Boekee, D. E., & Van der Lubbe, J. C. A. (1980). The R-norm information measure. *Information and Control*, *45*(2), 136–155. https://doi.org/10.1016/S0019-9958(80)90292-2

[5] C. E. Shannon. (1948). A mathematical Theory of Communication. *The Bell System Technical Journal*, *27*(3), 212–214.

[6] Daróczy, Z. (1970). Generalized information functions. *Information and Control*, *16*(1), 36–51. https://doi.org/10.1016/S0019-9958(70)80040-7

[7] Havrda, J., & Charvát, F. (1967). Quantification Method of Classification Processes. *Kybernetika*, *3*(1), 30–35.

[8] Khare, M., & Roy, S. (2007). Conditional Entropy and the Rokhlin Metric on an Orthomodular Lattice with Bayessian State. *International Journal of Theoretical Physics 2007 47:5*, *47*(5), 1386–1396. https://doi.org/10.1007/S10773-007-9581-1

[9] Kumar, S. (2009). Some more results on R-norm information measure. *Tamkang Journal of Mathematics*, *40*(1), 41–58. https://doi.org/10.5556/j.tkjm.40.2009.35

[10] Kumar, S., & Choudhary, A. (2012). *Generalized Parametric R-norm Information Measure*. Trends in Applied Sciences Research 7(5). https://mail.scialert.net/fulltext/?doi=tasr.2012.350.369&org=10

[11] *Zadeh L.a.-Fuzzy Sets (1965) | PDF*. (n.d.). Retrieved July 28, 2022, from https://www.scribd.com/doc/264710088/Zadeh-L-a-fuzzy-Sets-1965

[12] Zarenezhad, M. H., & Ebrahimzadeh, A. (2021). Conditional R-norm entropy and R-norm divergence in quantum logic. *JOURNAL OF MATHEMATICAL EXTENSION*, *16*(0), 1–20. https://doi.org/10.30495/JME.V0I0.1284

[13] Zhao, Y. X., & Ma, Z. H. (2007). Conditional entropy of partitions on quantum logic. *Communications in Theoretical Physics*, *48*(1), 11–13. https://doi.org/10.1088/0253-6102/48/1/003

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