Ordered Properties in Semirings

Dr. A. Rajeswari

Professor, Department of Mathematics, East West Institute of Technology, Bangalore, Karnataka, India
Email id:*rajeswari.bhat83[at]gmail.com*

Abstract: ***This paper contains some structures of semirings with a defined relation and see that they are always partially ordered semirings using some semigroup properties like regular, rectangular band etc.***

Keywords: Semiring, Partially ordered semirings, commutative semirings, rectangular band

AMS Mathematics Subject Classification (2010): **16Y30, 16Y99**

1. Introduction

The study of rings, which are special reveals that multiplicative structure are quite independent of their additive structures are abelian groups. However in semirings it is possible to derive the additive structures from their special multiplicative structures and vice versa. The partial order allows to simulate a number of basic notions and results of idempotent analysis at the purely algebraic level, since 1934, when the first abstract concept of this kind was introduced by Vandiver [1]. Semirings have been studied by various researchers in an attempt to broaden techniques coming from the semigroup theory or ring theory or in connection with applications. In recent times the study of partially ordered semigroups, groups, semirings, semimodules, rings and fields have been increasing widely. M. Sathyanarayana [2], J. Hanumanthachari [3], K.P. Shum [4], Jonathan S. Golan [5] are worth mentioning. Heinz Mitsch [6] defined natural partial order relation on a semigroup and proved that it is a totally ordered relation with respect to its natural partial order if and only if it is an idempotent semigroup. In this paper we extended his results in semirings and proved they are partially ordered semirings.

Definition 1.1: A triple (S,+,.) is said to be a semiring if S is a non-empty set and "+, ." are binary operations on S satisfying that
(i) (S, +) is a semigroup
(ii) (S, .) is a semigroup
(iii) a (b+c) = ab + ac and (b+c) a = ba+ca. for all a,b,c in S.

Examples:
(i) The set of natural numbers under the usual addition, multiplication
(ii) Every distributive lattice (L, ∧, ∨)
(iii) Any ring (R,+,.).

Definition 1.2: A semigroup is a non empty set S together with an associative binary operation from S x S -> S. The associative condition on S states that a(b c) = (a b)c. for a, b, c in S.

Definition 1.3: A semigroup (S , .) is said to be left (right) regular for any a in S there exists x in S such that xa^2 = a (a^2 x = a).

Definition 1.4: A system (S, ≤), where the relation '≤ 'on S satisfying the following axioms.
1) Reflexivity: a≤ a
2) Antisymmetry: a ≤ b, b ≤ a imply a = b
3) Transitivity : a ≤ b, b≤ c imply a ≤ c
4) Linearity : a ≤ b or b ≤ a
for all a, b, c in S, is called a totally (linearly) ordered set.
If (S, ≤) satisfies (3) alone then R is called ordered set .
If (1) and (2) are satisfied then itis called quasi – ordered set.
If (2) and (3) are satisfied then it is called pseudo – ordered set.
(1), (2) and (3) are together is called partially ordered set.

Examples: The set of natural number the usual multiplication and ordering

Theorem 1.5: Let (S, + , .) be a semiring in which (S, .) is left(right) regular band. If a relation 'β' defined by the rule a β b <=> a = xb = by , xa = a for alla, b in S and x,yin S^1 then (S, + , . , β) is apartially ordered semiring.

Proof: Let a, b in S and x,y in S^1

Define 'β' on S by the rule a β b <=> a = xb = by , xa = a.
For a = 1.a = a.1, a = 1. a ,
where '1' is the identity element in (S , .) =>a βa .
Therefore 'β' is reflexive.

Let a β b and b β a then a = xb = by , a = xa and b = ua = av, ub = b for some x,y , u,v in S^1
Now a = xb = x (av) = (xa) v= av = b. Hence 'β' is anti symmetric
Let a β b , b β c then a = xb = by , xa = a and b = uc = cv , ub = b for some x ,y ,u ,v in S^1
We want to prove that a β c
Let a = xb = x (uc) = (xu) c = sc (x, u in S^1 then xu is in S^1 for xu = s)
Similarly a = by = (cv) y = c (vy) = ct (v, y in S^1 then vy is in S^1 for vy = t)
Also (xu) a = (xu) (by) = x (ub) y = x by = x (by) = xa = a => sa = a => a = sc = ct, a = sa => a β c. Therefore 'β' is transitive
Therefore (S, β) is a partially ordered set.
Again aβb => a = xb = by, a = xa => ac = xbc = byc, ac = xac, for some c in S

=>ac = x(bc) = byc^2, ac = xac => ac = x (bc) = bc, ac = x (ac) (Since S is left regular, yc^2= c)
=>ac = x (bc) = bc^{2}y , ac = x (ac) => ac = x (bc) = (bc) y, ac = x (ac)
=>ac ≤ bc. Similarly we prove that caβcb

Therefore 'β' is compatible with respect to multiplication.
Now we prove β is compatible with respect to addition i.e, a + c β b + c
Let aβ b =>a = xb = by, a = xa
=> a + c = xb + c = by + c, a = xa + c
=> a + c = xb + x c^2 = by + c^{2}y, a = xa + xc^2
=> a + c = xb + xc = by + cy, a = xa + xc
=> a + c = x (b + c) = (b + c) y , a = x (a + c)
=> a+cβb+c
Similarly we prove c+aβc+b
Therefore (S, + , . , β) is a partially ordered semiring.

Definition 1.6: A semigroup (S , .) is said to be regular for each a in S there exists a unique element a^1 in S such that a a^{1}a= a.

Theorem 1.7: Let (S , + , .) be a semiring in which (S, .) is regular. A relation on this regular semigroup, by a ϕb<= > a = eb = bf for some e , f in E(S) where E(S) is a set of multiplicative idempotents in S and a, b in S. If (S , .) is permutable then (S , + , .) is a partially ordered semiring.

Proof: Let (S , + , .) be a semiring in which (S , .) is regular.
Let a , b ∊ S . Define a relation ϕ on S by
a ϕb< = >a = eb = bf for some e , f in E(S).
Since S is regular , for any a in S there exists a unique element a^1 in S such that a a^1 a = a.
=> a = (a a^1)a = a (a^1 a) => a = 1 . a = a .1 => a ϕ a .

Therefore 'ϕ' is reflexive .

Let a ϕ b and b ϕ a then a = eb = bf and b = ga = ah for e, f, g, h in E(S) and a ,b in S

Now a = eb = e (ah) = e (eb) h = (e b) h = a h = b =>a = b

Therefore 'ϕ' is anti symmetric

Let aϕ b and b ≤ c then a = eb = bf , b = gc = ch for all e, f, g,h in E(S)
Consider a = eb = e(gc)= (eg) c = s c, for eg = s ∊ E(S)
Similarly a = bf = ch f = c(hf) = ct, for hf = t ∊ E(S)
=> a = sc = ct for all s,t in E(S) => a ϕ c .
Therefore 'ϕ' is transitive.
Let aϕb => a = eb = bf => ac = ebc = bfc
=>ac = e(bc) = (bc) f (since S is pemutable)
=>ac ϕbc. Similarly ca ϕ cb.
Let a ϕ b => a = eb = bf
=> a + c = e.b + c = bf + c
=>a + c = eb + ec = bf + cf
=>a + c = e (b + c) = (b + c)f
=> a + c ϕ b + c . Similarly we prove c + a ϕ c + b.
Therefore (S, +, ., ϕ) is a partially ordered semiring.

Theorem 1.8: Let (S, + , .) be a commutative semiring in which (S ,.) is rectangular band . Define a relation ρ on a semigroup S as following a ρ b < = >
a^2 = ab = ba for all a , b in S. If (S , .) is right regular then (S ,+ ,., ρ) is a partial order semiring.

Proof: Define a relation ρ on a semigroup S as follows a ρ b < = > a^2 = ab = ba for all a, b in S.
For a^2 = a.a = a.a => a ρ a .Therefore 'ρ' is reflexive
Let a ρ b and b ρ a then a^2 = ab = ba, b^2 = ba = ab
consider a^2 = ab => a .a = ab => a(aba) = (aba) b (since S is rectangular band)
aba = a (ab) b (since S is right regular, aba = ba)
= ba ba (since S is comutative)
= b (aba) = b(ba)= b (ab)= bab => a= b

Hence 'ρ' is anti symmetric.

Let a ρ b and b ρ c then a^2 = ab = ba and b^2 = bc = cb
consider a^2= ab= (aba) b (since aba =a)= a (ab) b = a^2 b^2 = a^{2}bc = ab bc = a b^2 c = a (cb) c= a (cbc) =>a^2 = ac

Similarly we provea^2 = ca => a^2 = ca = ac => a ρ c.
Therefore 'ρ' is transitive
Let a ρ b => a^2 = ab = ba=> a^2 c^2 = ab c^2 = ba c^2
=> a^2 c^2 = a(bc) c = b(ac)c => a^2 c^2 = (ac) (bc) = (bc)(ac) => ac ρ bc . Similarly ca ρ cb
Let a ρ b => a^2 = ab = ba

Consider (a + c)^2 = (a +c) (a + c)
= a^2 + ac + ca + c^2
= ab + ac + ca + c.c
= a (b + c) + c (aba) + c .c
= a (b + c) + ca . ab + c.c
= a (b + c) + c a^2 b + c .c
= a (b + c) + c (ba) b + c .c
= a (b + c) + c (bab) + c.c
= a (b + c) + cb + c.c
= a (b + c) + c (b + c)
⇨ (a + c)^2 = (a + c) (b + c)

Similarly we prove (a + c) ^{2} = (b +c) (a + c)
=>a+cρb+c.

Similarly c+aρ c+b .

Therefore (S, + ,. , ρ) is a partially ordered semiring.

2. Conclusion

We have proved some structural properties of ordered properties in semirings

3. Acknowledgment

I would like to thank Visvesvaraya Technological University, Belagavi, and Karnataka, India for helping in my research work and the referee for his valuable time to review this paper.

References

[1] Vandiver : " Note on a simple type of algebra in which cancellation law of addition does not hold" Bull .Amer. Math. Soc .40 (1934) 914 - 920

[2] M.Sathyanarayana : " Naturally totally ordered semigroups" Pacific Journal of mathematics , Vol. 77, No .1 ,1978

[3] Hanumanthachari.J ,Venurju. K and H .J .Weinert : "Some results on partially ordered semirings and semigroups " Proceedings of the first international symposium on ordered algebraic structures , Luminy – Marseilles , 1984

[4] K. P.Shum : "Prime radical theorem on ordered semigroup " , Semigroup Form , 19 (1980) , No.1 , 87 -94

[5] J .S. Golan : " The theory of semirings with applications in mathematics and theoretical computer science , Vol 54 of pitman Monographs and surveys in pure and applied mathematics , Zongamann House, Burnt Mill , Harlow , Essex CM 20 JE. England : Longman Scientific and Technical , 1992

[6] HeinzMitsch : "Semigroups and their natural order " , Math . Slovaca ,44 (1994) , No.4, 445 – 462

[7] Chandrakala H K, Dr. A Rajeswari:- " Completely Regular Semiring", International Journal of Science and Research (IJSR) ISSN: 2319-7064 (2018): 0.28 | SJIF (2019): 7.583

[8] N. Sheela, A.Rajeswari:- "Anti-regular semiring", " International Journal of Research In Science & Engineering", Volume:3 Issue:3 May-june 2017.

[9] N.Sheela, A.Rajeswari:- "Structures of anti-inverse semirings" in Annals of Pure and Applied Mathematics, Vol.16,No.1,2018, 215-222.ISSN:2279-087X(P), 2279-0888 online, jan 2018.

[10] A. Rajeswari, "Structure of semirings satisfying identities", Proceedings of the International Conference on mathematical sciences, (2014)7-10.