International Journal of Science and Research (IJSR)
ISSN: 2319-7064
SJIF (2022): 7.942

Optimal Investment under Partial Information in a Portfolio with Three Financial Assets

Tatiagoum K. Eric*

Department of Mathematics and Computer sciences, University of Dschang
Department of Computer sciences and Telecommunications, Engineering Higher National School, University of Maroua (2014),
Department of Economics Policy and Analysis (2018), University of Dschang, Cameroon
P.O.Box. 242 Dschang, Cameroon.
erictatiagoum@yahoo.fr

Abstract: In this work, we solve the problem of optimal investment for the hedging of a European option with a portfolio made up of three financial assets: A risky asset of price S_t, an asset whose price S_t^* is a deterministic function of a stochastic interest rate r_t and finally, a non-risky asset of price S_t^0. We assume that the payoff Π of the option at maturity date T does not only depend on the strike price S_T of the underlying risky asset but also on an unobservable random variable B. We put ourselves in the situation where the short term interest rate $(r_t)_{0\leq t\leq T}$ is rather a stochastic variable. The two observable variables are: The price of the risky asset S_t and the interest rate r_t. We propose to determine a vector $(\hat{u})_{0\leq t\leq T} = (\hat{u}_t, \hat{u}_t^0)_{0\leq t\leq T}$ of optimal strategies giving at each date t before maturity the optimal amount of assets to invest in the portfolio. We base our determination on maximizing the utility of the terminal wealth of the portfolio. We first transform the problem into a full information problem using the theory of filtering with the stochastic partial differential equations (SPDE).

Keywords:Optimal investment, stochastic maximum principle, three assets , partial information.
2020 **MSC No:** $93E20, 60H15, 49K21, 49K45$

1 Introduction

Nowadays, derivatives have taken an increasingly important place at the global level in the risk profile and profitability of financial institutions. They are used by banking institutions both as risk management instruments and as sources of income. If the use of options is an effective instrument for risk management, calculate its price and cover its requirements is a fair deal for its issuer to account for the risks that it incurs in the bank with a payoff at uncertain character. Several authors have attempted to solve this problem (see [28, 5, 26, 8, 25, 3]).

In order to diversify risk and optimize gain, most portfolio managers spread their wealth across several financial assets. Unfortunately in theory the majority of the models of portfolio management(see [28, 5, 26, 18, 10, 12, 3, 16, 8, 25, 3]) consider the case to essentially consist of two financial assets.

As in incomplete but viable markets, in the absence of a strategy replicating an option, for the management of the portfolio or the price calculation, one is led either to the maximization of the usefulness of the terminal wealth of the portfolio (see [5, 28, 26, 10]), or to the end minimization of the risks (see [18, 16]). One of the main causes of inflation is the excessive increase in the money supply as a function of the interest rate in conventional monetary policies. Here we consider a stochastic interest rate and two observable variables. We solve the problem of optimal investment in a portfolio established on three financial assets and a stochastic interest rate for the hedging of a European option of maturity T. The hedging is done with a portfolio made up of three financial assets: A risky asset with price S_t, an asset whose price S_t^* is a deterministic function of a stochastic interest rate r_t and finally a non-risky asset with price S_t^0. We assume that the payoff Π of the option depends on the expiry price of the observable risky asset S_T and on an unobservable random event Z_T that we describe with the Vasicek model. As in [28, 26, 10, 5] using the improved Black-Scholes model, we assume that a factor μ of the risky asset's drift is a function of both the interest rate r_t and the unobservable variable Z_t. However here, we consider the general case where μ is not only one linear function has of this variables but any deterministic function. This is also contrary to certain articles [14, 28]) where it is rather the volatility that is the unobservable variable. We propose to determine an optimal strategy $(\hat{\bar{u}}_t)_{0\leq t\leq T} = (\hat{u}_t, \hat{u}_t^0)_{0\leq t\leq T}$ vector representing on each date before maturity the optimal quantity of risky assets and not risky to invest in a portfolio in the presence of the option and the same strategy $(\hat{\bar{u}}_t^*)_{0\leq t\leq T} = (\hat{u}_t^*, \hat{u}_t^{0*})_{0\leq t\leq T}$ without the option. Optimal control $(\hat{\bar{u}}_t)_{0\leq t\leq T} = (\hat{u}_t, \hat{u}_t^0)_{0\leq t\leq T}$ represents at each time $t \in [0, T]$ the quantity of risky and non-risky assets to invest in portfolio subject to the maximization of the utility of the terminal wealth of the portfolio and must be adapted to the natural filtration generated by the observable

*Corresponding author: Eric .K Tatiagoum.[1]erictatiagoum@yaoo.fr

variables S_t and r_t. This being seen as a problem under full information. Unfortunately the direct resolution of this problem can lead to the fact that this optimal control also depends on the unobservable variable $(Z_t)_{0\leq t\leq T}$; which makes it a problem under partial information. With a Black Scholes financial portfolio model, we base our hedging strategy on maximizing the utility of the portfolio's terminal wealth.

Use of the unobservable variable Z_t leads to consider the problem as being under partial information.

The authors in [26], [5], [10] considered this problem, but with a constant or deterministic interest rate, as the only observable variable the risky asset price S_t and a portfolio consisting only of two financial assets (risky asset and non-risky asset).

The following section 2 presents the problem, the model, the resolution- method. Section 3 presents the necessary mathematical tools. We go from partial information to complete information thanks to Girsanov's theorem. Indeed any control strategy $(\bar{u}_t)_{0\leq t\leq T}$ must be adapted to the generated σ - algebra by observable variables (S_t and r_t)). In section 4, we establish a stochastic maximum principle necessary to determine the optimal strategies and therefore the optimal portfolio using the Hamiltonian and the adjoint equations. Section 5 is devoted to the application of the previous tools to the resolution of the problem.

2 Motivations, problem,model

Any business is seen as a collection of opportunities for profitability and growth, risks and vulnerabilities. Consequently, the entrepreneur is called upon to achieve the strategic objectives that he has set for himself, taking into account these risks and these opportunities. However, the globalization of economies, the globalization of commercial transactions and trade, the expansion of activities, the financing of companies and the opening of markets have influenced a new dynamic in the business world, and companies have a permanent concern; that of making profits, of being competitive both internationally and locally. After the sale of an option, one of the major concerns for its issuer is to cover and meet the commitments made in this contingent asset. Therefore, he must invest optimally in his portfolio. In this paper, we consider the problem of an economic agent who is in a situation of selling-buying of a European option on a financial asset whose exercise price is S_T and the related payoff is Π. He sells the contingent asset at price x considered to be his initial wealth and buys the same derivative at price p^b from another economic agent. For the coverage of this option, it assumes that the payoff of the option is rather $\Pi\left(S_T, B(Z_T)+\bar{B}\right)$ which depends on the exercise price S_T and is subject to certain basic uncertain risks and events related to the market environment denoted $B(Z_T)+\bar{B}$ which is an unobservable random variable due for example to the cost of transactions (cost of transport, transit etc.), inflation, natural disasters etc. The financial agent holds in his portfolio at date t in addition to the risky asset whose price is S_t, a non-risky asset of price S_t^0, an asset whose price S_t^* is a deterministic function of a stochastic interest rate r_t.

Given a finite horizon T, we consider a complete filtered probability space $(\Omega, \mathfrak{F}, \{\mathfrak{F}_t\}_{0\in[0,T]}; \mathbb{P})$. Improving and generalizing the Black-Scholes model, we assume that a factor μ of the drift of the risky asset's price S_t is a function of the interest rate r_t and Z_t. This is less than and more realistic on financial markets. In fact, the action of the interest rate, risk and uncertainty general on the prices of products and financial products in particular is unavoidable. Moreover, according to Milton Friedman, a monetarist economist, one of the main causes of inflation is the increase in the money supply which is itself a function of the interest rate. Thus, we assume that part of the drift of our risky asset $\mu=\mu(r_t, Z_t)$ is a function of the interest rate r_t and the uncertainty Z_t. Joining the Black-Scholes model, we describe our dynamics of the price of the risked asset by:

$$dS_t = \mu(r_t, Z_t)S_t dt + \sigma_1 S_t dW_t^1, \;\; S_0 = s \tag{2.1}$$

Where W^1 is a Brownian motion on $(\Omega, \mathfrak{F}, \{\mathfrak{F}_t\}_{0\in[0,T]}; \mathbb{P})$
Taking the constant or deterministic interest rate is not a reality of short-term rates on the financial markets. In fact, they are volatile there. In current financial markets is possible that interest rate is been negative. We retain for the dynamic of our interest rate, Vasicek model:

$$dr_t = a(b-r_t)dt + \sigma_2 dW_t^2, \;\; r_0, \quad \sigma_2 > 0, \tag{2.2}$$

Where W^2 is a Brownian motion on $(\Omega, \mathfrak{F}, \{\mathfrak{F}_t\}_{0\in[0,T]}; \mathbb{P})$.
Throughout our study, we consider this model. On financial markets, the more time passes, the more economic agents have market information and tend to control and eliminate uncertainty. Therefore, the Vasicek model below takes this fact into account. The unobservable variable $(Z_t)_{0\leq t\leq T}$ represents all of the uncertain unobservable events in the market environment. It is assumed that over time, all of the information collected helps to reduce risk and uncertainty. Thus, Z_t tends to become constant over time. Z_t is described with the Vasicek model:

$$dZ_t = k(\beta - Z_t)dt + \sigma_3 dW_t^3, \;\; Z_0, \;\; \sigma_3 > 0 \tag{2.3}$$

which is a Gaussian process called Ornstein-Uhlenbeck process. k and β the constancies. In explicit form, we have

$$Z_t = \beta + e^{-kt}(Z_0 - \beta) + \sigma_3 \int_0^t e^{-k(t-s)} dW_s^3,$$

$$\mathbb{E}[Z_t] = \beta + e^{-kt}(Z_0 - \beta)$$

and

$$Var[Z_t] = \frac{\sigma_3^2}{2k}(1 - e^{-2kt}).$$

The observable variables from which we construct the control $\bar{u}_t = (u_t, u_t^0)$ are: The risky asset price S_t and the interest rate r_t. (W^1, W^2) and W^3 are Brownian motions on the filtered probability space $(\Omega, \mathfrak{F}, \{\mathfrak{F}_t\}_{0\in[0,T]}; \mathbb{P})$.
The price dynamics of the semi-risky asset is described by:

$$dS_t^* = r_t S_t^* dt, \;\; S_0^* > 0 \tag{2.4}$$

The non-risky asset dynamic is given by:

$$dS_t^0 = R_t^0 S_t^0 dt, \;\; S_0^0 > 0 \tag{2.5}$$

where R^0 is a deterministic function.
The wealth of the portfolio at each instant t is represented by the variable $X_t^{x,u}$ and its dynamics is given by:

$$\begin{aligned} dX_t^{x,u} &= u_t \frac{dS_t}{S_t} + u_t^0 \frac{dS_t^0}{S_t^0} + (X_t^{x,u} \quad u_t \quad u_t^0)\frac{dS_t^*}{S_t^*}, \quad X_0^{x,u} = x > 0 \\ &= \left[r_t X_t^{x,u} \quad u_t\,(r_t \quad \mu(r_t, Z_t)) \quad u_t^0(r_t \quad R_t^0)\right] dt + \sigma_1 u_t dW_t^1 \end{aligned} \tag{2.6}$$

At date $t \leq T$, the economic agent invests the quantity u_t in the risky asset, u_t^0 in the non-risky asset and the rest $X_t^{x,u} - u_t - u_t^0$ in the "semi-risky" asset considering x as its initial wealth.
Finally we have the system:

$$\begin{cases} dS_t = S_t\mu(r_t, Z_t)dt + S_t\sigma_1 dW_t^1, & S_0 = s \geq 0 \\ dr_t = a(b - r_t)dt + \sigma_2 dW_t^2, & r_0 > 0 \\ dZ_t = k(\beta \quad Z_t)dt + \sigma_3 dW_t^3, & Z_0 \\ dS_t^0 = R_t^0 S_t^0 dt, & S_0^0 > 0 \\ dS_t^* = r_t S_t^* dt, & S_0^* > 0 \\ dX_t^{x,u} = \left[r_t X_t^{x,u} - u_t\,(r_t - \mu(r_t, Z_t)) - u_t^0(r_t - R_t^0)\right] dt + \sigma_1 u_t dW_t^1, & X_0^{x,u} = x > 0 \end{cases} \tag{2.7}$$

where $\sigma_1 > 0, \sigma_2 > 0, \sigma_3 > 0$ are positive constancies. μ a deterministic function, a, b, k, β are constancies.

Problem:
For a European option of payoff $\Pi(S_T, B)$ and a utility function U, the economic agent sells the option at the price x. He receives x and uses it to reinsure himself with another economic agent by buying the same contract at price p^b. By buying this same option at price p^b his initial wealth becomes $x - p^b$. While seeking to maximize the expected utility of the terminal wealth of his portfolio, his problem of is: Determine the optimal management strategies $(\hat{\bar{u}}_t)_{0\leq t\leq T}$, $(\hat{\bar{u}}_t^*)_{0\leq t\leq T}$ vectors giving the quantities to invest in each of the assets respectively in the presence of the option and in its absence, in the aim of maximizing the utility of the terminal wealth of its portfolio. Say the following problem:
Determine optimal strategies $(\hat{\bar{u}}_t)_{0\leq t\leq T}$, $(\hat{\bar{u}}_t^*)_{0\leq t\leq T}$,so that:

$$V_\Pi(x \quad p^b) = \sup_{\bar{u}\in U_{ad}} \mathbb{E}[U(X_T^{x-p^b,\bar{u}} + \Pi(S_T, B))] = \mathbb{E}[U(X_T^{x-p^b,\hat{\bar{u}}} + \Pi(S_T, B))] \tag{2.8}$$

$$V_0(x) = \sup_{u\in U_{ad}} \mathbb{E}[U(X_T^{x,u^*})] = \mathbb{E}[U(X_T^{x,\hat{u}^*})] \tag{2.9}$$

With $U_{ad} \subset \mathbb{R}^2$ the set of admissible strategies $\bar{u}$, $\mathfrak{F}^{(s,r)}$- progressively measurable, contained in a closed set.
The optimal control $(\hat{u}_t)_{0\leq t\leq T}$ is the optimal strategy allowing the economic agent to cover the option with payoff $\Pi(S_T, B(Z_T) + \bar{B}))$ by maximizing the utility of the terminal wealth of its portfolio with the purchase of this option, $(\hat{\bar{u}}_t^*)_{0\leq t\leq T}$ this optimal strategy in the absence of this contingent asset. Since the observable variables are S_t and r_t, set of information at date t is the σ algebra $\mathfrak{F}_t = \mathfrak{F}^{(s,r)} = \sigma(s_{t_1}, r_{t_1}, 0 \leq t_1 \leq t)$ generated by all random variables s_{t_1} and $r_{t_1}, 0 \leq t_1 \leq t$.
By setting: $Y_t^1 = \log S_t, Y_t^2 = r_t$, we have $S_t = \exp Y_t^1$ and (2.7) becomes:

$$\begin{cases} dY_t^1 = \left[\mu(r_t, Z_t) - \frac{1}{2}\sigma_1^2\right] dt + \sigma_1 dW_t^1, & Y_0^1 = log(s) \\ dY_t^2 = a(b - r_t)dt + \sigma_2 dW_t^2, & Y_0^2 = r_0 \\ dZ_t = k\,(\beta \quad Z_t)\,dt + \rho_1\sigma_3 dW_t^1 + +\rho_2\sigma_3 dW_t^2 + \sigma_3\sqrt{1 \quad \rho_1^2}dW_t^{1^\perp} + \sigma_3\sqrt{1 \quad \rho_2^2}dW_t^{2^\perp}, & Z_0 \\ dX_t^{x,u} = \left[r_t X_t^{x,u} - u_t\,(r_t - \mu(r_t, Z_t)) - u_t^0(R_t^0 - r_t)\right] dt + \sigma_1 u_t dW_t^1, & X_0^{x,u} = x > 0 \end{cases} \tag{2.10}$$

where $\rho_1 = corr(W^1, W^3), \rho_2 = corr(W^2, W^3)$ are respectively the correlation coefficients between W^1 and W^3, W^2 and W^3. (W^1, W^2) and $(W^{1^\perp}, W^{2^\perp})$ independent.

With assumption $B = B(Z_T) + \bar{B}$ where B is a smooth function and $\bar{B}$ is a random variable independent of $\mathfrak{F}_T$. we can write (2.8) as follows:

$$\begin{aligned} V_{\Pi}(x) &= \sup_{u \in U_{ad}} \mathbb{E}\left[U(X_T^{x,u} + \Pi(S_T, B))\right] \\ &= \sup_{u \in U_{ad}} \mathbb{E}\left[\int_{\mathbb{R}} U\left(X_T^{x,u} + \Pi(S_T, B(Z_T) + \bar{b})\right) d\mathbb{P}_B\right] \text{ by independence of } \bar{B} \text{ and } Z_T. \\ &= \mathbb{E}\left[\int_{\mathbb{R}} U\left(X_T^{x,\hat{u}} + \Pi(S_T, B(Z_T) + b)\right) d\mathbb{P}_B\right] \end{aligned} \tag{2.11}$$

3 Passage from partial to full information

In this section, we go as in [2], [10], [5], [28], [26] using filtering theory with the Zakai equations to transform the control problem (2.10)-(2.11) from partial information to complete information problem. Unlike in [28] where one replaces only Z_t with its conditional expectation knowing $\mathfrak{F}_T^Y$here, we are going to use as in [2], [10], [5], [26] the property of iterated conditional expectation and replace any function $f(Z_T)$with its conditional expectation knowing $\mathfrak{F}_T^Y$, but here with two observable variables. Let us summarize a brief general result presented in [10], [5] and [26]. Given two independent Brownian movements W and $W^{\perp}$ respectively p and $q-$ dimensional, $(Y_t)_{0\le t\le T}$ the observable variable $n-$dimensional, $(Z_t)_{0\le t\le T}$ the variable not observable m dimensional with dynamics

$$\begin{cases} dY_t = h(t, Z_t, Y_t)dt + \sigma(t, Y_t)dW_t, \quad Y_0 = y & \text{(3.1a)} \\ dZ_t = g(t, Z_t, Y_t)dt + \alpha(t, Z_t, Y_t)dW(t) + \gamma(t, Z_t, Y_t)dW_t^{\perp}, \quad Z_0 = \varepsilon & \text{(3.1b)} \end{cases}$$

We assume the following:

i) $h(t,z,y) : [0,T] \times \mathbb{R}^m \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is globally continuous and of linear growth (in z and y: $|h(t,z,y)| \le k(1+\ z\ )+|\ y\ )$.

ii) $g(t,z,y) : [0,T] \times \mathbb{R}^m \times \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is uniformly continuous in z, y is bounded and twice continuously differentiable with bounded derivatives.

iii)$\sigma(t,y)\ :\ [0,T] \times \mathbb{R}^n \longrightarrow \mathcal{L}(\mathbb{R}^n, \mathbb{R}^p)$ is uniformly continuous, bounded, three times continuously differentiable with bounded derivative, satisfies the following: $\sigma\sigma^t \ge \lambda I$ [1] for all y and t, for some constant $\lambda > 0$ (uniform ellipticity condition).

iv) $\alpha(t,z,y) : [0,T] \times \mathbb{R}^m \times \mathbb{R}^n \longrightarrow \mathcal{L}\left(\mathbb{R}^m, \mathbb{R}^d\right), \gamma(t,z,y) : [0,T] \times \mathbb{R}^m \times \mathbb{R}^n \longrightarrow \mathcal{L}(\mathbb{R}^m, \mathbb{R}^q)$ are uniformly continuous and α is uniformly elliptic.

v) h, σ, g and γ are globally lipschitz of y and z.

Remark 3.1. .
As mentioned in [10], [5], [26], these general results can well be applied to our initial model although the drifts are not bounded. A standard localization of the argument can be used to have a linearly increasing drift.

Let $D_t = D(t, Y_t) = (\sigma\sigma^t)(t, Y_t)$ which we assume to be symmetrical and invertible. φ_t defined by :

$$d\varphi_t = \quad \varphi_t h^t(t, Z_t, Y_t) D_t^{\ \frac{1}{2}}(t, Y_t) dW_t, \quad \varphi_0 = 1^2$$

φ is a supermartingale with $\mathbb{E}[\varphi_t] = 1 \ \ \forall t \in [0,T]$. Therefore by Girsanov's theorem, we define a new probability measure $\tilde{\mathbb{P}}$ such that $\forall t \in [0,T]$:

$$\left.\frac{d\tilde{\mathbb{P}}}{d\mathbb{P}}\right|_{\mathfrak{F}_t} := \varphi_t \ \ (\ d\tilde{\mathbb{P}} = \varphi_t d\mathbb{P} \ \text{ on } \ \mathfrak{F}_t, \forall t \in [0,T]).$$

and there exists a Brownian motion $\tilde{W}$ under $\tilde{\mathbb{P}}$ such that: $dY_t = \sigma(t, Y_t) d\tilde{W}_t$

$$\begin{aligned} dZ_t &= \left[g(t, Z_t, Y_t) \quad \alpha^t(t, Z_t, Y_t) h^t(t, Z_t, Y_t) D_t^{\ \frac{1}{2}}\right] dt + \\ &\quad + \alpha^t(t, Z_t, Y_t) D_t^{\ \frac{1}{2}} dY_t + \gamma(t, Z_t, Y_t) dW^{\perp}(t). \end{aligned}$$

Let $(\tilde{Y}_t, t \in [0,T])$ be the process defined by:

$$d\tilde{Y}_t = D_t^{\ \frac{1}{2}} dY_t. \tag{3.2}$$

[1] σ^t denote the transposed of σ
[2] h^t the transposed vector of h

Then $\tilde{Y}$ is a Brownian motion under $\tilde{\mathbb{P}}$ and $\tilde{Y}$ and $W^{\perp}$ are two independent Brownian motions(see[2]). In addition, using the fact that D_t is invertible, we have : $\mathfrak{F}_t^Y = \mathfrak{F}_t^{\tilde{Y}}$.
Let us set

$$\begin{aligned} K_t =: \frac{1}{\rho_t} &= \exp\left\{\int_0^t h^t(s, Z_s, Y_s) D^{-\frac{1}{2}}(s) dW_s \right. \\ &\left. + \frac{1}{2}\int_0^t h^t(s, Z_s, Y(_s) D^{-1}(s) h(s, Z_s, Y(_s) d(s)\right\} \\ &= \exp\left\{\int_0^t h^t(s, Z_s, Y_s) D^{-1}(s) dY_s \right. \\ &\left. + \frac{1}{2}\int_0^t h^t(s, Z_s, Y_s) D^{-1}(s) h(s, Z_s, Y_s) d(s)\right\} \end{aligned}$$

Then K_t is a martingale.
Let $\phi = (\phi(t,z,w), (t,z,w) \in [0,T] \times \mathbb{R}^d \times \Omega)$, be a process such that for all $f \in C_0^\infty(\mathbb{R}^d)$ [3]. We have:

$$\tilde{E}\left[f(Z_t) K_t \middle| \mathfrak{F}_t^Y\right] = \int_{\mathbb{R}^d} f(z)\phi(t,z) dz, \tag{3.3}$$

$\tilde{E}$ its expectation under $\tilde{\mathbb{P}}$. Then $\phi(t,z)$ is called the unormalized conditional density of Z_t given $\mathfrak{F}_t^Y$ Applying the theorem of Îto to $(K_t f(Z_t))$and using the integration by parts, ϕ satisfy a backward stochastic partial differential equation precisely the following Zakai equation (see [10], [5], [26], [2]):

$$\begin{aligned} d\phi(t,z) = &\left[\frac{1}{2}\sum_{i,j}\sum \frac{\partial^2}{\partial z_i \partial z_j}\left\{\left[\gamma\gamma^{t4} + \alpha\alpha^t\right]_{i,j}\phi(t,z)\right\} - \sum_i \frac{\partial g_i \phi(t,z)}{\partial z_i}\right] dt + \\ &\left[h - \sum_i \frac{\partial}{\partial z_i}(\alpha^i \phi(t,z))\right] d\tilde{Y}_t, \quad \phi(0,z) = \xi(z) \\ =& L_Z^* \phi(t,z) dt + M^* \phi(t,z) d\tilde{Y}_t, \quad \phi(0,z) = \xi(z) \end{aligned} \tag{3.4}$$

where $\xi(z)$ is the density of Z_0

$$\begin{aligned} L_Z^*\phi(t,z) &= \frac{1}{2}\sum_{i,j}\sum \frac{\partial^2}{\partial z_i \partial z_j}\left\{\left[\gamma\gamma^t + \alpha\alpha^t\right]_{i,j}\phi(t,z)\right\} \quad \sum_i \frac{\partial\left(g_i \phi(t,z)\right)}{\partial z_i} \\ &= \mathfrak{L}\phi(t,z) + i\phi(t,z) \\ M^*\phi(t,z) &= h - \sum_i \frac{\partial}{\partial z_i}(\alpha^i \phi(t,z)) \end{aligned}$$

By combining (2.10), (2.11), (3.3) and (3.4) we transform the control problem under partial observation into a problem with SDE into a control problem (2.10) - (2.11) at full information with PSDE:

$$\begin{aligned} V_\Pi(x) = \sup_{u \in U_{ad}} \tilde{\mathbb{E}}\left[\int_{\mathbb{R}}\int_{\mathbb{R}} U(X_T^{x,u} + \Pi(S_T, B(z) + \bar{b})\phi(T,z)) d\mathbb{P}_B dz\right] \\ = \quad \tilde{\mathbb{E}}\left[\int_{\mathbb{R}}\int_{\mathbb{R}} U(X_T^{x,\hat{u}} + \Pi(S_T, B(z) + \bar{b}))\phi(T,z)) d\mathbb{P}_B dz\right] \end{aligned} \tag{3.5}$$

With $\phi(t,z)$ according to (3.4) solution of the stochastic PDE:

$$\begin{cases} d\phi(t,z) = & \left[\sigma_3^2 \frac{\partial^2 \phi}{\partial^2 z}(t,z) \quad k(\beta \quad z)\frac{\partial \phi}{\partial z}(t,z) + k\phi(t,z)\right] dt \\ & + \left[\mu(r_t, Z_t) \quad \frac{1}{2}\sigma_1^2 \quad \rho_1 \sigma_3 \frac{\partial \phi}{\partial z}(t,z)\right] d\tilde{Y}_t^1 + \left[a(b \quad r_t) \quad \sigma_3 \rho_2 \frac{\partial \phi}{\partial z}\right] d\tilde{Y}_t^2 \\ & = L_Z^*\phi(t,z) dt + M_1^*\phi(t,z) d\tilde{Y}_t^1 + M_2^*\phi(t,z) d\tilde{Y}_t^2 \\ \phi(0,z) & = \xi(z) \end{cases} \tag{3.6}$$

let's remind that $Y_t^1 = logS_t, Y_t^2 = r_t$, we have $S_t = \exp Y_t^1$, from (2.10):

$$\begin{cases} dS_t = \frac{\sigma_1^2 S_t}{2} dt + S_t dY_t^1, & S_0 = s \\ dr_t = dY_t^2, & r_0 \\ dX_t^{x,u} = \left[r_t X_t^{x,u} \quad u_t(r_t \quad \frac{1}{2}\sigma_1^2) \quad u_t^0(r_t \quad R_t^0)\right] dt + u_t dY_t^1, & X_0^{x,u} = x. \end{cases} \tag{3.7}$$

[3] $C_0^\infty(\mathbb{R}^d)$ = space of functions C^∞ on $\mathbb{R}^d$ with compact support

Finally by combining (2.10), (3.7), (3.6), we have:

$$\begin{cases} dS_t = \frac{\sigma_1^2 S_t}{2}dt + \sigma_1 S_t d\tilde{Y}_t^1, & S_0 = s \\ dr_t = \sigma_2 d\tilde{Y}_t^2, & r_0, \\ dX_t^{x,\bar{u}} = \left[r_t X_t^{x,\bar{u}} - u_t(r_t - \frac{1}{2}\sigma_1^2) - u_t^0(r_t - R_t^0)\right]dt + u_t\sigma_1 d\tilde{Y}_t^1, & X_0^{x,\bar{u}} = x \\ d\phi(t,z) = \left[\sigma_3^2 \frac{\partial^2\phi}{\partial^2 z}(t,z) - k(\beta - z)\frac{\partial\phi}{\partial z}(t,z) + k\phi(t,z)\right]dt & \\ \qquad + \left[\mu(r_t,z) - \frac{1}{2}\sigma_1^2 - \rho_1\sigma_3\frac{\partial\phi}{\partial z}(t,z)\right]d\tilde{Y}_t^1 + \left[a(b - r_t) - \sigma_3\rho_2\frac{\partial\phi}{\partial z}\right]d\tilde{Y}_t^2 & \phi(0,z) = \xi(z). \\ \qquad = L_Z^*\phi(t,z)dt + M_1^*\phi(t,z)d\tilde{Y}_t^1 + M_2^*\phi(t,z)d\tilde{Y}_t^2 & \end{cases} \tag{3.8}$$

4 Sufficient stochastic maximum principle

We will establish in this section, a stochastic maximum principle that we will use in the next section to determine the optimal amount to invest in the risky asset, the optimal value of the portfolio at each instant. Unlike as in [5, 26, 10], we will use two observable variables.

Let T be a fixed exercise date, $(\Omega, \mathfrak{F}, \{\mathfrak{F}_t\}_{0\in[0,T]}, \mathbb{P})$ a filtered probability space on which we have two Brownian motions W^1 and W^2. We consider the controlled diffusion below which describes the dynamics of the various state processes:

$$\begin{cases} dY_t^1 = b_1(t, Y_t^1, u_t)dt + \sigma_{11}(t, Y_t^1, u_t)dW_t^1 + \sigma_{12}(t, Y_t^1, u_t)dW_t^2, & Y_0^1 = y_0^1 \\ dY_t^2 = b_2(t, Y_t^2, u_t)dt + \sigma_{21}(t, Y_t^2, u_t)dW_t^1 + \sigma_{22}(t, Y_t^2, u_t)dW_t^2, & Y_0^2 = y_0^2 \\ dX_t = b_3(t, X_t, Y_t^1, Y_t^2, u_t)dt + \sigma_{31}(t, X_t, u_t)dW_t^1 + \sigma_{32}(t, X_t, u_t)dW_t^2, & X_0 = x \\ d\phi(t,z) = [L\phi(t,z) + b_4(t, z, \phi(t,z), \frac{\partial\phi}{\partial z}(t,z)), u_t]dt + \sigma_{41}(t, z, \phi(t,z), \frac{\partial\phi}{\partial z}(t,z), u_t)dW_t^1 & \\ \qquad + \sigma_{42}(t, z, \phi(t,z), \frac{\partial\phi}{\partial z}(t,z), u_t)dW_t^2, \ \ \phi(0,z) = \xi(z), \ z \in \mathbb{R} & \\ \qquad \lim_{\|z\|\to+\infty} \phi(t,z) = 0 \ \ \forall t \in [0,T] & \end{cases} \tag{4.1}$$

. where L is a linear differential operator. $b_1, b_2, b_3, b_4, \sigma_{11}, \sigma_{12}, \sigma_{21}, \sigma_{22}, \sigma_{31}, \sigma_{32}, \sigma_{41}, \sigma_{42}$ the given functions satisfying the conditions of existence and uniqueness of strong solutions of the above system, and L^* the formal adjoint of L.

Let f and g be given functions C^1 in their arguments. We consider the objective function:

$$\begin{aligned} J(u) =& \mathbb{E}\left[\int_0^T \int_{\mathbb{R}} \int_{\mathbb{R}} f(t, z, Y_t^1, Y_t^2, X_t, \phi(t,z), \bar{b}, u_t) dz d\mathbb{P}_{\bar{B}} dt \right. \\ & \left. + \int_{\mathbb{R}} \int_{\mathbb{R}} g(z, Y_T^1, Y_T^2, X_T, \phi(T,z), \bar{b}, u_T) dz d\mathbb{P}_{\mathbb{B}} \right] \end{aligned} \tag{4.2}$$

We note U_{ad} the set of admissible controls contained in the set of controls $\mathfrak{F}_t-$ predictable such that the above system has a single strong solution and

$\mathbb{E}\left[\int_0^T \int_{\mathbb{R}} \int_{\mathbb{R}} \left|f(t, z, Y_t^1, Y_t^2, X_t, \phi(t,z), \bar{b}, u_t)\right| dz d\mathbb{P}_{\bar{B}} dt + \int_{\mathbb{R}} \int_{\mathbb{R}} \left|g(z, Y_T^1, Y_T^2, X_T, \phi(T,z), \bar{b}, u_T)\right| dz d\mathbb{P}_{\mathbb{B}}\right] < \infty.$

Problem 4.1. Determine the value function

$$J(\hat{u}) = \sup_{u\in U_{ad}} J(u) \tag{4.3}$$

under conditions (4.1).

That is to say to seek the optimal control $\hat{u} \in U_{ad}$ which maximizes the objective function J

As such, we define the associated Hamiltonian by :

$$H : [0,T] \times \mathbb{R} \times \mathbb{R}^2 \times \mathbb{R} \times \mathbb{R}^2 \times \mathbb{R} \times \mathbb{R}^4 \times \mathbb{R}^4 \times \mathbb{R}^4 \longrightarrow \mathbb{R}$$

$H(t, z, y_1, y_2, x, \phi, \phi', u, p_1, p_2, p_3, p_4, q_1, q_2, q_3, q_4, q_1', q_2', q_3', q_4') = H(t, z, y, x, \phi, \phi', u, p, q, q')$

. with $y = (y_1, y_2), p = (p_1, p_2, p_3, p_4), q = (q_1, q_2, q_3, q_4), q' = (q_1', q_2', q_3', q_4')$.

$$\begin{aligned} H(t, z, y_1, y_2, x, \phi, \phi', u, p, q, q') =& \int_{\mathbb{R}} f(t, z, y_1, y_2, x, \phi, \bar{b}, u) d\mathbb{P}_B + b_1(t, y_1, u)p_1 + b_2(t, y_2, u)p_2 \\ & + b_3(t, y_1, y_2, x, u)p_3 + b_4(t, z, \phi, \phi', u)p_4 + \sigma_{11}(t, y_1, u)q_1 + \sigma_{21}(t, y_2, u)q_2 \\ & + \sigma_{31}(t, x, u)q_3 + \sigma_{41}(t, z, \phi, \phi', u)q_4 + \sigma_{12}(t, y_1, u)q_1' \\ & + \sigma_{22}(t, y_2, u)q_2' + \sigma_{32}(t, x, u)q_3' + \sigma_{42}(t, z, \phi, \phi', u)q_4' \end{aligned} \tag{4.4}$$

Where $\phi' = \frac{\partial\phi}{\partial z}$.We suppose that H is differentiable in the variables y_1, y_2, x, ϕ and ϕ'.
For $u \in U_{ad}$, we consider the adjoint processes satisfying the backward stochastic differential equations in the unknown $p_1(t,z), q_1(t,z), q_1'(t,z), p_2(t,z), q_2(t,z), q_2'(t,z), p_3(t,z), q_3(t,z), q_3'(t,z), p_4(t,z), q_4(t,z), q_4'(t,z) \in \mathbb{R}$ with the system of adjoint equations:

$$\begin{cases} -dp_1 = \frac{\partial H}{\partial y_1}(t,z)dt - q_1 dW_t^1 - q_1' dW_t^2, & p_1(T,z) = \int_{\mathbb{R}} \frac{\partial g}{\partial y_1}(z,\bar{b})d\mathbb{P}_{\bar{B}} \\ -dp_2 = \frac{\partial H}{\partial y_2}(t,z)dt - q_2 dW_t^1 - q_2' dW_t^2, & p_2(T,z) = \int_{\mathbb{R}} \frac{\partial g}{\partial y_2}(z,\bar{b})d\mathbb{P}_{\bar{B}} \\ -dp_3 = \frac{\partial H}{\partial x}(t,z)dt - q_3 dW_t^1 - q_3' dW_t^2, & p_3(T,z) = \int_{\mathbb{R}} \frac{\partial g}{\partial x}(z,\bar{b})d\mathbb{P}_{\bar{B}} \\ -dp_4 = \left[\frac{\partial H}{\partial \phi}(t,z) + L^* p_4(t,z) - \frac{\partial}{\partial z}\left(\frac{\partial H(t,z)}{\partial \phi'}\right)\right]dt - q_4 dW_t^1 - q_4' dW_t^2, & p_4(T,z) = \int_{\mathbb{R}} \frac{\partial g}{\partial \phi}(z,\bar{b})d\mathbb{P}_{\bar{B}} \\ \qquad , \lim_{\|x\| \to +\infty} p_4(t,z) = 0 \end{cases} \tag{4.5}$$

With short notations: $g(z,\bar{b}) = g(z, Y_T^1, Y_T^2, X_T, \phi(T,z), \bar{b}, u_T)$ and
$H(t,z) = H(t,z,Y_t^1, Y_t^2, X_t, \phi(t,z), \phi'(t,z), u_t, p(t,z), q(t,z), q'(t,z))$

Remark 4.2. .
If we suppose for example that the coefficients of the controlled diffusion, and our functions are fairly regular then there is existence and uniqueness of classical strong solutions of our backward stochastic differential equations as well as of the backward stochastic partial differential equation constituting the process of associated adjunct equations.

We have the following theorem:

Theorem 1. *(A Stochastic maximum principle)*
Let $\hat{u} \in U_{ad}$ with the corresponding solutions $\hat{Y}_t^1, \hat{Y}_t^2, \hat{X}_t, \hat{\phi}(t,z)$ of(4.1); $\hat{p} = (\hat{p}_1, \hat{p}_2, \hat{p}_3, \hat{p}_4), \hat{q} = (\hat{q}_1, \hat{q}_2, \hat{q}_3, \hat{q}_4), \hat{q}' = (\hat{q}_1', \hat{q}_2', \hat{q}_3', \hat{q}_4')$ *of* (4.3) *and* (4.5)*. Let the following conditions be satisfied:*

1. *The function $g : (y_1, y_2, x, \phi, u) \longmapsto g(z, y_1, y_2, x, \phi, \bar{b}, u)$ is concave in y_1, y_2, x and ϕ for all $z \in \mathbb{R}, \bar{b} \in \mathbb{R}, u \in U_{ad}$,*
2. $H(t,z,y_1,y_2,x,\hat{u},\phi,\phi',\hat{p},\hat{q},\hat{q}') = \sup_{u \in U_{ad}} H(t,z,y_1,y_2,x,\phi,\phi',u,\hat{p},\hat{q},\hat{q}')$ *exists* $\forall y_1, y_2, x, \phi, \phi'$,
3. *The function $h : (y_1, y_2, x, \phi, \phi') \longmapsto H(t,z,y_1,y_2,x,\phi,\phi',\hat{u},\hat{p},\hat{q},\hat{q}')$ is concave in y_1, y_2, x, ϕ and ϕ' for all $(t,z) \in [0,T] \times \mathbb{R}$,*
4. *With the integrability conditions below satisfied:*
$$\mathbb{E}\Big[\int_0^T \int_{\mathbb{R}} \Big\{ (\sigma_{11} - \hat{\sigma}_{11})^2 \hat{p}_1^2(t,z) + (\sigma_{12} - \hat{\sigma}_{12})^2 \hat{p}_1^2(t,z) + \left(Y_t^1 - \hat{Y}_t^1\right)^2 \hat{q}_1^2(t,z) + \left(Y_t^1 - \hat{Y}_t^1\right)^2 \hat{q'}_1^2(t,z) + (\sigma_{21} - \hat{\sigma}_{21})^2 \hat{p}_2^2(t,z) + (\sigma_{22} - \hat{\sigma}_{22})^2 \hat{p}_2^2(t,z) + \left(Y_t^2 - \hat{Y}_t^2\right)^2 \hat{q}_2^2(t,z) + \left(Y_t^2 \quad \hat{Y}_t^2\right)^2 \hat{q'}_2^2(t,z) + (\sigma_{31} \quad \hat{\sigma}_{31})^2 \hat{p}_3^2(t,z) + (\sigma_{32} \quad \hat{\sigma}_{32})^2 \hat{p}_3^2(t,z) + \left(X_t \quad \hat{X}_t\right)^2 \hat{q}_3^2(t,z) + \left(X_t - \hat{X}_t\right)^2 \hat{q'}_3^2(t,z) \Big\} dtdz\Big] < \infty. \textit{ and}$$
$$\mathbb{E}\Big[\int_0^T \int_{\mathbb{R}} \Big\{ (\sigma_{41} - \hat{\sigma}_{41})^2 \hat{p}_4^2(t,z) + (\sigma_{42} - \hat{\sigma}_{42})^2 \hat{p}_4^2(t,z) + \left(\phi(t,z) - \hat{\phi}_1(t,z)\right)^2 \hat{q}_4^2(t,z) + \left(\phi(t,z) \quad \hat{\phi}_1(t,z)\right)^2 \hat{q'}_4^2(t,z) \Big\} dtdz\Big] < \infty.$$

Then the control $\hat{u}_t$ is optimal for the problem (4.3).

Proof. .
. Let us show that $J(\hat{u}) \geq J(u) \ \forall u \in U_{ad}$.
Recall that

$$\begin{aligned} H(t,z,y_1,y_2,x,\phi,\phi',u,p,q,q') = &\int_{\mathbb{R}} f(t,z,y_1,y_2,x,\phi,\bar{b},u)d\mathbb{P}_B + b_1(t,y_1,u)p_1 + b_2(t,y_2,u)p_2 \\ &+ b_3(t,y_1,y_2,x,u)p_3 + b_4(t,z,\phi,\phi',u)p_4 + \sigma_{11}(t,y_1,u)q_1 + \sigma_{21}(t,y_2,u)q_2 \\ &+ \sigma_{31}(t,x,u)q_3 + \sigma_{41}(t,z,\phi,\phi',u)q_4 + \sigma_{12}(t,y_1,u)q_1' \\ &+ \sigma_{22}(t,y_2,u)q_2' + \sigma_{32}(t,x,u)q_3' + \sigma_{42}(t,z,\phi,\phi',u)q_4' \end{aligned}$$

Let's pose
$H(t,z) = H(t,z,Y_t^1,Y_t^2,X_t,u_t,\phi(t,z),\phi'(t,z),\hat{p},\hat{q},\hat{q}')$,
$\hat{H}(t,z) = H(t,z,\hat{Y}_t^1,\hat{Y}_t^2,\hat{X}^t,\hat{u}_t,\hat{\phi}(t,z),\hat{\phi}'(t,z),\hat{p},\hat{q},\hat{q}')$,

$f(t,z,\bar{b}) = f(t,z,Y_t^1,Y_t^2,X_t,\phi(t,z),\bar{b},u_t), \hat{f}(t,z,\bar{b}) = f(t,z,\hat{Y}_t^1,\hat{Y}_t^2,\hat{X}_t,\hat{\phi}(t,z),\bar{b},\hat{u}),$
$g(z,b) = g(z,Y_T^1,Y_T^2,X_T,\phi(T,z),\bar{b},u), \hat{g}(z,\bar{b}) = g(z,\hat{X}_T,\hat{Y}_T^1,\hat{Y}_T^2,\hat{\phi}(T,z),b,u)$

$$b_1(t) = b_1(t,Y_t^1,u_t) \;\; \hat{b}_1(t) = \hat{b}_1(t,\hat{Y}_t^1,\hat{u}_t)$$
$$b_2(t) = b_2(t,Y_t^1,u_t) \;\; \hat{b}_2(t) = \hat{b}_2(t,\hat{Y}_t^1,\hat{u}_t)$$
$$b_3(t) = b_1(t,X_t,Y_t^1,Y_t^2,u_t) \;\; \hat{b}_3(t) = \hat{b}_3(t,\hat{X}_t,\hat{Y}_t^1,\hat{Y}_t^2,\hat{u}_t)$$
$$b_4(t,z) = b_4(t,z,\phi(t,z),\phi^{'}(t,z)),u_t) \;\; \hat{b}_4(t,z) = \hat{b}_4(t,z,\hat{\phi}(t,z),\hat{\phi}^{'}(t,z),\hat{u}(t))$$
$$\sigma_{11}(t) = \sigma_{11}(t,Y_t^1,u_t) \;\; \hat{\sigma}_{11}(t) = \hat{\sigma}_{11}(t,\hat{Y}_t^1,\hat{u}_t)$$
$$\sigma_{12}(t) = \sigma_{12}(t,Y_t^1,u_t) \;\; \hat{\sigma}_{12}(t) = \hat{\sigma}_{12}(t,\hat{Y}_t^1,\hat{u}_t)$$

$$\sigma_{21}(t) = \sigma_{21}(t,Y_t^2,u_t) \;\; \hat{\sigma}_{21}(t) = \hat{\sigma}_{21}(t,\hat{Y}_t^2,\hat{u}_t)$$
$$\sigma_{22}(t) = \sigma_{22}(t,Y_t^2,u_t) \;\; \hat{\sigma}_{22}(t) = \hat{\sigma}_{22}(t,\hat{Y}_t^2,\hat{u}_t)$$
$$\sigma_{31}(t) = \sigma_{31}(t,X_t,u_t) \;\; \hat{\sigma}_{31}(t) = \hat{\sigma}_{31}(t,\hat{X}_t,\hat{u}_t)$$
$$\sigma_{32}(t) = \sigma_{32}(t,X_t,u_t) \;\; \hat{\sigma}_{32}(t) = \hat{\sigma}_{32}(t,\hat{X}_t,\hat{u}_t)$$
$$\sigma_{41}(t,z) = \sigma_{41}(t,z,\phi(t,z),\phi^{'}(t,z),u_t) \;\; \hat{\sigma}_{41}(t,z) = \hat{\sigma}_{41}(t,z,\hat{\phi}(t,z),\hat{\phi}^{'}(t,z),\hat{u}_t)$$
$$\sigma_{42}(t,z) = \sigma_{42}(t,z,\phi(t,z),\phi^{'}(t,z),u_t) \;\; \hat{\sigma}_{42}(t) = \hat{\sigma}_{42}(t,z,\hat{\phi}(t,z),\hat{\phi}^{'}(t,z),\hat{u}_t)$$

$\hat{p}_1 = \hat{p}_1(t,z), \hat{p}_2 = \hat{p}_2(t,z), \hat{p}_3 = \hat{p}_3(t,z), \hat{p}_4 = \hat{p}_4(t,z), \hat{q}_1 = \hat{q}_1(t,z), \hat{q}_2 = \hat{q}_2(t,z), \hat{q}_3 = \hat{q}_3(t,z), \hat{q}_4 = \hat{q}_4(t,z), \hat{q'}_1 = \hat{q'}_1(t,z), \hat{q'}_2 = \hat{q'}_2(t,z), \hat{q'}_3 = \hat{q'}_3(t,z), \hat{q'}_4 = \hat{q'}_4(t,z)$, we have :

$$\begin{aligned}\int_{\mathbb{R}} f(t,z,\bar{b})d\mathbb{P}_B &= H(t,z) - b_1(t)\hat{p}_1 - b_2(t)\hat{p}_2 - b_3(t)\hat{p}_3 - b_4(t)\hat{p}_4 - \sigma_{11}\hat{q}_1 \\ &- \sigma_{21}\hat{q}_2 - \sigma_{31}\hat{q}_3 - \sigma_{41}\hat{q}_4 - \sigma_{12}\hat{q'}_1 - \sigma_{22}\hat{q'}_2 - \sigma_{32}\hat{q}'_3 + \sigma_{42}\hat{q}'_4\end{aligned} \tag{4.6}$$

$$\begin{aligned}\int_{\mathbb{R}} \hat{f}(t,z,\bar{b})d\mathbb{P}_B &= \hat{H}(t,z) - \hat{b}_1(t)\hat{p}_1 - \hat{b}_2(t)\hat{p}_2 - \hat{b}_3(t)\hat{p}_3 - \hat{b}_4(t)\hat{p}_4 - \hat{\sigma}_{11}\hat{q}_1 \\ &\quad \hat{\sigma}_{21}\hat{q}_2 \quad \hat{\sigma}_{31}\hat{q}_3 \quad \hat{\sigma}_{41}\hat{q}_4 \quad \hat{\sigma}_{12}\hat{q'}_1 \quad \hat{\sigma}_{22}\hat{q'}_2 \quad \hat{\sigma}_{32}\hat{q}'_3 + \hat{\sigma}_{42}\hat{q}'_4\end{aligned} \tag{4.7}$$

$$\begin{aligned}J(\hat{u}) - J(u) &= \mathbb{E}\left[\int_0^T \int_{\mathbb{R}} \int_{\mathbb{R}} \big(\hat{f}(t,z,\bar{b}) - f(t,z,\bar{b})\big)dtdzd\mathbb{P}_{\mathbb{B}} + \int_{\mathbb{R}} \int_{\mathbb{R}} \big(\hat{g}(z,\bar{b}) - g(z,\bar{b})\big)dzd\mathbb{P}_{\mathbb{B}}\right] \\ &= I_1 + I_2\end{aligned} \tag{4.8}$$

From (4.6), (4.7) and (4.12) we have:

$$\begin{aligned}I_1 &= \mathbb{E}\left[\int_0^T \int_{\mathbb{R}} \int_{\mathbb{R}} \big(\hat{f}(t,z,\bar{b}) - f(t,z,\bar{b})\big)dtdzd\mathbb{P}_{\mathbb{B}}\right] \\ &= \mathbb{E}\Bigg[\int_0^T \int_{\mathbb{R}} \Big\{\hat{H}(t,z) - H(t,z) - \big(\hat{b}_1(t) - b_1(t)\big)\hat{p}_1(t,z) - \big(\hat{b}_2(t) - b_2(t)\big)\hat{p}_2(t,z) \\ &\quad \big(\hat{b}_3(t) \quad b_3(t)\big)\hat{p}_3(t,z) \quad \big(\hat{b}_4(t,z) \quad b_4(t,z)\big)\hat{p}_4(t,z) \quad \big(\hat{\sigma}_{11}(t) \quad \sigma_{11}(t)\big)\hat{q}_1(t,z) \quad \big(\hat{\sigma}_{21}(t) \quad \sigma_{21}(t)\big)\hat{q}_2(t,z) \\ &\quad \big(\hat{\sigma}_{31}(t) \quad \sigma_{31}(t)\big)\hat{q}_3(t,z) \quad \big(\hat{\sigma}_{41}(t,z) \quad \sigma_{41}(t,z)\big)\hat{q}_4(t,z) \quad \big(\hat{\sigma}_{12}(t) \quad \sigma_{12}(t)\big)\hat{q'}_1(t,z) \\ &- \big(\hat{\sigma}_{22}(t) - \sigma_{22}(t)\big)\hat{q'}_2(t,z) - \big(\hat{\sigma}_{32}(t) - \sigma_{32}(t)\big)\hat{q'}_3(t,z) - \big(\hat{\sigma}_{42}(t,z) - \sigma_{42}(t,z)\big)\hat{q'}_4(t,z)\Big\}dtdz\Bigg],\end{aligned} \tag{4.9}$$

$$\begin{aligned}I_2 &= \mathbb{E}\left[\int_{\mathbb{R}} \int_{\mathbb{R}} \big\{\hat{g}(z,\bar{b}) - g(z,\bar{b})\big\}dzd\mathbb{P}_{\mathbb{B}}\right] \\ &\geq \mathbb{E}\left[\int_{\mathbb{R}} \int_{\mathbb{R}} \Big\{\frac{\partial \hat{g}}{\partial x}(z,\bar{b})\big(\hat{X}_T - X_T\big) + \frac{\partial \hat{g}}{\partial y_1}(z,\bar{b})\big(\hat{Y}_T^1 - Y_T^1\big) + \frac{\partial \hat{g}}{\partial y_2}(z,\bar{b})\big(\hat{Y}_T^2 - Y_T^2\big) + \frac{\partial \hat{g}}{\partial \phi}(z,\bar{b})\big(\hat{\phi}(T,z) - \phi(T,z)\big)\Big\}dzd\mathbb{P}_{\mathbb{B}}\right]\end{aligned}$$

according to the concavity of g in $\hat{X}, \hat{Y}_1, \hat{Y}_2$ and $\hat{\phi}$

Paper ID: SR22717163401 DOI: 10.21275/SR22717163401

$$= \mathbb{E}\Big[\int_{\mathbb{R}} \Big\{ \big(\hat{X}_T \quad X_T\big) \int_{\mathbb{R}} \frac{\partial \hat{g}}{\partial x}(z,\bar{b}) d\mathbb{P}_{\mathbb{B}} + \big(\hat{Y}^1_T \quad Y^1_T\big) \int_{\mathbb{R}} \frac{\partial \hat{g}}{\partial y_1}(z,b) d\mathbb{P}_{\mathbb{B}}$$

$$+ \big(\hat{Y}^2_T \quad Y^2_T\big) \int_{\mathbb{R}} \frac{\partial \hat{g}}{\partial y_2}(z,b) d\mathbb{P}_{\mathbb{B}} + \big(\hat{\phi}(T,z) \quad \phi(T,z)\big) \int_{\mathbb{R}} \frac{\partial \hat{g}}{\partial \phi}(z,\bar{b}) d\mathbb{P}_{\mathbb{B}} \Big\} dz\Big] \quad \text{since } X, Y_1, Y_1 \text{ et } \phi \text{ thus depend on } b$$

$$= \mathbb{E}\Big[\int_{\mathbb{R}} \Big\{ \big(\hat{Y}^1_T \quad Y^1_T\big)\hat{p}_1(T,z) + \big(\hat{Y}^2_T \quad Y^2_T\big)\hat{p}_2(T,z) + \big(\hat{X}_T \quad X_T\big)\hat{p}_3(T,z) + \big(\hat{\phi}(T,z) \quad \phi(T,z)\big)\hat{p}_4(T,z) \Big\} dz\Big]$$

hence from(4.5) $\hat{p}_1(T,z) = \int_{\mathbb{R}} \frac{\partial \hat{g}}{\partial y_1}(z,b) d\mathbb{P}_B$, $\hat{p}_2(T,z) = \int_{\mathbb{R}} \frac{\partial \hat{g}}{\partial y_2}(z,\bar{B}) d\mathbb{P}_B$, $\hat{p}_3(T,z) = \int_{\mathbb{R}} \frac{\partial \hat{g}}{\partial x}(z,\bar{b}) d\mathbb{P}_B$,

and $\hat{p}_4(T,z) = \int_{\mathbb{R}} \frac{\partial \hat{g}}{\partial \phi}(z,\bar{b}) d\mathbb{P}_B$,

$$= \mathbb{E}\Big[\int_{\mathbb{R}}\int_0^T \Big\{ d\big(\hat{Y}^1_t - Y^1_t\big)\hat{p}_1(t,z) + \big(\hat{Y}^1_t - Y^1_t\big)d\hat{p}_1(t,z) + d < \hat{Y}^1_t - Y^1_t, \hat{p}_1(t,z) > + d\big(\hat{Y}^2_t - Y^2_t\big)\hat{p}_2(t,z)$$

$$+ \big(\hat{Y}^2_t \quad Y^2_t\big)d\hat{p}_2(t,z) + d < \hat{Y}^2_t \quad Y^2_t, \hat{p}_2(t,z) > + d\big(\hat{X}_t \quad X_t\big)\hat{p}_3(t,z) + \big(\hat{X}_t \quad X_t\big)d\hat{p}_3(t,z) + d < \hat{X}_t \quad X_t; \hat{p}_3(t,z) >$$

$$+ d\big(\hat{\phi}(t,z) - \phi(t,z)\big)\hat{p}_4(t,z) + \big(\hat{\phi}(t,z) - \phi(t,z)\big)d\hat{p}_4(t,z) + d < \hat{\phi}(t,z) - \phi(t,z), \hat{p}_4(t,z) > \Big\} dz dt\Big]$$

$$= \mathbb{E}\Big[\int_{\mathbb{R}}\int_0^T \Big\{ \Big((\hat{b}_1(t) \quad b_1(t))dt + (\hat{\sigma}_{11}(t) \quad \sigma_{11}(t))dW_1(t) + (\hat{\sigma}_{12}(t) \quad \sigma_{12}(t))dW^2_t\Big)\hat{p}_1(t,z)$$

$$+ \Big((\hat{b}_2(t) \quad b_2(t))dt + (\hat{\sigma}_{21}(t) \quad \sigma_{21}(t))dW^1_t + (\hat{\sigma}_{22}(t) \quad \sigma_{22}(t))dW^2_t\Big)\hat{p}_2(t,z)$$

$$+ \Big((\hat{b}_3(t) - b_3(t))dt + (\hat{\sigma}_{31}(t) - \sigma_{31}(t))dW^1_t + (\hat{\sigma}_{32}(t) - \sigma_{32}(t))dW^2_t\Big)\hat{p}_3(t,z)$$

$$\Big((L(\hat{\phi}(t,z) \quad \phi(t,z)) + (\hat{b}_4(t,z) \quad b_4(t,z))dt + (\hat{\sigma}_{41}(t,z) \quad \sigma_{41}(t,z))dW^1_t$$

$$+ \big(\hat{\sigma}_{42}(t,z) - \sigma_{42}(t,z))dW^2_t\big)\hat{p}_4(t,z) + \big(\hat{Y}^1_t - Y^1_t\big)(-\frac{\partial \hat{H}}{\partial y_1}(t,z)dt + \hat{q}_1(t,z)dW^1_t + \hat{q}'_1(t,z)dW^2_t)$$

$$+ \big(\hat{Y}^2_t - Y^2_t\big)(-\frac{\partial \hat{H}}{\partial y_2}(t,z)dt + \hat{q}_2(t,z)dW^1_t + \hat{q}'_2(t,z)dW^2_t)$$

$$+ \big(\hat{X}_t \quad X_t\big)(\quad \frac{\partial \hat{H}}{\partial x}(t,z)dt + \hat{q}_3(t,z)dW^1_t + \hat{q}'_3(t,z)dW^2_t)$$

$$+ \big(\hat{\phi}(t,z) - \phi(t,z)\big)\Big(\big(-\frac{\partial \hat{H}}{\partial \phi}(t,z) - L^*\hat{p}_4(t,z) + \frac{\partial}{\partial z}\left(\frac{\partial \hat{H}(t,z)}{\partial \phi'}\right)dt + \hat{q}_4(t,z)dW^1_t + \hat{q}'_4(t,z)dW^2_t\Big)$$

$$+ (\hat{\sigma}_{11}(t) - \sigma_{11}(t))\hat{q}_1(t,z)dt + (\hat{\sigma}_{12}(t) - \sigma_{12}(t))\hat{q}'_1(t,z)dt + (\hat{\sigma}_{21}(t) - \sigma_{21}(t))\hat{q}_2(t,z)dt$$

$$+ (\hat{\sigma}_{22}(t) - \sigma_{22}(t))\hat{q}'_2(t,z)dt + (\hat{\sigma}_{31}(t) - \sigma_{31}(t))\hat{q}_3(t,z)dt + (\hat{\sigma}_{32}(t) - \sigma_{32}(t))\hat{q}'_3(t,z)dt$$

,

$$+ (\hat{\sigma}_{41}(t,z) \quad \sigma_{41}(t,z))\hat{q}_4(t,z) + (\hat{\sigma}_{42}(t,z) \quad \sigma_{42}(t,z))\hat{q}'_4(t,z)\Big\} dz\Big]$$

$$= \mathbb{E}\Big[\int_{\mathbb{R}}\int_0^T \Big\{ \Big(\hat{b}_1(t) - b_1(t)\Big)\hat{p}_1(t,z) + \Big(\hat{b}_2(t) - b_2(t)\Big)\hat{p}_2(t,z) + \Big(\hat{b}_3(t) - b_3(t)\Big)\hat{p}_3(t,z)dt$$

$$+ \Big(\hat{b}_4(t) \quad b_4(t)\Big)\hat{p}_4(t,z) + L\big(\hat{\phi}(t,z) \quad \phi(t,z)\big)\hat{p}_4(t,z) + \big(\hat{Y}^1_t \quad Y^1_t\big)(\quad \frac{\partial \hat{H}}{\partial y_1}(t,z)) + \big(\hat{Y}^2_t \quad Y^2_t\big)(\quad \frac{\partial \hat{H}}{\partial y_2}(t,z))$$

$$, + \big(\hat{X}_t \quad X_t\big)(\quad \frac{\partial \hat{H}}{\partial x}(t,z)) + \big(\hat{\phi}(t,z) \quad \phi(t,z)\big)(\quad \frac{\partial \hat{H}}{\partial \phi}(t,z) \quad L^*\hat{p}_4(t,z) + \frac{\partial}{\partial z}\left(\frac{\partial \hat{H}(t,z)}{\partial \phi'}\right)) \tag{4.10}$$

$$+ \big(\hat{\sigma}_{11}(t) \quad \sigma_{11}(t)\big)\hat{q}_1(t,z) + (\hat{\sigma}_{12}(t) \quad \sigma_{12}(t))\hat{q}'_1(t,z) + (\hat{\sigma}_{21}(t) \quad \sigma_{21}(t))\hat{q}_2(t,z)$$

$$+ (\hat{\sigma}_{22}(t) \quad \sigma_{22}(t))\hat{q}'_2(t,z) + (\hat{\sigma}_{31}(t) \quad \sigma_{31}(t))\hat{q}_3(t,z) + (\hat{\sigma}_{32}(t) \quad \sigma_{32}(t))\hat{q}'_3(t,z)$$

$$+ (\hat{\sigma}_{41}(t,z) - \sigma_{41}(t,z))\hat{q}_4(t,z) + (\hat{\sigma}_{42}(t,z) - \sigma_{42}(t,z))\hat{q}'_4(t,z)\Big\} dt dz\Big]$$

From (4.9) and (4.10) we have:

$$J(\hat{u}) \quad J(u) \geq I_1 + I_2$$

$$= \mathbb{E}\Big[\int_0^T \int_{\mathbb{R}} \Big\{ \hat{H}(t,z) - H(t,z) - \big(\hat{b}_1(t) - b_1(t)\big)\hat{p}_1(t,z) - \big(\hat{b}_2(t) - b_2(t)\big)\hat{p}_2(t,z)$$

$$
\begin{aligned}
&\big(\hat{b}_3(t) \quad b_3(t)\big)\hat{p}_3(t,z) \quad \big(\hat{b}_4(t,z) \quad b_4(t,z)\big)\hat{p}_4(t,z) \quad \big(\hat{\sigma}_{11}(t) \quad \sigma_{11}(t)\big)\hat{q}_1(t,z)\\
&-\big(\hat{\sigma}_{21}(t)-\sigma_{21}(t)\big)\hat{q}_2(t,z)-\big(\hat{\sigma}_{31}(t)-\sigma_{31}(t)\big)\hat{q}_3(t,z)-\big(\hat{\sigma}_{41}(t,z)-\sigma_{41}(t,z)\big)\hat{q}_4(t,z)\\
&-\big(\hat{\sigma}_{12}(t)-\sigma_{12}(t)\big)\hat{q'}_1(t,z)-\big(\hat{\sigma}_{22}(t)-\sigma_{22}(t)\big)\hat{q'}_2(t,z)-\big(\hat{\sigma}_{32}(t)-\sigma_{32}(t)\big)\hat{q'}_3(t,z)\\
&-\big(\hat{\sigma}_{42}(t,z)-\sigma_{42}(t,z)\big)\hat{q'}_4(t,z)+\big(\hat{b}_1(t)-b_1(t)\big)\hat{p}_1(t,z)+\big(\hat{b}_2(t)-b_2(t)\big)\hat{p}_2(t,z)\\
&+\big(\hat{b}_3(t) \quad b_3(t)\big)\hat{p}_3(t,z)+\big(\hat{b}_4(t) \quad b_4(t)\big)\hat{p}_4(t,z)+L\left(\hat{\phi}(t,z) \quad \phi(t,z)\right)\hat{p}_4(t,z)\\
&-\big(\hat{Y}^1_t-Y^1_t\big)\frac{\partial\hat{H}}{\partial y_1}(t,z)-\big(\hat{Y}^2_t-Y^2_t\big)\frac{\partial\hat{H}}{\partial y_2}(t,z)-\big(\hat{X}_t-X_t\big)\frac{\partial\hat{H}}{\partial x}(t,z)\\
&-\big(\hat{\phi}(t,z)-\phi(t,z)\big)\frac{\partial\hat{H}}{\partial\phi}(t,z)-\big(\hat{\phi}(t,z)-\phi(t,z)\big)L^*\hat{p}_4(t,z)\\
&+\big(\hat{\phi}(t,z) \quad \phi(t,z)\big)\frac{\partial}{\partial z}\left(\frac{\partial\hat{H}(t,z)}{\partial\phi'}\right)+(\hat{\sigma}_{11}(t) \quad \sigma_{11}(t))q_1(t,z)+(\hat{\sigma}_{12}(t) \quad \sigma_{12}(t))q'_1(t,z)\\
&+(\hat{\sigma}_{21}(t) \quad \sigma_{21}(t))q_2(t,z)+((\hat{\sigma}_{22}(t) \quad \sigma_{22}(t))q'_2(t,z)+(\hat{\sigma}_{31}(t) \quad \sigma_{31}(t))q_3(t,z)\\
&+(\hat{\sigma}_{32}(t)-\sigma_{32}(t))q^{'}_3(t,z)+(\hat{\sigma}_{41}(t,z)-\sigma_{41}(t,z))q_4(t,z)+(\hat{\sigma}_{42}(t,z)-\sigma_{42}(t,z))q^{'}_4(t,z)\Big\}dtdz\Big]
\end{aligned}
$$

$$
\begin{aligned}
&=\mathbb{E}\left[\int_0^T\int_{\mathbb{R}}\Big\{\hat{H}(t,z) \quad H(t,z) \quad \big(\hat{Y}_1(t) \quad Y_1(t)\big)\frac{\partial\hat{H}}{\partial y_1}(t,z) \quad \big(\hat{Y}_2(t) \quad Y_2(t)\big)\frac{\partial\hat{H}}{\partial y_2}(t,z)\right.\\
&\quad \big(\hat{X}(t) \quad X_t\big)\frac{\partial\hat{H}}{\partial x}(t,z) \quad \big(\hat{\phi}(t,z) \quad \phi(t,z)\big)\frac{\partial\hat{H}}{\partial\phi}(t,z)+\big(\hat{\phi}(t,z) \quad \phi(t,z)\big)\frac{\partial}{\partial z}\left(\frac{\partial\hat{H}(t,z)}{\partial\phi'}\right)\\
&\left.+L\left(\hat{\phi}(t,z) \quad \phi(t,z)\right)\hat{p}_4(t,z) \quad \big(\hat{\phi}(t,z) \quad \phi(t,z)\big)L^*\hat{p}_4(t,z)\Big\}dtdz\right]\\
&=\mathbb{E}\left[\int_0^T\int_{\mathbb{R}}\Big\{\hat{H}(t,z)-H(t,z)-\big(\hat{Y}_1(t)-Y_1(t)\big)\frac{\partial\hat{H}}{\partial y_1}(t,z)-\big(\hat{Y}_2(t)-Y^2_t\big)\frac{\partial\hat{H}}{\partial y_2}(t,z)\right.\\
&-\big(\hat{X}(t)-X_t\big)\frac{\partial\hat{H}}{\partial x}(t,z)-\big(\hat{\phi}(t,z)-\phi(t,z)\big)\frac{\partial\hat{H}}{\partial\phi}(t,z)+\big(\hat{\phi}(t,z)-\phi(t,z)\big)\frac{\partial}{\partial z}\left(\frac{\partial\hat{H}(t,z)}{\partial\phi'}\right)\\
&\left.+L^*\left(\hat{\phi}(t,z)-\phi(t,z)\right)\hat{p}_4(t,z)-\big(\hat{\phi}(t,z)-\phi(t,z)\big)L^*\hat{p}_4(t,z)\Big\}dtdz\right]
\end{aligned}
$$

,

$$
\begin{aligned}
&=\mathbb{E}\left[\int_0^T\int_{\mathbb{R}}\Big\{\hat{H}(t,z) \quad H(t,z) \quad \frac{\partial\hat{H}}{\partial y_1}(t,z)\big(\hat{Y}^1_t \quad Y^1_t\big) \quad \frac{\partial\hat{H}}{\partial y_2}(t,z)\big(\hat{Y}^2_t \quad Y^2_t\big)\right.\\
&\left.\quad \frac{\partial\hat{H}}{\partial x}(t,z)\big(\hat{X}_t \quad X_t\big) \quad \frac{\partial\hat{H}}{\partial\phi}(t,z)\big(\hat{\phi}(t,z) \quad \phi(t,z)\big) \quad \frac{\partial\hat{H}}{\partial\phi^{'}}(t,z)\big(\hat{\phi}'(t,z) \quad \phi'(t,z)\big)\Big\}dtdz\right]
\end{aligned}
\tag{4.11}
$$

According to (4.11) and the concavity of h we have :

$$
\begin{aligned}
J(\hat{u})-J(u)\geq\;&\mathbb{E}\left[\int_0^T\int_{\mathbb{R}}\Big\{\hat{H}(t,z)-H(t,z,X,Y^1,Y^2,\phi,\phi^{'},\hat{u},\hat{p},\hat{q})-\frac{\partial\hat{H}}{\partial y_1}(t,z)\big(\hat{Y}^1_t-Y^1_t\big)\right.\\
&-\frac{\partial\hat{H}}{\partial y_2}(t,z)\big(\hat{Y}^2_t-Y^2_t\big)-\frac{\partial\hat{H}}{\partial x}(t,z)\big(\hat{X}_t-X_t\big)-\frac{\partial\hat{H}}{\partial\phi}(t,z)\big(\hat{\phi}(t,z)-\phi(t,z)\big)\\
&\left.-\frac{\partial\hat{H}}{\partial\phi^{'}}(t,z)\big(\hat{\phi}'(t,z)-\phi(t,z)\big)\Big\}dtdz\right]\\
\geq\;&0
\end{aligned}
\tag{4.12}
$$

.

5 Optimal Investment

From (3.8), section 3 we have:

$$\begin{cases} dS_t = \frac{\sigma_1^2 S_t}{2}dt + \sigma_1 S_t d\tilde{Y}_t^1, & S_0 = s \\ dr_t = \sigma_2 d\tilde{Y}_t^2, & r_0 \\ dX_t^{x,\bar{u}} = \left[r_t X_t^{x,\bar{u}} - u_t(r_t - \frac{1}{2}\sigma_1^2) - u_t^0(r_t - R_t^0)\right]dt + u_t\sigma_1 d\tilde{Y}_t^1, & X_0^{x,\bar{u}} = x \\ d\phi(t,z) = \left[\sigma_3^2 \frac{\partial^2\phi}{\partial^2 z}(t,z) - k(\beta - z)\frac{\partial\phi}{\partial z}(t,z) + k\phi(t,z)\right]dt & \\ \qquad + \left[\mu(r_t,z) - \frac{1}{2}\sigma_1^2 - \rho_1\sigma_3\frac{\partial\phi}{\partial z}(t,z)\right]d\tilde{Y}_t^1 + \left[a(b-r_t) - \sigma_3\rho_2\frac{\partial\phi}{\partial z}(t,z)\right]d\tilde{Y}_t^2 & \phi(0,z) = \xi(z) \\ \qquad = L_Z^*\phi(t,z)dt + M_1^*\phi(t,z)d\tilde{Y}_t^1 + M_2^*\phi(t,z)d\tilde{Y}_t^2 & \end{cases}$$

We note that only the process $X_t^{x,\bar{u}}$ depends on the control $\bar{u}$.

$$\begin{aligned} V_\Pi(x) &= \sup_{u\in U_{ad}} \mathbb{E}[U\left(X_T^{x,\bar{u}} + \Pi(S_T,B)\right)] \\ &= \sup_{u\in U_{ad}} \tilde{\mathbb{E}}\left[\int_{\mathbb{R}}\int_{\mathbb{R}} \phi(T,z)U\left(X_T^{x,\bar{u}} + \Pi(S_T, B(z)+\bar{b})\right)d\mathbb{P}_{\bar{B}}dz\right] \\ &= \tilde{\mathbb{E}}\left[\int_{\mathbb{R}}\int_{\mathbb{R}} \phi(T,z)U\left(X_T^{x,\hat{\bar{u}}} + \Pi(S_T, B(z)+\bar{b})\right)d\mathbb{P}_{\bar{B}}dz\right] \end{aligned} \tag{5.1}$$

The objective function to be maximized is given by:

$$J(u) = \tilde{\mathbb{E}}\left[\int_{\mathbb{R}}\int_{\mathbb{R}} \left[U\left(X_T^{x,\bar{u}} + \Pi(S_T, B(z)+\bar{b})\right)\right]\phi(T,z)d\mathbb{P}_{\bar{B}}dz\right]$$

Applying the theorem 1 with Hamiltonian:

$$\begin{aligned} H(t,z,s,r,x,\phi,\phi',u,p,q,q') &= \frac{\sigma_1^2 s}{2}p_1 + \left[rx - u(r - \frac{1}{2}\sigma_1^2) - u^0\left(r - R^0\right)\right]p_3 \\ &+ \left[-k(\beta - z)\frac{\partial\phi}{\partial z}(t,z) + k\phi(t,z)\right]p_4 + \sigma_1 s q_1 + \sigma_1 u q_3 \\ &+ \left[\mu(r_t,z) - \frac{1}{2}\sigma_1^2 - \rho_1\sigma_3\frac{\partial\phi}{\partial z}\right]q_4 + \sigma_2 q_2' + \left[a(b-r) - \sigma_3\rho_2\frac{\partial\phi}{\partial z}(t,z)\right]q_4' \end{aligned}$$

The adjoint equations are given by:

$$\begin{cases} dp_1(t,z) &= \left[\frac{\sigma_1^2}{2}p_1(t,z) + \sigma_1 q_1(t,z)\right]dt \quad q_1(t,z)d\tilde{Y}_t^1 \quad q_1'(t,z)d\tilde{Y}_t^2 \\ p_1(T,z) &= \phi(T,z)\int_{\mathbb{R}}\frac{\partial U}{\partial S}\left(X_T^{x,u} + \Pi(S_T, B(z)+\bar{b})\right)d\mathbb{P}_B \end{cases} \tag{5.2}$$

$$\begin{cases} -dp_2(t,z) &= \left[(x - u - u^0)p_3(t,z) + \frac{\partial\mu}{\partial r}(r,z)q_4(t,z) - aq_4'(t,z)\right]dt - q_2(t,z)d\tilde{Y}_t^1 - q_2'(t,z)d\tilde{Y}_t^2 \\ p_2(T,z) &= \phi(T,z)\int_{\mathbb{R}}\frac{\partial U}{\partial r}\left(X_T^{x,\bar{u}} + \Pi(S_T, B(z)+\bar{b})\right)d\mathbb{P}_B \end{cases} \tag{5.3}$$

$$\begin{cases} -dp_{3(t,z)} &= rp_3(t,z)dt - q_3(t,z)d\tilde{Y}_t^1 - q_3'(t,z)d\tilde{Y}_t^2 \\ p_3(T,z) &= \phi(T,z)\int_{\mathbb{R}}\frac{\partial U}{\partial X}(X_T^{x,\bar{u}} + \Pi(S_T, B(z)+\tilde{b}))d\mathbb{P}_B \end{cases} \tag{5.4}$$

$$\begin{cases} dp_4(t,z) &= \left[\sigma_3^2\frac{\partial^2 p_4(t,z)}{\partial^2 z}\right]dt \quad q_4(t,z)d\tilde{Y}_t^1 \quad q_4'(t,z)d\tilde{Y}_t^2 \\ p_4(T,z) &= \int_{\mathbb{R}}\frac{\partial U}{\partial\phi}(X_T^{x,u} + \Pi(S_T, B(z)+\tilde{b}))d\mathbb{P}_{\bar{B}} \end{cases} \tag{5.5}$$

Let μ such that (3.8), (5.3) and (5.5) each have a unique strong solution, $\hat{\bar{u}} = (\hat{u}, \hat{u}^0)$ be an optimal control with the optimal processes corresponding $\hat{X}, \hat{S}, \hat{r}, \hat{\phi}, \hat{\phi}'$ and the corresponding adjoining equations with solutions: $\hat{p}_1, \hat{p}_2, \hat{p}_3, \hat{p}_4, \hat{q}_1, \hat{q}_2, \hat{q}_3, \hat{q}_4, \hat{q}_1', \hat{q}_2', \hat{q}_3', \hat{q}_4'$. U and Π are concave. H is a linear function of u. Thus the first order optimality condition below:
First order optimality condition:

$$\frac{\partial H}{\partial u}(t,z) = 0 \Longleftrightarrow (r_t - \frac{1}{2}\sigma_1^2)\hat{p}_3(t,z) = \sigma_1\hat{q}_3(t,z) \tag{5.6}$$

We will now use these conditions with the adjoint equation (5.4) to determine the optimal control $\hat{u}$.
Recall that this linear equation is:

$$\begin{cases} dp_{3(t,z)} &= rp_3(t,z)dt \quad q_3(t,z)d\tilde{Y}_t^1 \quad q_3'(t,z)d\tilde{Y}_t^2 \\ p_3(T,z) &= \phi(T,z)\int_{\mathbb{R}}\frac{\partial U}{\partial X}(X_T^{x,u} + \Pi(S_T, B(z)+\tilde{b}))d\mathbb{P}_B \end{cases} \tag{5.7}$$

Let us find a solution of (5.7)) in the form:

$$\hat{p}_3(t,z) = \rho(t, r_t, \phi(t,z))X_t^{x,\hat{u}} + \psi(t, S_t, r_t, \phi_t) \tag{5.8}$$

where ψ is a C^2 function in each of its variables with the differential form:

$$d\psi(t, S_t, r_t, \phi_t) = \Theta_t dt + \beta_t^1 d\tilde{Y}_t^1 + \beta_t^2 d\tilde{Y}_t^2. \tag{5.9}$$

$$\begin{aligned} d\rho(t, r_t, \phi(t,z)) = &\left[\frac{\partial \rho}{\partial t} + \left(\sigma_3^2 \frac{\partial^2 \phi}{\partial^2 z}(t,z) \quad k(\beta \quad z)\frac{\partial \phi}{\partial z}(t,z) + k\phi(t,z)\right)\frac{\partial \rho}{\partial \phi} + \frac{1}{2}\sigma_2^2 \frac{\partial^2 \rho}{\partial^2 r}\right. \\ &+ \frac{1}{2}\left(\left(\mu(r_t,z) \quad \frac{1}{2}\sigma_1^2 \quad \rho_1\sigma_3\frac{\partial \phi}{\partial z}(t,z)\right)^2 + \left(a(b \quad r_t) \quad \sigma_3\rho_2\frac{\partial \phi}{\partial z}(t,z)\right)^2\right)\frac{\partial^2 \rho}{\partial^2 \phi} \\ &\left.+ \sigma_2\left(a(b-r_t) - \sigma_3\rho_2\frac{\partial \phi}{\partial z}(t,z)\right)\frac{\partial^2 \rho}{\partial r \partial \phi}\right]dt + \left(\mu(r_t,z) - \frac{1}{2}\sigma_1^2 - \rho_1\sigma_3\frac{\partial \phi}{\partial z}(t,z)\right)\frac{\partial \rho}{\partial \phi}d\tilde{Y}_t^1 \\ &+ \left[\sigma_2\frac{\partial \rho}{\partial r} + \left(a(b-r_t) - \sigma_3\rho_2\frac{\partial \phi}{\partial z}(t,z)\right)\frac{\partial \rho}{\partial \phi}\right]d\tilde{Y}_t^2 \end{aligned} \quad . \tag{5.10}$$

We have:

$$\begin{aligned} d\hat{p}_3(t,z) = & d\rho(t, r_t, \phi(t,z))X_t^{x,\hat{u}} + \rho(t, r_t, \phi(t,z))dX_t^{x,\hat{u}} + \sigma_1\hat{u}_t\left(\mu(r_t,z) \quad \frac{1}{2}\sigma_1^2 \quad \rho_1\sigma_3\frac{\partial \phi}{\partial z}(t,z)\right)\frac{\partial \rho}{\partial \phi}. \\ &+ \Theta_t dt + \beta_t^1 d\tilde{Y}_t^1 + \beta_t^2 d\tilde{Y}_t^2 \end{aligned}$$

$$\begin{aligned} d\hat{p}_3(t,z) = &\left[\left[\frac{\partial \rho}{\partial t} + \left(\sigma_3^2 \frac{\partial^2 \phi}{\partial^2 z}(t,z) \quad k(\beta \quad z)\frac{\partial \phi}{\partial z}(t,z) + k\phi(t,z)\right)\frac{\partial \rho}{\partial \phi} + \frac{1}{2}\sigma_2^2 \frac{\partial^2 \rho}{\partial^2 r}\right.\right. \\ &+ \frac{1}{2}\left(\left(\mu(r_t,z) \quad \frac{1}{2}\sigma_1^2 \quad \rho_1\sigma_3\frac{\partial \phi}{\partial z}(t,z)\right)^2 + \left(a(b \quad r_t) \quad \sigma_3\rho_2\frac{\partial \phi}{\partial z}(t,z)\right)^2\right)\frac{\partial^2 \rho}{\partial^2 \phi} \\ &\left.+ \sigma_2\left(a(b-r_t) - \sigma_3\rho_2\frac{\partial \phi}{\partial z}(t,z)\right)\frac{\partial^2 \rho}{\partial r \partial \phi} + r_t\rho(t, r_t, \phi(t,z))\right]X_t^{x,\hat{u}} - \rho(t, r_t, \phi(t,z))\hat{u}_t(r_t - \frac{1}{2}\sigma_1^2) \\ &\left.-u_t^0\rho(t, r_t, \phi(t,z))(r_t - R_t^0) + \sigma_1\hat{u}_t\left(\mu(r_t,z) - \frac{1}{2}\sigma_1^2 - \rho_1\sigma_3\frac{\partial \phi}{\partial z}(t,z)\right)\frac{\partial \rho}{\partial \phi} + \Theta_t\right]dt \\ &+ \left[\left[\mu(r_t,z) \quad \frac{1}{2}\sigma_1^2 \quad \rho_1\sigma_3\frac{\partial \phi}{\partial z}(t,z)\right]\frac{\partial \rho}{\partial \phi}X_t^{x,\hat{u}} + \rho(t, r_t, \phi(t,z))\hat{u}_t\sigma_1 + \beta_t^1\right]d\tilde{Y}_t^1 \\ &+ \left[\left[\sigma_2\frac{\partial \rho}{\partial r} + \left(a(b \quad r_t) \quad \sigma_3\rho_2\frac{\partial \phi}{\partial z}(t,z)\right)\frac{\partial \rho}{\partial \phi}\right]X_t^{x,\hat{u}} + \beta_t^2\right]d\tilde{Y}_t^2. \end{aligned} \tag{5.11}$$

On the other hand, according to (5.7) and (5.8) , we have:

$$d\hat{p}_3(t,z) = \quad r_t\left[\rho(t, r_t, \phi(t,z))X_t^{x,\hat{u}} + \psi(t, S_t, r_t, \phi_t)\right]dt + \hat{q}_3 d\tilde{Y}_t^1 + \hat{q}_3' d\tilde{Y}_t^2 \tag{5.12}$$

By identiffication of (5.11) and (5.12) we have:

$$\begin{cases} \left[\frac{\partial \rho}{\partial t} + \left(\sigma_3^2 \frac{\partial^2 \phi}{\partial^2 z}(t,z) \quad k(\beta \quad z)\frac{\partial \phi}{\partial z}(t,z) + k\phi(t,z)\right)\frac{\partial \rho}{\partial \phi} + \frac{1}{2}\sigma_2^2 \frac{\partial^2 \rho}{\partial^2 r}\right. \\ +\frac{1}{2}\left(\left(\mu(r_t,z) \quad \frac{1}{2}\sigma_1^2 \quad \rho_1\sigma_3\frac{\partial \phi}{\partial z}(t,z)\right)^2 + \left(a(b \quad r_t) \quad \sigma_3\rho_2\frac{\partial \phi}{\partial z}(t,z)\right)^2\right)\frac{\partial^2 \rho}{\partial^2 \phi} \\ \left.+ \sigma_2\left(a(b-r_t) - \sigma_3\rho_2\frac{\partial \phi}{\partial z}(t,z)\right)\frac{\partial^2 \rho}{\partial r \partial \phi} + r_t\rho(t, r_t, \phi(t,z))\right]X_t^{x,\hat{u}} - \rho(t, r_t, \phi(t,z))\hat{u}_t(r_t - \frac{1}{2}\sigma_1^2) \\ -u_t^0\rho(t, r_t, \phi(t,z))(r_t - R_t^0) + \sigma_1\hat{u}_t\left(\mu(r_t,z) - \frac{1}{2}\sigma_1^2 - \rho_1\sigma_3\frac{\partial \phi}{\partial z}(t,z)\right)\frac{\partial \rho}{\partial \phi} + \Theta_t \\ \quad = -r_t\rho(t, r_t, \phi(t,z))X_t^{x,\hat{u}} - r_t\psi(t, S_t, r_t, \phi(t,z)) & \text{(5.13a)} \\ \left[\mu(r_t,z) - \frac{1}{2}\sigma_1^2 - \rho_1\sigma_3\frac{\partial \phi}{\partial z}(t,z)\right]\frac{\partial \rho}{\partial \phi}X_t^{x,\hat{u}} + \rho(t, r_t, \phi(t,z))\hat{u}_t\sigma_1 + \beta_t^1 = \hat{q}_3(t,z) & \text{(5.13b)} \\ \left[\sigma_2\frac{\partial \rho}{\partial r} + \left(a(b-r_t) - \sigma_3\rho_2\frac{\partial \phi}{\partial z}(t,z)\right)\frac{\partial \rho}{\partial \phi}\right]X_t^{x,\hat{u}} + \beta_t^2 = \hat{q}_3'(t,z). \end{cases}$$

$$(5.13a)$$
$$\Updownarrow$$

$$\hat{u}_t = \frac{\left[\frac{\partial\rho}{\partial t} + L_Z^*\phi(t,z)\frac{\partial\rho}{\partial\phi} + \frac{1}{2}\sigma_2^2\frac{\partial^2\rho}{\partial^2 r} + \frac{1}{2}\left((M_1^*\phi(t,z))^2 + (M_2^*\phi(t,z))^2\right)\frac{\partial^2\rho}{\partial^2\phi} + \sigma_2 M_2^*\phi(t,z)\frac{\partial^2\rho}{\partial r\partial\phi} + 2r_t\rho(t,r_t,\phi(t,z))\right]X_t^{x,\hat{u}} + r_t\psi(t,S_t,r_t,\phi_t) + \Theta_t \quad u_t^0\rho(t,r_t,\phi(t,z))(r_t \quad R_t^0)}{\rho(t,r_t,\phi(t,z))\left(r_t \quad \frac{1}{2}\sigma_1^2\right) \quad \sigma_1 M_1^*\phi(t,z)\frac{\partial\rho}{\partial\phi}}. \tag{5.14}$$

We have by (5.13b) and the first order optimality condition (5.6):

$$\hat{u}_t = \frac{\left[\left(r_t - \frac{1}{2}\sigma_1^2\right)\rho(t,r_t,\phi(t,z) - \sigma_1 M_1^*\phi(t,z)\frac{\partial\rho}{\partial\phi}\right]X_t^{x,\hat{u}} + \left(r_t - \frac{1}{2}\sigma_1^2\right)\psi(t,S_t,r_t,\phi_t) - \sigma_1\beta_t^1}{\sigma_1^2\rho(t,r_t,\phi(t,z))} \tag{5.15}$$

The equations satisfied by $\rho(t,r_t,\phi(t,z))$ and $\psi(t,S_t,r_t,\phi_t)$ are obtained by identifying (5.14) and (5.15) as a first degree polynomial in $X_t^{x,\hat{u}}$. Thus, we have:

$$\begin{cases} \dfrac{\frac{\partial\rho}{\partial t} + L_Z^*\phi(t,z)\frac{\partial\rho}{\partial\phi} + \frac{1}{2}\sigma_2^2\frac{\partial^2\rho}{\partial^2 r} + \frac{1}{2}\left((M_1^*\phi(t,z))^2 + (M_2^*\phi(t,z))^2\right)\frac{\partial^2\rho}{\partial^2\phi} + \sigma_2 M_2^*\phi(t,z)\frac{\partial^2\rho}{\partial r\partial\phi} + 2r_t\rho(t,r_t,\phi(t,z))}{\rho(t,r_t,\phi(t,z))\left(r_t - \frac{1}{2}\sigma_1^2\right) - \sigma_1 M_1^*\phi(t,z)\frac{\partial\rho}{\partial\phi}} \\ = \dfrac{(r_t \quad \frac{1}{2}\sigma_1^2)\rho(t,r_t,\phi(t,z)) \quad \sigma_1 M_1^*\phi(t,z)\frac{\partial\rho}{\partial\phi}}{\sigma_1^2\rho(t,r_t,\phi(t,z))} & (5.16a) \\ \dfrac{\Theta_t + r_t\psi(t,S_t,r_t,\phi_t) - u_t^0\rho(t,r_t,\phi(t,z))(r_t - R_t^0)}{\rho(t,r_t,\phi(t,z))\left(r_t - \frac{1}{2}\sigma_1^2\right) - \sigma_1 M_1^*\phi(t,z)\frac{\partial\rho}{\partial\phi}} = \dfrac{\left(r_t - \frac{1}{2}\sigma_1^2\right)\psi(t,S_t,r_t,\phi_t) - \sigma_1\beta_t^1}{\sigma_1^2\rho(t,r_t,\phi(t,z))} & (5.16b) \end{cases}$$

$$(5.16a)$$
$$\Updownarrow$$

$$\begin{aligned} &\frac{\partial\rho}{\partial t} + L_Z^*\phi(t,z)\frac{\partial\rho}{\partial\phi} + \frac{1}{2}\sigma_2^2\frac{\partial^2\rho}{\partial^2 r} + \frac{1}{2}\left((M_1^*\phi(t,z))^2 + (M_2^*\phi(t,z))^2\right)\frac{\partial^2\rho}{\partial^2\phi} + \sigma_2 M_2^*\phi(t,z)\frac{\partial^2\rho}{\partial r\partial\phi} + 2r_t\rho(t,r_t,\phi(t,z)) \\ &= \left(\frac{(r_t \quad \frac{1}{2}\sigma_1^2)}{\sigma_1^2} - \frac{M_1^*\phi(t,z)\frac{\partial\rho}{\partial\phi}}{\sigma_1\rho(t,r_t,\phi(t,z))}\right)\left[(r_t - \frac{1}{2}\sigma_1^2)\rho(t,r_t,\phi(t,z)) - \sigma_1 M_1^*\phi(t,z)\frac{\partial\rho}{\partial\phi}\right] \end{aligned} \tag{5.17}$$

(5.17) is a parabolic PDE.With quadratic utility function $U(x) = -(x-\alpha_0)^2$ and terminal condition:

$$\rho(T,r_T,\phi(T,z)) = \quad 2\phi(T,z). \tag{5.18}$$

If for example we take $\frac{\partial\rho}{\partial\phi} = 0$, by Feynman-Kac representation formula, we have ρ solution of backward stochastic differential equation:

$$dY_t = \left[\left(\frac{(r_t \quad \frac{1}{2}\sigma_1^2)^2}{\sigma_1^2} - 2r_t\right)Y_t\right]dt + Z_t^2 d\tilde{Y}_t^2, \quad Y_T = -2\phi(T,z) \tag{5.19}$$

$$\rho(t,r) = \exp\left(\int_0^t\left(\frac{(r_s \quad \frac{1}{2}\sigma_1^2)^2}{\sigma_1^2} - 2r_s\right)ds\right)\mathbb{E}\left[-2\phi(T,z)\exp\left(-\int_0^T\left[\left(\frac{r_s \quad \frac{1}{2}\sigma_1^2}{\sigma_1}\right)^2 - 2r_s\right]ds\right) \; r_t = r\right] \tag{5.20}$$

Equation of ψ

$$(5.16b)$$
$$\Updownarrow$$

$$\begin{aligned} r_t\psi(t,S_t,r_t,\phi(t,z)) + \Theta_t = &\left[\frac{(r_t \quad \frac{1}{2}\sigma_1^2)}{\sigma_1^2} - \frac{M_1^*\phi(t,z)\frac{\partial\rho}{\partial\phi}}{\sigma_1\rho(t,r_t,\phi(t,z))}\right]\left[(r_t - \frac{1}{2}\sigma_1^2)\psi(t,S_t,r_t,\phi(t,z)) - \sigma_1\beta_t^1\right] \\ &+ u_t^0\rho(t,r_t,\phi(t,z))(r_t - R_t^0) \end{aligned}$$

$$\begin{aligned} \Theta_t + \left[\frac{(r_t - \frac{1}{2}\sigma_1^2)}{\sigma_1} \quad \frac{M_1^*\phi(t,z)\frac{\partial\rho}{\partial\phi}}{\rho(t,r_t,\phi(t,z))}\right]\beta_t^1 = &\left[\frac{(r_t - \frac{1}{2}\sigma_1^2)^2}{\sigma_1^2} \quad \frac{(r_t - \frac{1}{2}\sigma_1^2)M_1^*\phi(t,z)\frac{\partial\rho}{\partial\phi}}{\sigma_1\rho(t,r_t,\phi(t,z))} \quad r_t\right]\psi(t,S_t,r_t,\phi(t,z)) \\ &+ u_t^0\rho(t,r_t,\phi(t,z))(r_t - R_t^0) \end{aligned} \tag{5.21}$$

Let's calculate Θ_t and β^1

$$\begin{aligned}
d\psi(t,S_t,r_t,\phi(t,z)) =& \frac{\partial\psi}{\partial t}(t,S_t,r_t,\phi(t,z))dt + \left[\frac{\sigma_1^2 S_t}{2}dt + \sigma_1 S_t d\tilde{Y}_t^1\right]\frac{\partial\psi}{\partial S}(t,S_t,r_t,\phi(t,z)) \\
&+ \sigma_2\frac{\partial\psi}{\partial r}(t,S_t,r_t,\phi(t,z))d\tilde{Y}_t^2 + \left[\left[\sigma_3^2\frac{\partial^2\phi}{\partial^2 z}(t,z) - k(\beta - z)\frac{\partial\phi}{\partial z}(t,z) + k\phi(t,z)\right]dt\right. \\
&+ \left.\left[\mu(r_t,z) - \frac{1}{2}\sigma_1^2 - \rho_1\sigma_3\frac{\partial\phi}{\partial z}(t,z)\right]d\tilde{Y}_t^1 + \left[a(b-r_t) - \sigma_3\rho_2\frac{\partial\phi}{\partial z}(t,z)\right]d\tilde{Y}_t^2\right]\frac{\partial\psi}{\partial\phi}(t,S_t,r_t,\phi(t,z)) \\
&+ \frac{1}{2}\sigma_1^2 S_t^2\frac{\partial^2\psi}{\partial^2 S}(t,S_t,r_t,\phi(t,z))dt + \frac{1}{2}\sigma_2^2\frac{\partial^2\psi}{\partial^2 r}(t,S_t,r_t,\phi(t,z))dt \\
&+ \frac{1}{2}\left[\left(\mu(r_t,z) - \frac{1}{2}\sigma_1^2 - \rho_1\sigma_3\frac{\partial\phi}{\partial z}(t,z)\right)^2 + \left(a(b-r_t) - \sigma_3\rho_2\frac{\partial\phi}{\partial z}(t,z)\right)^2\right]\frac{\partial^2\psi}{\partial^2\phi}(t,S_t,r_t,\phi(t,z))dt \\
&+ \sigma_1 S_t\left[\mu(r_t,z) - \frac{1}{2}\sigma_1^2 - \rho_1\sigma_3\frac{\partial\phi}{\partial z}(t,z)\right]\frac{\partial^2\psi}{\partial S\partial r}(t,S_t,r_t,\phi(t,z))dt \\
&+ \sigma_2\left[a(b-r_t) - \sigma_3\rho_2\frac{\partial\phi}{\partial z}(t,z)\right]\frac{\partial^2\psi}{\partial r\partial\phi}(t,r_t,r_t,\phi(t,z))dt
\end{aligned} \tag{5.22}$$

$$\begin{aligned}
\Theta_t =& \frac{\partial\psi}{\partial t}(t,S_t,r_t,\phi(t,z)) + \frac{\sigma_1^2 S_t}{2}\frac{\partial\psi}{\partial S}(t,S_t,r_t,\phi(t,z)) \\
&+ \left[\sigma_3^2\frac{\partial^2\phi}{\partial^2 z}(t,z) \quad k(\beta \quad z)\frac{\partial\phi}{\partial z}(t,z) + k\phi(t,z)\right]\frac{\partial\psi}{\partial\phi}(t,S_t,r_t,\phi(t,z)) \\
&+ \frac{1}{2}\sigma_1^2 S_t^2\frac{\partial^2\psi}{\partial^2 S}(t,S_t,r_t,\phi(t,z)) + \frac{1}{2}\sigma_2^2\frac{\partial^2\psi}{\partial^2 r}(t,S_t,r_t,\phi(t,z)) \\
&+ \frac{1}{2}\left[\left(\mu(r_t,z) \quad \frac{1}{2}\sigma_1^2 \quad \rho_1\sigma_3\frac{\partial\phi}{\partial z}(t,z)\right)^2 + \left(a(b \quad r_t) \quad \sigma_3\rho_2\frac{\partial\phi}{\partial z}(t,z)\right)^2\right]\frac{\partial^2\psi}{\partial^2\phi}(t,S_t,r_t,\phi(t,z)) \\
&+ \sigma_1 S_t\left[\mu(r_t,z) \quad \frac{1}{2}\sigma_1^2 \quad \rho_1\sigma_3\frac{\partial\phi}{\partial z}(t,z)\right]\frac{\partial^2\psi}{\partial S\partial r}(t,S_t,r_t,\phi(t,z)) \\
&+ \sigma_2\left[a(b-r_t) - \sigma_3\rho_2\frac{\partial\phi}{\partial z}(t,z)\right]\frac{\partial^2\psi}{\partial r\partial\phi}(t,S_t,r_t,\phi(t,z))
\end{aligned}$$

$$\beta_t^1 = \sigma_1 S_t\frac{\partial\psi}{\partial S}(t,S_t,r_t,\phi(t,z)) + \left[\mu(r_t,z) \quad \frac{1}{2}\sigma_1^2 \quad \rho_1\sigma_3\frac{\partial\phi}{\partial z}(t,z)\right]\frac{\partial\psi}{\partial\phi}(t,S_t,r_t,\phi(t,z))$$

Let $\phi''(t,z) = \frac{\partial^2\phi}{\partial^2 z}(t,z), \phi'(t,z) = \frac{\partial\phi}{\partial z}(t,z)$.
After calculations, we obtain with utility function $U(x) = \quad (x \quad \alpha_0)^2$, ψ solution of the PDE:

$$\begin{aligned}
&\frac{\partial\psi}{\partial t}(t,S,r,\phi) + \frac{\sigma_1^2 S^2}{2}\frac{\partial^2\psi}{\partial^2 S}(t,S,r,\phi) + \frac{\sigma_2^2}{2}\frac{\partial^2\psi}{\partial^2 r}(t,S,r,\phi) + \frac{1}{2}\left[(M_1^*\phi(t,z))^2 + (M_2^*\phi(t,z))^2\right]\frac{\partial^2\psi}{\partial^2\phi}(t,S,r,\phi) \\
&+ \sigma_1 S M_1^*\phi(t,z)\frac{\partial^2\psi}{\partial S\partial\phi}(t,S,r,\phi) + \sigma_2 M_2^*\phi(t,z)\frac{\partial^2\psi}{\partial r\partial\phi}(t,S,r,\phi) + S\left[r \quad \frac{\sigma_1 M_1^*\phi(t,z)\frac{\partial\rho}{\partial\phi}}{\rho(t,r,\phi(t,z))}\right]\frac{\partial\psi}{\partial S}(t,S,r,\phi) \\
&+ \left[L_Z^*\phi(t,z) + M_1^*\phi(t,z)\left[\frac{(r-\frac{1}{2}\sigma_1^2)}{\sigma_1} \quad \frac{M_1^*\phi(t,z)\frac{\partial\rho}{\partial\phi}}{\rho(t,r,\phi(t,z))}\right]\right]\frac{\partial\psi}{\partial\phi}(t,S,r,\phi) \\
&= \left[\frac{\left(r \quad \frac{1}{2}\sigma_1^2\right)^2}{\sigma_1^2} - \frac{M_1^*\phi(t,z)(r \quad \frac{1}{2}\sigma_1^2)\frac{\partial\rho}{\partial\phi}}{\sigma_1\rho(t,r,\phi)} - r\right]\psi(t,S,r,\phi) + u_t^0\rho(t,r_t,\phi(t,z))(r_t - R_t^0)
\end{aligned} \tag{5.23}$$

with terminal condition

$$\psi(T,S,r,\phi) = \quad 2\int_{\mathbb{R}}\phi(T,z)\left(\Pi(S_T,B(z)+b) \quad \alpha_0\right)d\mathbb{P}_b \tag{5.24}$$

Let $M_\rho^*\phi(t,z) = \frac{M_1^*\phi(t,z)\frac{\partial\rho}{\partial\phi}}{\rho(t,r,\phi(t,z))}$. Considering the elliptic operator associated with the state processes given by:

$$\mathcal{L}_\phi^* = \frac{1}{2}\left[(M_1^*\phi)^2 + (M_2^*\phi)^2\right]\partial^2_{\phi\phi} + \left[L_Z^*\phi + M_1^*\phi\left(\frac{(r-\frac{1}{2}\sigma_{1)}^2}{\sigma_1} \quad M_\rho^*\phi\right)\right]\partial_\phi \tag{5.25}$$

(5.23) becomes:

$$\psi_t+\frac{\sigma_1^2S^2}{2}\psi_{SS}+\frac{\sigma_2^2}{2}\psi_{rr}+<\mathcal{L}_\phi^*,\psi>+<M_1^*\phi,\ \sigma_1SD_\phi\psi_S>+<M_2^*\phi,\ \sigma_2D_\phi\psi_r>+<S(r-\sigma_1M_\rho^*\phi),\ \psi_S>$$
$$+<r-\left(\frac{r\quad\frac{1}{2}\sigma_1^2}{\sigma_1}\right)^2+\left(\frac{r\quad\frac{1}{2}\sigma_1^2}{\sigma_1}\right)M_\rho^*\phi,\psi>+u_t^0\rho(t,r_t,\phi(t,z))(r_t-R_t^0)=0 \tag{5.26}$$

With D_ϕ the partial derivative with respect to ϕ.
By Feynman-Kac representation formula, under regularity assumptions we have ψ solution of BSDEs:

$$\begin{aligned}Y_t^{s,S,r,\phi}=&-2\int_{\mathbb{R}}\phi(T,z)\left(\Pi(S_T,B(z)+\bar{b})-\alpha_0\right)d\mathbb{P}_{\bar{b}}\\&+\int_t^T\left[\left(r_u-\left(\frac{r_u-\frac{1}{2}\sigma_1^2}{\sigma_1}\right)^2+\left(\frac{r_u-\frac{1}{2}\sigma_1^2}{\sigma_1}\right)M_\rho^*\phi\right)Y_u^{s,S,r,\phi}+\left(\frac{r_u-\frac{1}{2}\sigma_1^2}{\sigma_1^2}-M_\rho^*\phi\right)Z_u^1\right)\\&-u_t^0\rho(t,r_t,\phi(t,z))(r_t-R_t^0)]du-\int_t^TZ_u^1d\tilde{Y}_u^1-\int_t^TZ_u^2d\tilde{Y}_u^2,\ \ 0\le t\le T.\end{aligned} \tag{5.27}$$

where $S=S_t,r=r_t,\phi=\phi(t,z)$.

$$\psi(t,S,r,\phi)=Y_t^{t,S,r,\phi}$$

$$\begin{aligned}\psi(t,S,r,\phi)=&\exp\left\{\int_0^t\left(r_u-\left(\frac{r_u-\frac{1}{2}\sigma_1^2}{\sigma_1}\right)^2+\left(\frac{r_u-\frac{1}{2}\sigma_1^2}{\sigma_1}\right)M_\rho^*\phi\right)du-\frac{1}{2}\int_0^t\left(\frac{r_u-\frac{1}{2}\sigma_1^2}{\sigma_1^2}-M_\rho^*\phi\right)^2du\right.\\&\left.+\int_0^t\left(\frac{r_u-\frac{1}{2}\sigma_1^2}{\sigma_1^2}-M_\rho^*\phi\right)d\tilde{Y}_t^1\right\}\times\tilde{\mathbb{E}}\left[\left(-2\int_{\mathbb{R}}\phi(T,z)\left(\Pi(S_T,B(z)+\bar{b})-\alpha_0\right)d\mathbb{P}_{\bar{b}}\right)\right.\\&\exp\left\{-\int_0^T\left(r_u-\left(\frac{r_u-\frac{1}{2}\sigma_1^2}{\sigma_1}\right)^2+\left(\frac{r_u-\frac{1}{2}\sigma_1^2}{\sigma_1}\right)M_\rho^*\phi\right)du-\frac{1}{2}\int_0^T\left(\frac{r_u-\frac{1}{2}\sigma_1^2}{\sigma_1^2}-M_\rho^*\phi\right)^2du\right.\\&\left.-\int_0^T\left(\frac{r_u-\frac{1}{2}\sigma_1^2}{\sigma_1^2}-M_\rho^*\phi\right)d\tilde{Y}_t^1\right\}-\int_t^T\left(u_u^0\rho(u,r_u,\phi(u,z))(r_u-R_u^0)\right)\\&\exp\left\{\int_0^u\left(r_s-\left(\frac{r_s-\frac{1}{2}\sigma_1^2}{\sigma_1}\right)^2+\left(\frac{r_s-\frac{1}{2}\sigma_1^2}{\sigma_1}\right)M_\rho^*\phi\right)ds-\frac{1}{2}\int_0^u\left(\frac{r_s-\frac{1}{2}\sigma_1^2}{\sigma_1^2}-M_\rho^*\phi\right)^2ds\right.\\&\left.\left.+\int_0^u\left(\frac{r_u-\frac{1}{2}\sigma_1^2}{\sigma_1^2}-M_\rho^*\phi\right)d\tilde{Y}_s^1\right\}du|S_t=S,r_t=r,\phi(t,z)=\phi\right].\end{aligned} \tag{5.28}$$

Optimal control in the presence of the option is given by:

$$\begin{aligned}\hat{u}_t=&\frac{r_t-\frac{1}{2}\sigma_1^2}{\sigma_1^2\rho(t,r_t,\phi(t,z))}\exp\left\{\int_0^t\left(r_u-\left(\frac{r_u-\frac{1}{2}\sigma_1^2}{\sigma_1}\right)^2+\left(\frac{r_u-\frac{1}{2}\sigma_1^2}{\sigma_1}\right)M_\rho^*\phi\right)du-\frac{1}{2}\int_0^t\left(\frac{r_u-\frac{1}{2}\sigma_1^2}{\sigma_1^2}-M_\rho^*\phi\right)^2du\right.\\&\left.+\int_0^t\left(\frac{r_u-\frac{1}{2}\sigma_1^2}{\sigma_1^2}-M_\rho^*\phi\right)d\tilde{Y}_t^1\right\}\times\tilde{\mathbb{E}}\left[\left(-2\int_{\mathbb{R}}\phi(T,z)\left(\Pi(S_T,B(z)+\bar{b})-\alpha_0\right)d\mathbb{P}_{\bar{b}}\right)\right.\\&\exp\left\{-\int_0^T\left(r_u-\left(\frac{r_u-\frac{1}{2}\sigma_1^2}{\sigma_1}\right)^2+\left(\frac{r_u-\frac{1}{2}\sigma_1^2}{\sigma_1}\right)M_\rho^*\phi\right)du-\frac{1}{2}\int_0^T\left(\frac{r_u-\frac{1}{2}\sigma_1^2}{\sigma_1^2}-M_\rho^*\phi\right)^2du\right.\\&\left.-\int_0^T\left(\frac{r_u-\frac{1}{2}\sigma_1^2}{\sigma_1^2}-M_\rho^*\phi\right)d\tilde{Y}_t^1\right\}-\int_t^T\left(u_u^0\rho(u,r_u,\phi(u,z))(r_u-R_u^0)\right)\\&\exp\left\{\int_0^u\left(r_s-\left(\frac{r_s-\frac{1}{2}\sigma_1^2}{\sigma_1}\right)^2+\left(\frac{r_s-\frac{1}{2}\sigma_1^2}{\sigma_1}\right)M_\rho^*\phi\right)ds-\frac{1}{2}\int_0^u\left(\frac{r_s-\frac{1}{2}\sigma_1^2}{\sigma_1^2}-M_\rho^*\phi\right)^2ds\right.\\&\left.\left.+\int_0^u\left(\frac{r_u-\frac{1}{2}\sigma_1^2}{\sigma_1^2}-M_\rho^*\phi\right)d\tilde{Y}_s^1\right\}du|S_t=S,r_t=r,\phi(t,z)=\phi\right].\\&-\frac{1}{\sigma_1\rho(t,r_t,\phi(t,z))}\left[\sigma_1S_t\frac{\partial\psi}{\partial S}(t,S_t,r_t,\phi(t,z))+\left[\mu(r_t,z)-\frac{1}{2}\sigma_1^2-\rho_1\sigma_3\frac{\partial\phi}{\partial z}(t,z)\right]\frac{\partial\psi}{\partial\phi}(t,S_t,r_t,\phi(t,z))\right]\\&+\frac{\left[(r_t-\frac{1}{2}\sigma_1^2)\,\rho(t,r_t,\phi(t,z)-\sigma_1M_1^*\phi(t,z)\frac{\partial\rho}{\partial\phi}\right]}{\sigma_1^2\rho(t,r_t,\phi(t,z))}X_t^{x,\hat{u}}\quad\forall t\in[0,T]\end{aligned} \tag{5.29}$$

In the absence of the option, $\Pi(S_T, B(z)+B) = 0$ and the function ψ does not depend on S_t therefore, the optimal control is given by:

$$\begin{aligned}
\hat{u}_t^0 =& \frac{r_t - \frac{1}{2}\sigma_1^2}{\sigma_1^2 \rho(t, r_t, \phi(t,z))} \exp\left\{ \int_0^t \left(r_u - \left(\frac{r_u - \frac{1}{2}\sigma_1^2}{\sigma_1}\right)^2 + \left(\frac{r_u - \frac{1}{2}\sigma_1^2}{\sigma_1}\right) M_\rho^* \phi) \right) du - \frac{1}{2}\int_0^t \left(\frac{r_u - \frac{1}{2}\sigma_1^2}{\sigma_1^2} - M_\rho^*\phi\right)^2 du \right. \\
&\left. + \int_0^t \left(\frac{r_u - \frac{1}{2}\sigma_1^2}{\sigma_1^2} - M_\rho^*\phi\right) d\tilde{Y}_t^1 \right\} \times \tilde{\mathbb{E}}\left[\left(-2\int_{\mathbb{R}} \phi(T,z)\left(\Pi(S_T, B(z)+\bar{b}) - \alpha_0\right) d\mathbb{P}_{\bar{b}} \right) \right. \\
&\exp\left\{ -\int_0^T \left(r_u - \left(\frac{r_u - \frac{1}{2}\sigma_1^2}{\sigma_1}\right)^2 + \left(\frac{r_u - \frac{1}{2}\sigma_1^2}{\sigma_1}\right) M_\rho^* \phi) \right) du - \frac{1}{2}\int_0^T \left(\frac{r_u - \frac{1}{2}\sigma_1^2}{\sigma_1^2} - M_\rho^*\phi\right)^2 du \right. \\
&\left. - \int_0^T \left(\frac{r_u - \frac{1}{2}\sigma_1^2}{\sigma_1^2} - M_\rho^*\phi\right) d\tilde{Y}_t^1 \right\} - \int_t^T \left(u_u^0 \rho(u, r_u, \phi(u,z))(r_u - R_u^0) \right) \\
&\exp\left\{ \int_0^u \left(r_s - \left(\frac{r_s - \frac{1}{2}\sigma_1^2}{\sigma_1}\right)^2 + \left(\frac{r_s - \frac{1}{2}\sigma_1^2}{\sigma_1}\right) M_\rho^* \phi) \right) ds - \frac{1}{2}\int_0^u \left(\frac{r_s - \frac{1}{2}\sigma_1^2}{\sigma_1^2} - M_\rho^*\phi\right)^2 ds \right. \\
&\left.\left. + \int_0^u \left(\frac{r_u - \frac{1}{2}\sigma_1^2}{\sigma_1^2} - M_\rho^*\phi\right) d\tilde{Y}_s^1 \right\} du | S_t = S, r_t = r, \phi(t,z) = \phi \right]. \\
&- \left[\frac{\mu(r_t, z) - \frac{1}{2}\sigma_1^2 - \rho_1\sigma_3 \frac{\partial\phi}{\partial z}(t,z)}{\sigma_1 \rho(t, r_t, \phi(t,z))}\right] \frac{\partial \psi_0}{\partial \phi}(t, S_t, r_t, \phi(t,z)) + \frac{\left[\left(r_t - \frac{1}{2}\sigma_1^2\right)\rho(t, r_t, \phi(t,z) - \sigma_1 M_1^*\phi(t,z)\frac{\partial\rho}{\partial\phi}\right]}{\sigma_1^2 \rho(t, r_t, \phi(t,z))} X_t^{x,\hat{u}} \\
&\forall t \in [0,T]
\end{aligned} \tag{5.30}$$

this time with ψ_0 solution of the PDE:

$$\begin{aligned}
&\frac{\partial \psi_0}{\partial t}(t,S,r,\phi) + \frac{\sigma_2^2}{2}\frac{\partial^2 \psi_0}{\partial^2 r}(t,S,r,\phi) + \frac{1}{2}\left[(M_1^*\phi(t,z))^2 + (M_2^*\phi(t,z))^2\right]\frac{\partial^2\psi_0}{\partial^2\phi}(t,S,r,\phi) + \sigma_2 M_2^*\phi(t,z)\frac{\partial^2\psi_0}{\partial r \partial\phi}(t,S,r,\phi) \\
&+ \left[L_Z^*\phi(t,z) + M_1^*\phi(t,z)\left[\frac{(r - \frac{1}{2}\sigma_1^2)}{\sigma_1} - \frac{M_1^*\phi(t,z)\frac{\partial\rho}{\partial\phi}}{\rho(t,r,\phi(t,z))}\right]\right]\frac{\partial\psi_0}{\partial\phi}(t,S,r,\phi) \\
&= \left[\frac{\left(r - \frac{1}{2}\sigma_1^2\right)^2}{\sigma_1^2} - \frac{M_1^*\phi(t,z)(r - \frac{1}{2}\sigma_1^2)\frac{\partial\rho}{\partial\phi}}{\sigma_1\rho(t,r,\phi)} - r\right]\psi_0(t,S,r,\phi) + u_t^0\rho(t,r_t,\phi(t,z))(r_t - R_t^0)
\end{aligned} \tag{5.31}$$

with terminal condition

$$\psi_0(T, r_T, \phi_T) = 2\alpha_0\phi(T,z). \tag{5.32}$$

In other words

$$\begin{aligned}
\psi_{0_t} + \frac{\sigma_2^2}{2}\psi_{0_{rr}} + < \mathcal{L}_\phi^*, \psi_0 > + < M_2^*\phi,\ \sigma_2 D_\phi \psi_{0_r} > + < r \quad \left(\frac{r - \frac{1}{2}\sigma_1^2}{\sigma_1}\right)^2 + \left(\frac{r - \frac{1}{2}\sigma_1^2}{\sigma_1}\right) M_\rho^*\phi, \psi_0 > \\
+ u_t^0\rho(t, r_t, \phi(t,z))(r_t - R_t^0) = 0
\end{aligned} \tag{5.33}$$

$$\frac{\partial H}{\partial u^0} = (R_t^0 - r_t)p_3(t,z)$$

H is a linear function from u^0 with coefficient $(R_t^0 \quad r_t)\hat{p}_3(t,z)$ thus a harmonic function of u^0, so according to the maximum theorem reaches the maximum on the bounds. Assume $u_t^0 \in [0, U_0^{max}]$ the set of all possible values of u_t^0 where $U_0^{max} \in \mathbb{R}$. The equation satisfied by $p_3(t,z)$ is a linear BSDE. A representation of the solution in explicit form is given:

$$p_3(t,z) = \quad 2\exp\left(\int_0^t \quad r_s ds\right)\mathbb{E}\left[\phi(T,z)\exp\left(\int_0^T r_s ds\right)\int_{\mathbb{R}}\left(X_T^{x,\hat{u}} + \Pi(S_T, B(z) + \bar{b} \quad \alpha_0)\right) d\mathbb{P}_b | \mathfrak{F}_t\right] \tag{5.34}$$

Thus an optimal investment strategy in non-risky assets is given by:

$$\hat{u}_t^0 = \begin{cases} 0 & \text{if}(R_t^0 \quad r_t)\mathbb{E}\left[\phi(T,z)\exp\left(\int_0^T r_s ds\right)\int_{\mathbb{R}}\left(X_T^{x,\hat{u}} + \Pi(S_T, B(z) + \bar{b} \quad \alpha_0)\right) d\mathbb{P}_b \ \mathfrak{F}_t\right] \geq 0 \\ U_0^{max} & \text{if}(R_t^0 - r_t)\mathbb{E}\left[\phi(T,z)\exp\left(\int_0^T r_s ds\right)\int_{\mathbb{R}}\left(X_T^{x,\hat{u}} + \Pi(S_T, B(z) + \bar{b} - \alpha_0)\right) d\mathbb{P}_b \ \mathfrak{F}_t\right] < 0 \end{cases} \tag{5.35}$$

An optimal strategy for investing in risk-free assets is to invest as much as possible when the return R_t^0 of this asset is greater than the stochastic interest rate r_t which is the yield of "semi-risky" assets and mathematical expectation

$$\mathbb{E}\left[\phi(T,z)\exp\left(\int_0^T r_s ds\right)\int_{\mathbb{R}}\left(X_T^{x,\hat{u}}+\Pi(S_T,B(z)+\bar{b}\quad\alpha_0)\right)d\mathbb{P}_b\ \mathfrak{F}_t\right]$$

is negative.

Theorem 2. *(Optimal control)*
An optimal control $\hat{\hat{u}}$ (resp $\hat{\hat{u}}^0$) to the terminal utility maximization problem under partial information (2.7), (2.8)*(resp* (2.9)*) is given by* (5.29), (5.23) *and* (5.35) *(resp by* (5.30), (5.31) *and* (5.35)*).*

6 Conclusion

In this article, for the hedging of a European option with a portfolio made up of three financial assets, we have given an optimal investment strategy in each of these assets. We have assumed that: the interest rate is a stochastic variable with the dynamic of Vasicek model, the payoff of the option does not depend only on the horizon price of the risky asset but also an unobservable variable. Given the dynamics of the portfolio, we have given the optimal strategies for investing in risky and non-risky assets. The passage of partial information to full information is done using the filtering theory with the Girsanov theorem and the Zakai equations. We used a stochastic maximum principle established with the backward stochastic differential equations. This was to determine optimal control which was an optimal portfolio management strategy. We mainly used the adjunct equations of the stochastic maximum principle with the Hamiltonian, the first order optimality condition. In the future, we intend to generalize these results of higher dimensions with a portfolio made up of several risky, non-risky assets, semi-risky as well as with other dynamics of interest rates and digital simulations.

References

[1] AN, T. T. K., PROSKE, F. AND RUBTSOV, M. (2007). A spde maximum principle for stochastic differential games under partial information with application to optimal portfolios on fixed income markets. *Dept. of Math./CMA University of Oslo*.

[2] BENSOUSSAN, A. (2004). *Stochastic Control of Partially Observable Systems* cambridge university press ed. Cambridge, University Paris Dauphine.

[3] BOUGHAMOURA, W., PANDEY, A. N. AND TRABELSI, F. (2011). Pricing and hedging of asian option under jumps. *IAENG International Journal of Applied Mathematics* **41,**.

[4] BURKHOLDER, D. L., PARDOUX, E. AND SZNITMAN, A.-S. (1989). Ecole d'eté de probabilités de saint-flour xix - 1989. In *Lecture Notes in Mathematics*. ed. C.-F. Paul-Louis Hennequin Université Blaise Pascal. Université Blaise Pascal, Clermont-Ferrand. Springer-Verlag.

[5] CARMONA, R. AND LUDKOVSKI, M. (2004). Convenience yield model with partial observations and exponential utility. *Technical report, Princeton University* https:// www.princeton.edu/rcarmona/download/fe/IJTAF2.pdf.

[6] CETIN, U., JARROW, R., PROTTER, P. AND YILDIRIM, Y. (2004). Modeling credit risk with partial information. *The Annals of Applied Probability,Cornell University* **14,** 1167–1178.

[7] DEUGOUE, G. AND WOUKENG, J. L. (2015). Sigma-convergence of semilinear stochastic wave equations. *arXiv preprintarXiv:1510.06624*.

[8] FÄOLLMER, H. AND SCHWEIZER, M. (1991). Hedging of contingent claims under incomplete information. *Applied Stochastic Analysis", Stochastics Monographs* **5,** 389–414.

[9] FOUETIO, A. AND WOUKENG, J. L. (2018). Homogenization of hyperbolic damped stochastic wave equations. *Acta Mathematica Sinica, English Series* **34,** 233–254.

[10] GIBSON, R. AND SCHWARTZ., E. S. (1990). Stochastic convenience yield and the pricing of oil contingent claims. *Journal of finance* **14,** 959–976.

[11] HAUSSMANN, U. G. (1985). L'equation de zakai et le problème séparé du contrôle optimal stochastique. In *Séminaire de probabilité(Strasbourg)*. ed. Springer. vol. 19. pp. 37–62.

[12] HULL, J. AND WHITE, A. (1987). The pricing of options on assets with stochastic volatilities. *The Journal Of Finance* **XVII,**.

[13] KOHLMANN, M. AND XIONG, D. (1991). The mean-variance hedging of a default option with partial information. *National Natural Science Foundation of China under Grant*.

[14] LASKRY, J. M. AND LIONS, P.-L. (1999). Contrôle stochastique avec informations partielles et application à la finance. *C. R. Acad. Sci. Paris* **328,** 1003–1010.

[15] LIM, T.-S. (2010). Quelques applications du contrôle stochastique aux risques de défaut et de liquidité. *PhD thesis.* Université Paris Diderot-Paris VII UFR de Mathématiques.

[16] MATARAMVURA, S. AND ØKSENDAL, B. (2008). Risk minimizing portfolios and hjbi equations for stochastic differential games. *Stochastics An International Journal of Probability and Stochastic Processes* **80,** 317–337.

[17] MONOYIOS, M. (2008). *Optimal investment and hedging under partial information* tutorial lectures for the session on inverse and partial information problems ed. Mathematical Institute, University of Oxford www.maths.ox.ac.uk/ monoyios.

[18] ØKSENDAL, B. AND SULEM, A. (2012). Risk indifference pricing in jump diffusion markets. *Mathematical Finance* **19,** 619–637.

[19] PHAM, H. (2003). *Contrôle Optimal Stochastique et Application en Finance*. Université de Paris 7.

[20] PHAM, H. (2007). *Optimisation et contrôle stochastique appliqués à la finance* vol. 61. Mathematiques et Applications, http://fribok.blogspot.com/.

[21] PLATEN, E. AND RUNGGALDIER, W. J. (2007). A benchmark approach to portfolio optimization under partial information. *Quantitative Finance Research Centre* **191,**.

[22] PRATO, G. D. AND ZABCZYK, J. (2014). *Stochastic Equations In Infinite Dimensions* second edition ed. Cambridge University Press.

[23] RAZAFIMANDIMBY, P., M, S. AND WOUKENG, J. L. (2012). Homogenization of a stochastic nonlinear reaction-diffusion equation with a large reaction term: The almost periodic framework. *J. Math. Anal. Appl* **394,** 186–212.

[24] RAZAFIMANDIMBY, P. A. AND WOUKENG, J. L. (2013). Homogenization of non linear stochastic partial differential equations in a general ergodic environment. *Stochastic Anal. Appl* **31,** 755–784.

[25] RUNGGALDIER, W. (2008). *Optimization problems in finance under full and partial information* tutorial for the special semester on stochastics ed. University of Padova, Italy www.math.unipd.it/runggaldier.

[26] SOCGNIA, V. K. AND PAMEN, O. M. (2017.). A maximum principle for controlled stochastic factor model. *ESAIM: Control, Optimisation and Calculus of Variations* **24,** 495–517.

[27] WOUKENG, J. L. (2015). Homogenization in algebra with mean value. banach. *J. Math. Anal* **9,** 142–182.

[28] WU, M., LU, J., HUANG, N. J. AND CHENGDU (2016). On european option pricing under partial information. *Applications of Mathematics* **61,** 61–77.