Application of Legendre Wavelets on Second Order Differential Equations

Mani Sharma[1], Dr. Chitra Singh[2]

[1]Research Scholar, Rabindranath Tagore University Bhopal (M.P), India
mssharmamani108[at]gmail.com

[2]Associate Professor Mathematics, Rabindranath Tagore University Bhopal (M.P), India
chitra.singh[at]aisectuniversity.ac.in

Abstract: ***In this research paper, we procreate a framework to get ordinary differential equations (ODEs) involving fractional order derivatives using the techniques of Legendre wavelets. By the properties of Legendre wavelets we analyze the application of second order linear differential equations to the solution of algebraic equations. It demonstrates by the illustration and proves the validity and applicability of the technique.***

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1. Introduction

In ordinary differential equations involving a fractional derivatives are used to various systems of calculus which an important engineering application such as viscoelastic damping [1–3]. Other application of fractional derivatives in control theory [4] and application of other various field are found in [5, 6].

In these years, mathematicians and physicists have intent the considerable effort to find numerical and analytical methods for solving fractional differential equations. In numerical and analytical methods have included the finite difference method [7–9], Adomian decomposition method [10–14], variation AL iteration method [15–18], homotopy perturbation method [19–22], generalized differential transform method [23–26], homotopy analysis method [27, 28], and other methods [1, 29].

On this research paper we extend the application of the Legendre wavelet approximations to solve linear and nonlinear differential equations of fractional order.

There are many different types of definitions of fractional calculus. For example, the Riemann–Liouvilleintegral operator [5] of order n is defined by

$$(J^n f)(x) = \begin{cases} \frac{1}{\Gamma(n)} \int_0^x \frac{f(t)}{(x-t)^{1-n}} dt, & n \geq 0, t > 0 \\ f(x), & n = 0 \end{cases} \quad (1)$$

And its fractional Derivative of order $\alpha (\alpha \geq 0)$ is used

$$(D_1^n f)(x) = \left(\frac{d}{dx}\right)^m (J^{m-n} f)(x),\ n \geq 0, m-1 < \alpha < m \quad (2)$$

Where n is an integer. For Riemann–Liouville definition, one has

$$J^n x^m = \frac{\Gamma(m+1)}{\Gamma(m+1+n)} x^{m+n} \quad (3)$$

$$(D^n f)(x) = \begin{cases} \frac{1}{\Gamma(m-n)} \int_0^x \frac{f^{(m)}(t)}{(x-t)^{n-m+1}} dt & (n \geq 0, m-1 < n < m) \\ \frac{\partial^m f(x)}{\partial x^m} & n = m \end{cases} \quad (4)$$

Where m is an integer.

$$(J^n D^n f) = f(x) - \sum_{k=0}^{m-1} \frac{f^{(k)} 0^+}{k!} x^k \quad , \quad (x \geq 0, m-1 < n < m) \quad (5)$$

Where m is an integers.

Legendre Wavelet theory is a vast area in mathematical research work. In Section 2, Describe the properties of wavelets and Legendre wavelets.. In Section 3 ,solve the considering numerical examples in our research work.

2. The following properties of Legendre Wavelet are:

Wavelets constitute a family of functions which constructed from the dilation and translation of a single function is called the mother wavelet. The dilation parameter is a and the translation parameter $b$it vary continuously, following family of continuous wavelets as [18]

$$\psi_{a,b}(t) = |a|^{\frac{-1}{2}} \psi\left(\frac{t-b}{a}\right) \qquad a, b \in R, a \neq 0. \quad (6)$$

The restricted parameters of a and b with discrete values as $a = a_0^k$, $b = nb_0 a_0^k$, $a_0 > 1, b_0 > 0$ and n and k are positive integers, The following family of discrete wavelets are as

$$\psi_{k,n}(t) = |a|^{\frac{k}{2}}\psi\,(a_0^k t - nb_0) \tag{7}$$

Where $\psi_{k,n}(t)$ have a wavelet basis for $L^2(R)$. When $a_0 = 2$ and $b_0 = 1$, and also$\psi_{k,n}(t)$ forms an orthonormal basis [33].

Legendre wavelet $\psi_{(n,m)}(t) = \psi(k,\hat{n},m,t)$ have four arguments $\hat{n} = 2n-1, n = 1,2,3 \ldots \ldots, 2^{k-1}, k$assume any positive integer, and m is the order for Legendre polynomials and t is the normalize time.

In interval [0,1] by

$$\psi_{nm}(t) = \begin{cases} \sqrt{(m+1)/2}\; 2^{k+1/2} L_m(2^{k+1}t - (2n+1)\frac{n}{2^k} \le t \le \frac{n+1}{2^k} \\ 0 \qquad\qquad\qquad\qquad\qquad otherwise, \end{cases} \tag{8}$$

Where $m = 0,1 \ldots \ldots \ldots .M\ and\ n = 0,1, \ldots \ldots 2^{k-1}$. These are the coefficient of orthonormality . Hence $L_m(t)$ are the Legendre polynomials of order m. Function $f(t)$ defined over [0,1] by the terms of Legendre wavelets as

$$f(t) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} c_{nm}\,\varphi_{nm}(t)\,, \tag{9}$$

Where $c_{nm} = \left(f(t), \varphi_{nm}(t)\right)$, it denotes the inner product. If the infinite series (9) istruncated ,then it can be written as

$$f(t) = \sum_{n=0}^{2^{k-1}}\sum_{m=0}^{M-1} c_{nm}\,\psi_{nm}(t) = C^T\psi(t), \tag{10}$$

Where C and $\varphi(t)$ are $2^k(M+1)$ matrices given by

$$C = \left[c_{0,0}, c_{0,1,\ldots\ldots}, c_{0,M,\ldots\ldots\ldots}, c_{2,M,\ldots\ldots\ldots}, c_{(2^k-1),0,} c_{(2^k-1),1,\ldots\ldots}, c_{(2^k-1),M}\right]^T, \tag{11}$$

$$\psi(t) = \left[\varphi_{0,0}, \varphi_{0,1,\ldots\ldots}, \varphi_{0,M,\ldots\ldots\ldots}, \varphi_{(2^k-1),0,} \varphi_{(2^k-1),1,\ldots\ldots}, \varphi_{(2^k-1),M}\right]^T \tag{12}$$

3. Application in Linear differential equation on Legendre wavelet

Consider the linear second order differential equation

$$y''(x) + f_1(x)y'(x) + f_2(x)y(x) = g(x) \tag{3.1}$$

$$\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} - xy = 1 \tag{3.2}$$

With the initial condition

$$y(0) = 0$$

$$y'(0) = B0$$

Or boundary conditions

$$y(0) = 0$$
$$y'(1) = 0$$

To solve problem (3.1),we approximate $y(x), f_1(x), f_2(x) and\ g(x)$ by the Legendre wavelet as

$$y(x) = C^T\psi(x) = x \tag{3.3}$$
$$f_1(x) = F_1^T\psi(x) = 2x \tag{3.4}$$
$$f_2(x) = F_2^T\psi(x) = 0 \tag{3.5}$$
$$g(x) = G^T\psi(x) = 1 \tag{3.6}$$

By using operational matrix of nth derivative $\frac{d^n\psi(x)}{dx^n} = D^n\varphi(x)$ where D^n is nth power of matrix D then

$$y'(x) = C^TD\psi^T(x) = 1 \tag{3.7}$$
$$y''(x) = C^TD^2\psi^T(x) = 0 \tag{3.8}$$

Employing Eqs. (3.7) and (3.8) , the residual $R(x)$ for (3.1)

$$R(x) = C^TD^2\psi(x) + F_1^T\psi(x)C^TD\psi^T(x) + F_2^T\psi(x)C^T\psi^T(x) = G^T\psi(x)$$

In equation (3.2) transform in Legendre wavelet

$$R(x) = C^TD^2\psi(x) + F_1^T\psi(x)C^TD\psi^T(x) + F_2^T\psi(x)C^T\psi^T(x) = G^T\psi(x)$$

$$R(x) = 2x.1 = 1$$
$$R(x) = x = \frac{1}{2}$$

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