Bayesian Estimation of Change Point of Burr Type III Distribution under Precautionary Loss Function

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Abstract: ***Alterations or sudden changes within a sequence of temporal observations always create disturbance to data analysis. The problem to detect this alterations or changes in any temporal data may allow researchers to identify the abnormality in every sequence. The Bayesian method proposed by Basu and Ebrahimi (1991) have greatly plays important role to find the Bayes Estimators of the parameters of any sequence and analysis of change point problems through Bayesian Technique. In this paper Bayes estimators the Change point and the parameters of Burr type III distribution are obtained under Precautionary Loss Function using Inverted Gamma Prior as natural conjugate prior. We study Bayesian analysis for change point problem with R programming. The result provides accurate change point and posterior means estimation.***

Keywords: Bayesian analysis; change point problem; Burr type III distribution Precautionary Loss Function using Inverted Gamma Prior

1. Introduction

In this estimation approach, the parameter θ in the model distributions $p_\theta(x)$is treated as a random variable with some prior distribution $\pi(\theta)$. The estimator for θ is defined as a value depending on the data and minimizing the expected loss function or the maximal loss function, where the loss function is denoted as $l\left(\theta, \hat{\theta}(X)\right)$. The usual loss function includes the quadratic loss $\left(\theta - \hat{\theta}(X)\right)^2$, the absolute loss $\left|\theta - \hat{\theta}(X)\right|$ etc. It often turns out that $\hat{\theta}(X)$ can be determined from the posterior distribution of $P(\theta|X) = P(X|\theta)\, P(\theta)/P(X)$.

In decision theory the loss criterion is specified in order to obtain best estimator. The simplest form of loss function is squared error loss function (SELF) which assigns equal magnitudes to both positive and negative errors. However this assumption may be inappropriate in most of the estimation problems. Some time overestimation leads to many serious consequences. In such situation many authors found the asymmetric loss functions, appropriate. There are several loss functions which are used to deal such type of problem. In this research work we have considered some of the asymmetric loss function named precautionary loss functions (PLF) suggested by Norstorm (1996). Such asymmetric loss functions are also studied by Basu, A.P. and Ebrahimi, N. (1991), Goldstein, M. (1998), Perlman, M., &Balug, M. (Eds) (1997), Pandya et. al. (1994), Shah, J.B. & Patel, M.N. (2007) and Singh, U.

1.1 Precautionary Loss

Norstrom (1996) introduced an alternative asymmetric precautionary loss function and also presented a general class of precautionary loss functions with quadratic loss function as a special case. These loss function approach infinitely near the origin to prevent underestimation and thus giving a conservative estimators , especially when , low failure rates are being estimated. These estimators are very useful and simple asymmetric precautionary loss function is

$$L\left(\hat{\theta}, \theta\right) = \frac{\left(\hat{\theta} - \theta\right)^2}{\hat{\theta}} \qquad (1.2.1)$$

In a Bayesian setup, the unknown parameter is viewed as random variable. The uncertainty about the true value of parameter is expressed by a prior distribution. The parametric inference is made using the posterior distribution which is obtained by incorporating the observed data in to the prior distribution using the Bayes theorem, The first theorem of inference. Hence we update the prior distribution in the light of observed data. Thus the uncertainty about the parameter prior to the experiment is represented by the prior distribution and the same, after the experiment, is represented by the posterior distribution. The various statistical models are considered are as;

1.2 Natural Conjugate Prior (NCP)

The various prior distributions are considered for different situations, like non-informative, when no information about the parameter is available, Natural Conjugate Prior (NCP), when post and prior distribution of parameter belong to same distribution family, etc. Hence the appropriate distribution chosen is Natural Conjugate Prior. If there is no inherent reason to prefer one prior probability distribution over another, a conjugate prior is sometimes chosen for simplicity. A conjugate prior is defined as a prior distribution belonging to some parametric family, for which the resulting posterior distribution also belongs to the same family. This is an important property. Since the Bayes estimator, as well as its statistical properties (variance, confidence interval, etc.), can all be derived from the posterior distribution.

In each case we observe that the statistical analysis based on the sufficient statistic will be effective as the one based on the entire data set $\underline{x}$.

As in frequentist framework, sufficient statistic plays an important role in Bayesian inference in constructing a family of prior distributions known as Natural Conjugate Prior (NCP). The family of prior distributions $g(\theta)$, $\theta \in \Omega$, is called a natural conjugate family if the corresponding posterior distribution belongs to the same family as g(θ). De Groot (1970) has outlined a simple and elegant method of constructing a conjugate prior for a family of distributions f (x|θ) which admits a sufficient statistic.

One of the fundamental problems in Bayesian analysis is that of the choice of prior distribution g(θ) of θ. The non informative and natural conjugate prior distributions are which in practice, Box and Tiao (1973) and Jeffrey (1961) provide a thorough discussion on non informative priors.

Both De Groot (1970) and Raffia & Schlaifer (1961) provide proof that when a sufficient statistics exist a family of conjugate prior distributions exists.

The most widely used prior distribution of θ is the inverted Gamma distribution with the parameters 'a' and 'b' (> 0) with p.d.f. given by

$$g(\theta) = \begin{cases} \frac{b^a}{\Gamma a}\theta^{-(\alpha+1)}e^{-b/\theta} ; \ \theta > 0 ; \ (a,b) > 0, \\ 0 \qquad , \text{otherwise}. \end{cases} \qquad (1.3.1)$$

The main reason for general acceptability is the mathematical tractability resulting from the fact that the inverted Gamma distribution is conjugate prior of θ Raffia & Schlaifer (1961), Bhattacharya (1967) and others have found that the inverted Gamma can also be used for practical reliability applications.

In this paper the Bayesian estimation of change point 'm' and scale parameter 'γ' of three parameter of Generalized Compound Rayleigh Distribution (G.C.R.D.) and also the change point 'm' and scale parameter 'θ' of Exponentiated Inverted Weibull distribution is done by using Precautionary Loss Function (PLF) and Natural conjugate Prior distribution as Inverted Gamma prior. The sensitivity analysis of Bayesian estimates of change point and the parameters of the distributions have been done by using R-programming.

1.3 Burr Type III Distribution

Burr type III distribution with two parameters was first introduced in the literature of Burr (1942) for modelling lifetime data or survival data. It is more flexible and includes a variety of distributions with varying degrees of skewness and kurtosis. Burr type III distribution with two parameters β and θ, which is denoted by (β, θ). Burr type III , has also been applied in areas of statistical modelling such as forestry , meteorology, and reliability (Mokhlis (2005)).

The Probability Density Function and the Cumulative Distribution Function of Burr III are given by, respectively,

$$f(x;\ \theta,\beta) = \theta\, \beta\, x^{-(\beta+1)}\left(1+x^{-\beta}\right)^{-(\theta+1)};\ x>0,\theta,\beta>0 \qquad (1.4.1)$$

And the Cumulative distribution function

$$F(x;\ \theta,\beta) = \left(1+x^{-\beta}\right)^{-\theta}; x>0, \theta>0, \beta>0 \quad (1.4.2)$$

Reliability function is

$$R(t;\ \theta,\beta) = 1-\left(1+t^{-\beta}\right)^{-\theta};\ t>0, \theta>0, \beta>0 \quad (1.4.3)$$

Note that Burr type XII distribution can be derived from Burr type III distribution by replacing X with $\frac{1}{X}$. The usefulness and properties of Burr distribution are discussed by Burr and Cislak (1968). Abd-Elfattah and Alharbey (2012) considered a Bayesian estimation for Burr type III distribution based on double censoring.

1.4 Bayesian Estimation of Change Point in Burr Type III Distribution under Precautionary Loss Function (PLF)

A sequence of independent life times $x_1, x_2, \dots x_m, x_{(m+1)}, \dots x_n\ (n \geq 3)$ were observed from Burr Type III Distribution with parameter θ, β. But it was found that there was a change in the system at some point of time 'm' and it is reflected in the sequence after 'x_m' which results change in a sequence as well as parameter value θ. The Bayes estimate of θ and 'm' are derived for symmetric and asymmetric loss function under inverted Gamma prior as natural conjugate prior.

1.4.1 Likelihood, Prior, Posterior and Marginal

Let $x_1, x_2, \dots \dots, x_n$, $(n \geq 3)$ be a sequence of observed discrete life times. First let observations $x_1, x_2, \dots \dots, x_n$ have come from Burr Type III Distribution with probability density function as

$$f(x,\theta,\beta) = \theta\beta x^{-(\beta+1)}\left(1+x^{-\beta}\right)^{-(\theta+1)}(x,\theta,\beta > 0)(1.5.1.1)$$

Let 'm' is change point in the observation which breaks the distribution in two sequences as

$$(x_1, x_2, \dots \dots \dots x_m) \& x_{(m+1)}, x_{(m+2)}, \dots \dots x_n$$

The probability density functions of the above sequences are

$$f_1(x) = \theta_1\beta_1 x^{-(\beta_1+1)}\left(1+x^{-\beta_1}\right)^{-(\theta_1+1)}; (1.5.1.2)$$

Where $x_1, \dots, x_m > 0; \theta_1, \beta_1, > 0$

$$f_2(x) = \theta_2\beta_2 x^{-(\beta_2+1)}\left(1+x^{-\beta_2}\right)^{-(\theta_2+1)} \quad ; (1.5.1.3)$$

Where $x_{(m+1)}, x_{(m+2)}, \dots, x_n;\ \theta_2, \beta_2 > 0$

The likelihood functions of probability density function of the sequence are

$$L_1(x|\theta_1,\beta_1) = \prod_{j=1}^{m} f(x_j|\theta_1,\beta_1)$$

$$L_1(x|\theta_1,\beta_1) = \theta_1{}^m \beta_1{}^m \prod_{j=1}^{m} \frac{x_j{}^{-(\beta_1+1)}}{\left(1+x_j{}^{-\beta_1}\right)} e^{-\theta_1} \sum_{j=1}^{m} \log\left(1+x_j{}^{-\beta_1}\right)$$

$$L_1(x|\theta_1,\beta_1,) = (\theta_1\beta_1)^m U_1 e^{-\theta_1 T_{3m}} \ (1.5.1.4)$$

Where

$$U_1 = \prod_{j=1}^{m} \frac{x_j{}^{-(\beta_1+1)}}{\left(1+x_j{}^{-\beta_1}\right)}$$

$$T_{3m} = \sum_{j=1}^{m} \log\left(1+x_j{}^{-\beta_1}\right)$$

$$L_2(x|\theta_2,\beta_2) = \prod_{j=(m+1)}^{n} f(x_j|\theta_2,\beta_2)$$

$$= \theta_2^{(n-m)}\beta_2^{(n-m)} \prod_{j=(m+1)}^{n} \frac{x_j^{-(\beta_2+1)}}{(1+x_j^{-\beta_2})} e^{-\theta_2 \sum_{j=1}^{m} \log(1 + x_j^{-\beta_2})}$$

$$L_2(x|\theta_2,\beta_2) = (\theta_2\beta_2)^{(n-m)} U_2 e^{-\theta_2(T_{3n}-T_{3m})} \quad (1.5.1.5)$$

Where

$$U_2 = \prod_{j=m+1}^{n} \frac{x_j^{-(\beta_2+1)}}{(1+x_j^{-\beta_2})}$$

and $T_{3n} - T_{3m} = \sum_{j=(m+1)}^{n} \log(1 + x_j^{-\beta_2})$

The joint likelihood function is given by

$$L(\theta_1,\theta_2|\underline{x}) \propto (\theta_1\beta_1)^m U_1 e^{-\theta_1 T_{3m}} (\theta_2\beta_2)^{n-m} U_2 e^{-\theta_2(T_{3n}-T_{3m})} \quad (1.5.1.6)$$

Suppose the marginal prior distribution of θ_1 and θ_2 are natural conjugate prior

$$\pi_1(\theta_1,\underline{x}) = \frac{b_1^{a_1}}{\Gamma a_1}\theta_1^{(a_1-1)} e^{-b_1\theta_1}; a_1, b_1 > 0, \theta_1 > 0 \quad (1.5.1.7)$$

$$\pi_2(\theta_2,\underline{x}) = \frac{b_2^{a_2}}{\Gamma a_2}\theta_2^{(a_2-1)} e^{-b_2\theta_2};\ a_2, b_2 > 0, \theta_2 > 0 \quad (1.5.1.8)$$

The joint prior distribution of θ_1, θ_2 and change point 'm' is

$$\pi(\theta_1,\theta_2,m) \propto \frac{b_1^{a_1}}{\Gamma a_1}\frac{b_2^{a_2}}{\Gamma a_2}\theta_1^{(a_1-1)} e^{-b_1\theta_1}\theta_2^{(a_2-1)} e^{-b_2\theta_2} \quad (1.5.1.9)$$

Where $\theta_1, \theta_2 > 0\ \&\ m = 1,2, \dots\dots (n-1)$

The joint posterior density of θ_1, θ_2 and m say $\rho(\theta_1,\theta_2,m/\underline{x})$ is obtained by using equations (1.5.1.6)&(1.5.1.9)

$$\rho(\theta_1,\theta_2,m|\underline{x}) = \frac{\mathrm{L}(\theta_1,\theta_2/\underline{x})\pi(\theta_1,\theta_2,m)}{\sum_m \iint_{\theta_1\theta_2} \mathrm{L}(\theta_1,\theta_2/\underline{x})\pi(\theta_1,\theta_2,m)d\theta_1 d\theta_2} \quad (1.5.1.10)$$

$$\rho(\theta_1,\theta_2,m|\underline{x}) = \frac{\theta_1^{(m+a_1-1)} e^{-\theta_1(T_{3m}+b_1)}\theta_2^{(n-m+a_2-1)} e^{-\theta_2(T_{3n}-T_{3m}+b_2)}}{\sum_m \int_0^\infty e^{-\theta_1(T_{3m}+b_1)}\,\theta_1^{(m+a_1-1)}\,d\theta_1 \int_0^\infty \theta_2^{(n-m+a_2-1)} e^{-\theta_2(T_{3n}-T_{3m}+b_2)}\,d\theta_2}$$

Assuming $\theta_1(T_{3m}+b_1) = x\ \&\theta_2(T_{3n}-T_{3m}+b_2) = y$

$$\theta_1 = \frac{x}{(T_{3m}+b_1)} \& \theta_2 = \frac{y}{T_{3n}-T_{3m}+b_2}$$

$$d\theta_1 = \frac{dx}{(T_{3m}+b_1)} \&\ d\theta_2 = \frac{dy}{T_{3n}-T_{3m}+b_2}$$

$$\rho(\theta_1,\theta_2,m|\underline{x}) = \frac{\theta_1^{(m+a_1-1)} e^{-\theta_1(T_{3m}+b_1)}\theta_2^{(n-m+a_2-1)} e^{-\theta_2(T_{3n}-T_{3m}+b_2)}}{\sum_m \int_0^\infty e^{-x}\frac{x^{(m+a_1-1)}}{(T_{3m}+b_1)^{(m+a_1-1)}}\frac{dx}{(T_{3m}+b_1)}\int_0^\infty \mathrm{e}^{-\mathrm{y}}\frac{\mathrm{y}^{(n-m+a_2-1)}}{(T_{3n}-T_{3m}+b_2)^{(n-m+a_2-1)}}\frac{dy}{(T_{3n}-T_{3m}+b_2)}}\rho(\theta_1,\theta_2,m|\underline{x})$$

$$= \frac{e^{-\theta_1(T_{3m}+b_1)}\theta_1^{(m+a_1-1)} e^{-\theta_2(T_{3n}-T_{3m}+b_2)}\theta_2^{(n-m+a_2-1)}}{\sum_m \frac{\Gamma(m+a_1)}{(T_{3m}+b_1)^{(m+a_1)}}\frac{\Gamma(n-m+a_2)}{(T_{3n}-T_{3m}+b_2)^{(n-m+a_2)}}}$$

$$\rho(\theta_1,\theta_2,m|\underline{x}) = \frac{e^{-\theta_1(T_{3m}+b_1)}\theta_1^{(m+a_1-1)} e^{-\theta_2(T_{3n}-T_{3m}+b_2)}\theta_2^{(n-m+a_2-1)}}{\xi(a_1,a_2,b_1,b_2,m,n)} \quad (1.5.1.11)$$

Where $\xi(a_1,a_2,b_1,b_2,m,n) = \sum_{m=1}^{n-1}\left[\frac{\Gamma(m+a_1)}{(T_{3m}+b_1)^{m+a_1}}\frac{\Gamma(n-m+a_2)}{(T_{3n}-T_{3m}+b_2)^{(n-m+a_2)}}\right]$

The Marginal posterior distribution of change point 'm' using the equations (1.5.1.6), (1.5.1.7)&(1.5.1.8)

$$\rho(m|\underline{x}) = \frac{\mathrm{L}(\theta_1,\theta_2/\underline{x})\,\pi(\theta_1)\,\pi(\theta_2)}{\sum_m \mathrm{L}(\theta_1,\theta_2/\underline{x})\,\pi(\theta_1)\,\pi(\theta_2)} \quad (1.5.1.12)$$

On solving which gives

$$\rho(m|\underline{x}) = \frac{\theta_1^{(m+a_1-1)} e^{-\theta_1(T_{3m}+b_1)}\theta_2^{(n-m+a_2-1)} e^{-\theta_2(T_{3n}-T_{3m}+b_2)}}{\sum_m \theta_1^{(m+a_1-1)} e^{-\theta_1(T_{3m}+b_1)}\theta_2^{(n-m+a_2-1)} e^{-\theta_2(T_{3n}-T_{3m}+b_2)}}$$

$$\rho(m|\underline{x}) = \frac{\int_0^\infty e^{-\theta_1(T_{3m}+b_1)}\theta_1^{(m+a_1-1)}\,d\theta_1 \int_0^\infty e^{-\theta_2(T_{3n}-T_{3m}+b_2)}\theta_2^{(n-m+a_2-1)}\,d\theta_2}{\sum_m \int_0^\infty e^{-\theta_1(T_{3m}+b_1)}\theta_1^{(m+a_1-1)}\,d\theta_1 \int_0^\infty e^{-\theta_2(T_{3n}-T_{3m}+b_2)}\theta_2^{(n-m+a_2-1)}\,d\theta_2}$$

Assuming $\theta_1(T_{3m}+b_1) = y\quad \&\theta_2(T_{3n}-T_{3m}+b_2) = z$

$$\theta_1 = \frac{y}{(T_{3m}+b_1)} \& \theta_2 = \frac{z}{T_{3n}-T_{3m}+b_2}$$

$$d\theta_1 = \frac{dy}{(T_{3m}+b_1)} \& d\theta_2 = \frac{z}{T_{3n}-T_{3m}+b_2}$$

$$\rho(m|\underline{x}) = \frac{\int_0^\infty e^{-y}\frac{y^{(m+a_1-1)}}{(T_{3m}+b_1)^{(m+a_1-1)}}\frac{dy}{(T_{3m}+b_1)}\int_0^\infty \mathrm{e}^{-\mathrm{z}}\frac{\mathrm{z}^{(n-m+a_2-1)}}{(T_{3n}-T_{3m}+b_2)^{(n-m+a_2-1)}}\frac{dz}{(T_{3n}-T_{3m}+b_2)}}{\sum_m \int_0^\infty e^{-y}\frac{y^{(m+a_1-1)}}{(T_{3m}+b_1)^{(m+a_1-1)}}\frac{dy}{(T_{3m}+b_1)}\int_0^\infty \mathrm{e}^{-\mathrm{z}}\frac{\mathrm{z}^{(n-m+a_2-1)}}{(T_{3n}-T_{3m}+b_2)^{(n-m+a_2-1)}}\frac{dz}{(T_{3n}-T_{3m}+b_2)}}$$

$$\rho(m|\underline{x}) = \frac{\frac{\Gamma(m+a_1)}{(T_{3m}+b_1)^{(m+a_1)}}\frac{\Gamma(n-m+a_2)}{(T_{3n}-T_{3m}+b_2)^{(n-m+a_2)}}}{\xi(a_1,a_2,b_1,b_2,m,n)} \quad (1.5.1.13)$$

The marginal posterior distribution of θ_1, using equations (1.5.1.6) and (1.5.1.7)

$$\rho(\theta_1|\underline{x}) = \frac{L(\theta_1,\theta_2/\underline{x})\,\pi(\theta_1)}{\int_0^\infty L(\theta_1,\theta_2/\underline{x})\,\pi(\theta_1)d\theta_1}$$

On solving which gives

$$\rho(\theta_1|\underline{x}) = \frac{\sum_m \int_0^\infty (\theta_1\beta_1)^m U_1 e^{-\theta_1 T_{3m}} (\theta_2\beta_2)^{n-m} U_2 e^{-\theta_2(T_{3n}-T_{3m})} \frac{b_1^{a1}}{\Gamma a_1}\theta_1^{(a_1-1)} e^{-b_1\theta_1} \frac{b_2^{a2}}{\Gamma a_2}\theta_2^{(a_2-1)} e^{-b_2\theta_2} d\theta_2}{\sum_m \iint_0^\infty (\theta_1\beta_1)^m U_1 e^{-\theta_1 T_{3m}} (\theta_2\beta_2)^{n-m} U_2 e^{-\theta_2(T_{3n}-T_{3m})} \frac{b_1^{a1}}{\Gamma a_1}\theta_1^{(a_1-1)} e^{-b_1\theta_1} \frac{b_2^{a2}}{\Gamma a_2}\theta_2^{(a_2-1)} e^{-b_2\theta_2} d\theta_1 d\theta_2}$$

$$\rho(\theta_1|\underline{x}) = \frac{\sum_m e^{-\theta_1(T_{3m}+b_1)}\theta_1^{(m+a_1-1)} \int_0^\infty e^{-\theta_2(T_{3n}-T_{3m}+b_2)}\theta_2^{(n-m+a_2-1)}\,d\theta_2}{\sum_m \int_0^\infty e^{-\theta_1(T_{3m}+b_1)}\theta_1^{(m+a_1-1)}\,d\theta_1 \int_0^\infty e^{-\theta_2(T_{3n}-T_{3m}+b_2)}\theta_2^{(n-m+a_2-1)}\,d\theta_2}$$

Assuming $\theta_1(T_{3m}+b_1) = y$ &$\theta_2(T_{3n}-T_{3m}+b_2) = z$

$$\theta_1 = \frac{y}{(T_{3m}+b_1)} \;\&\theta_2 = \frac{z}{T_{3n}-T_{3m}+b_2}$$

$$d\theta_1 = \frac{dy}{(T_{3m}+b_1)} \;\&d\theta_2 = \frac{dz}{T_{3n}-T_{3m}+b_2}$$

$$\rho(\theta_1|\underline{x}) = \frac{\sum_m e^{-\theta_1(T_{3m}+b_1)}\theta_1^{(m+a_1-1)} \frac{\Gamma(n-m+a_2)}{(T_{3n}-T_{3m}+b_2)^{(n-m+a_2)}}}{\sum_m \frac{\Gamma(m+a_1)}{(T_{3m}+b_1)^{(m+a_1)}} \frac{\Gamma(n-m+a_2)}{(T_{3n}-T_{3m}+b_2)^{(n-m+a_2)}}}$$

$$\rho(\theta_1|\underline{x}) = \frac{\sum_m e^{-\theta_1(T_{3m}+b_1)}\theta_1^{(m+a_1-1)} \frac{\Gamma(n-m+a_2)}{(T_{3n}-T_{3m}+b_2)^{(n-m+a_2)}}}{\xi(a_1,a_2,b_1,b_2,m,n)} \quad (1.5.1.14)$$

The marginal posterior distribution of θ_2, using the equation (1.5.1.6)&(1.5.1.8) is

$$\rho(\theta_2|\underline{x}) = \frac{L(\theta_1,\theta_2/\underline{x})\,\pi(\theta_2)}{\int_0^\infty L(\theta_1,\theta_2/\underline{x})\,\pi(\theta_2)\,d\theta_2}$$

$$\rho(\theta_2|\underline{x}) = \frac{\sum_m \int_0^\infty (\theta_1\beta_1)^m U_1 e^{-\theta_1 T_{3m}} (\theta_2\beta_2)^{n-m} U_2 e^{-\theta_2(T_{3n}-T_{3m})} \frac{b_1^{a1}}{\Gamma a_1}\theta_1^{(a_1-1)} e^{-b_1\theta_1} \frac{b_2^{a2}}{\Gamma a_2}\theta_2^{(a_2-1)} e^{-b_2\theta_2} d\theta_1}{\sum_m \iint_0^\infty (\theta_1\beta_1)^m U_1 e^{-\theta_1 T_{3m}} (\theta_2\beta_2)^{n-m} U_2 e^{-\theta_2(T_{3n}-T_{3m})} \frac{b_1^{a1}}{\Gamma a_1}\theta_1^{(a_1-1)} e^{-b_1\theta_1} \frac{b_2^{a2}}{\Gamma a_2}\theta_2^{(a_2-1)} e^{-b_2\theta_2} d\theta_1 d\theta_2}$$

$$\rho(\theta_2|\underline{x}) = \frac{\sum_m e^{-\theta_2(T_{3n}-T_{3m}+b_2)}\theta_2^{(n-m+a_2-1)} \int_0^\infty e^{-\theta_1(T_{3m}+b_1)}\theta_1^{(m+a_1-1)}\,d\theta_1}{\sum_m \int_0^\infty e^{-\theta_1(T_{3m}+b_1)}\theta_1^{(m+a_1-1)}\,d\theta_1 \int_0^\infty e^{-\theta_2(T_{3n}-T_{3m}+b_2)}\theta_2^{(n-m+a_2-1)}\,d\theta_2}$$

Assuming $\theta_1(T_{3m}+b_1) = y$ &$\theta_1 = \frac{y}{(T_{3m}+b_1)}$

$$\rho(\theta_2|\underline{x}) = \frac{\sum_m e^{-\theta_2(T_{3n}-T_{3m}+b_2)}\theta_2^{(n-m+a_2-1)} \int_0^\infty e^{-y}\frac{y^{(m+a_1-1)}}{(T_{3m}+b_1)^{(m+a_1-1)}}\frac{dy}{(T_{3m}+b_1)}}{\sum_m \int_0^\infty e^{-y}\frac{y^{(m+a_1-1)}}{(T_{3m}+b_1)^{(m+a_1-1)}}\frac{dy}{(T_{3m}+b_1)} \int_0^\infty e^{-z}\frac{z^{(n-m+a_2-1)}}{(T_{3n}-T_{3m}+b_2)^{(n-m+a_2-1)}}\frac{dz}{(T_{3n}-T_{3m}+b_2)}}$$

$$\rho(\theta_2|\underline{x}) = \frac{\sum_m \frac{\Gamma(m+a_1)}{(T_{3m}+b_1)^{(m+a_1)}} e^{-\theta_2(T_{3n}-T_{3m}+b_2)}\theta_2^{(n-m+a_2-1)}}{\xi(a_1,a_2,b_1,b_2,m,n)} \quad (1.5.1.15)$$

1.4.2 Bayes Estimators under Precautionary Loss Function (PLF)

The Precautionary loss function is given by

$$L_3(\hat{\theta},\theta) = \frac{(\hat{\theta}-\theta)^2}{\hat{\theta}} \qquad (1.5.2.1)$$

The Bayes estimator of θ under precautionary Loss Function is obtain by solving the equation;

$$\frac{\partial}{\partial\hat{\theta}} E_\rho\left[L_3(\hat{\theta},\theta)\right] = 0$$

$$\Rightarrow \hat{\theta}_{BP} = \left[E_\rho(\theta^2)\right]^{1/2} \qquad (1.5.2.2)$$

The Bayes estimate $\hat{m}_{BP}$ of m using the marginal posterior from equation (1.5.1.14) is

$$\hat{m}_{BP} = \left[E_\rho(m^2)\right]^{1/2}$$

$$\hat{m}_{BP} = \left[\frac{\sum_m m^2 \frac{\Gamma(m+a_1)}{(T_{3m}+b_1)^{(m+a_1)}} \frac{\Gamma(n-m+a_2)}{(T_{3n}-T_{3m}+b_2)^{(n-m+a_2)}}}{\xi(a_1,a_2,b_1,b_2,m,n)}\right]^{1/2} \quad (1.5.2.3)$$

The Bayes estimator $\hat{\theta}_{1BP}$ of θ_1 under PLF using the marginal posterior from equation (1.5.1.14) is

$$\hat{\theta}_{1BP} = \left[E_\rho\left({\theta_1}^2\right)\right]^{1/2}$$

$$\hat{\theta}_{1BP} = \left[\frac{\sum_m \frac{\Gamma(n-m+a_2)}{(T_{3n}-T_{3m}+b_2)^{(n-m+a_2)}} \int_0^\infty e^{-\theta_1(T_{3m}+b_1)} {\theta_1}^{(m+a_1+1)} d\theta_1}{\xi(a_1,a_2,b_1,b_2,m,n)}\right]^{1/2}$$

Assuming $\theta_1(T_{3m}+b_1) = y \quad \& \theta_1 = \frac{y}{(T_{3m}+b_1)}$

Then

$$\hat{\theta}_{1BP} = \left[\frac{\sum_m \frac{\Gamma(n-m+a_2)}{(T_{3n}-T_{3m}+b_2)^{(n-m+a_2)}} \int_0^\infty e^{-y} \frac{y^{(m+a_1+1)}}{(T_{3m}+b_1)^{(m+a_1+1)}} \frac{dy}{(T_{3m}+b_1)}}{\xi(a_1,a_2,b_1,b_2,m,n)}\right]^{1/2}$$

$$\hat{\theta}_{1BP} = \left[\frac{\sum_m \frac{\Gamma(m+a_1+2)}{(T_{3m}+b_1)^{(m+a_1+2)}} \frac{\Gamma(n-m+a_2)}{(T_{3n}-T_{3m}+b_2)^{(n-m+a_2)}}}{\xi(a_1,a_2,b_1,b_2,m,n)}\right]^{1/2}$$

$$\hat{\theta}_{1BP} = \left[\frac{\xi[(a_1+2),a_2,b_1,b_2,m,n]}{\xi(a_1,a_2,b_1,b_2,m,n)}\right]^{1/2} \quad (1.5.2.4)$$

The Bayes estimator $\hat{\theta}_{2BP}$ of θ_2 under PLF using the marginal posterior from equation (1.5.1.15) is

$$\hat{\theta}_{2BP} = \left[E_\rho\left({\theta_2}^2\right)\right]^{1/2}$$

$$\hat{\theta}_{2BP} = \left[\frac{\sum_m \frac{\Gamma(m+a_1)}{(T_{3m}+b_1)^{(m+a_1)}} \int_o^\infty e^{-\theta_2(T_{3n}-T_{3m}+b_2)} {\theta_2}^{(n-m+a_2+1)} d\theta_2}{\xi(a_1,a_2,b_1,b_2,m,n)}\right]^{1/2}$$

Assuming $\theta_2(T_{3n}-T_{3m}+b_2) = y \quad \& \theta_2 = \frac{y}{(T_{3n}-T_{3m}+b_2)}$

Then

$$\hat{\theta}_{2BP} = \left[\frac{\sum_m \frac{\Gamma(m+a_1)}{(T_{3m}+b_1)^{(m+a_1)}} \int_0^\infty \mathrm{e}^{-y} \frac{\mathrm{y}^{(n-m+a_2+1)}}{(T_{3n}-T_{3m}+b_2)^{(n-m+a_2+1)}} \frac{dy}{(T_{3n}-T_{3m}+b_2)}}{\xi(a_1,a_2,b_1,b_2,m,n)}\right]^{1/2}$$

$$\hat{\theta}_{2BP} = \left[\frac{\sum_m \frac{\Gamma(m+a_1)}{(T_{3m}+b_1)^{(m+a_1)}} \frac{\Gamma(n-m+a_2+2)}{(T_{3n}-T_{3m}+b_2)^{(n-m+a_2+2)}}}{\xi(a_1,a_2,b_1,b_2,m,n)}\right]^{1/2}$$

$$\hat{\theta}_{2BP} = \left[\frac{\xi[a_1,(a_2+2),b_1,b_2,m,n]}{\xi(a_1,a_2,b_1,b_2,m,n)}\right]^{1/2} \quad (1.5.2.5)$$

Numerical Comparison for Burr Type III Distribution

As in chapter 2 we have generated 20 random observations from Burr Type III distribution with parameter θ = 2 and β = 0.5. The observed data mean is μ = 1.8829 and variance σ^2 = 23.8886. Let the change in sequence is at 11th observation, so the means and variances of both sequences ($x_1, x_2, \ldots, x_m$) and ($x_{(m+1)}, x_{(m+2)}, \ldots, x_n$) are μ_1= 0.8277, μ_2= 3.2668, σ_1^2 = 0.7281 and σ_2^2= 51.8509. If the target value of μ_1 is unknown, its estimating ($\hat{\mu}_1$) is given by the mean of first m sample observation given m=11, μ = 0.8277.

Sensitivity Analysis of Bayes Estimates

In this section we have studied the sensitivity of the Bayes estimates with respect to changes in the parameters of prior distribution a_1, b_1, a_2 and b_2. The means and variances of the prior distribution are used as prior information in computing these parameters. Then with these parameter values we have computed the Bayes estimates of m, θ_1 and θ_2 under PLF considering different set of values of $(a_1, b_1)$and (a_2, b_2).We have also considered the different sample sizes n=10(10)30. The Bayes estimates of the change point 'm' and the parameters θ_1 and θ_2 are given in table-5.3 under PLF. Their respective mean squared errors (M.S.E's) are calculated by repeating this process 1000 times and presented in same table in small parenthesis under the estimated values of parameters. All these values appears to be robust with respect to correct choice of prior parameter values and appropriate sample size. All the estimators perform better with sample size n=20 and (a_1=1.8,1.9)(b_1=2.3,2.4),(a_2=1.3,1.4) and (b_2=1.55, 1.65).Similarly the Bayes estimates of PLF are presented in table 5.2 appears to be sensitive with wrong choice of prior parameters and sample size. All the calculations are done by R- programming. From the below two table we conclude that –

The Bayes estimates of the parameters θ_1 and θ_2 of Burr Type III obtained with loss function PLF have more or less same numerical values. The respective M.S.E's shows that the Bayes estimates uniformly smaller for $\hat{\theta}_{1BP}$ and $\hat{\theta}_{2BP}$ under PLF except of $\hat{m}_{BP}$. The Bayes estimates of the parameters are robust uniformly with all values of prior parameters as and all sample size.

Table 1.1: Bayes Estimates of m,θ_1&$\theta_2$for Burr Type III and their respective M.S.E.'s Under PLF

(a_1, b_1)	(a_2, b_2)	N	$\hat{m}_{BP}$	$\hat{\theta}_{1BP}$	$\hat{\theta}_{2BP}$
(1.25,1.50)	(1.50,1.60)	10	5.3682 **(16.1368)**	0.6411 **(0.3374)**	0.6479 **(0.0119)**
		20	10.2381 **(93.3795)**	0.5447 **(0.1218)**	0.8243 **(0.2334)**
		30	15.3143 **(195.4584)**	0.9061 **(0.0290)**	0.6031 **(0.0368)**
(1.50,1.75)	(1.70,1.80)	10	5.4855 **(14.6901)**	0.7828 **(0.0001)**	0.6574 **(0.1592)**
		20	11.2830 **(82.3266)**	0.8828 **(0.2390)**	0.8064 **(0.1734)**
		30	14.9025 **(227.2012)**	0.9301 **(0.0417)**	0.6859 **(0.0451)**
(1.75,2.0)	(1.90,2.0)	10	5.4091 **(11.3318)**	0.7551 **(0.1802)**	0.7881 **(0.0065)**
		20	11.6450 **(103.7665)**	0.8941 **(0.0015)**	0.6386 **(0.0409)**
		30	18.1470 **(177.9180)**	0.5218 **(0.0129)**	0.7267 **(0.0825)**
(2.0,2.25)	(2.10,2.20)	10	5.4949 **(8.2752)**	0.7759 **(0.4825)**	0.6723 **(0.0002)**
		20	11.1375 **(89.0678)**	1.0261 **(0.0665)**	1.1985 **(0.0624)**
		30	17.0109 **(305.2535)**	0.7120 **(0.1028)**	0.8602 **(0.0126)**
(2.25,2.50)	(2.30,2.40)	10	5.3972 **(11.5300)**	1.3856 **(.2231)**	0.9654 **(0.0305)**
		20	11.2967 **(68.8955)**	0.6913 **(0.0032)**	0.7222 **(0.2218)**
		30	16.1228 **(285.1916)**	0.9316 **(0.0014)**	0.6857 **(0.0233)**
(2.50,2.75)	(2.50,2.60)	10	5.5867 **(14.1249)**	0.8565 **(3.3404)**	0.9124 **(0.0291)**
		20	11.3819 **(82.1639)**	0.7271 **(0.1431)**	0.6055 **(0.0152)**
		30	16.8922 **(96.9484)**	0.6598 **(0.0979)**	0.5785 **(0.4466)**

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