Stability of Second Order Partial Differential Equation

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Abstract: ***In this paper, we prove the Hyers-Ulam-Rassias stability of second order partial differential equation:***

$p(x,t)u_{xx}(x,t)+p_x(x,t)u_x(x,t)+ q(x,t)u_x(x,t)+ q_x(x,t)u(x,t)=g(x,t,u(x,t))$.

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AMS subject classification: 35B35; 26D10.

1. Introduction

In 1940, S. M. Ulam's [14] presented a famous talk to the Mathematics Club of the University of Wisconsin, where he discussed a number of important unsolved problems. One of them was concerned with the stability of group homomorphismand D. H. Hyers [5] gave partial solution to it in 1941. Thereafter numbers of authors have studied the stability of solutions of differential equations [3, 6, 7] and partial differential equations [8, 9]. This is now known as Hyers-Ulam (HU) stability and its various extensions has been named with additional word. One such extension is Hyers Ulam Rassias (HUR) stability. In [10] and [11], HURstability for linear differential operators of n^{th}order with non-constant coefficients was studied.HUR stability for special types of non-linear equations have been studied in [1, 2, 12]. HUR stability of second order partial differential equation have been studied in [13]. In 2011, Gordji et al. [4], proved the HUR stability of non-linear partial differential equations by using Banach's Contraction Principle. In this paper, by using the result of [4], we prove the HUR stability of second order partial differential equation:

$$p(x,t)u_{xx}(x,t)+p_x(x,t)u_x(x,t) + q(x,t)u_x(x,t)+ q_x(x,t)u(x,t)=g(x,t,u(x,t)). \quad (1.1)$$

Here p, $q : J \times J \to \mathbb{R}^+$ be a differentiable function at least once w. r. t. both the arguments and $p(x,t)\neq 0, q(x,t)\neq 0 \, \forall x,t\in J, g: J\times J\times \mathbb{R}\to\mathbb{R}$ be a continuous function and $J=[a, b]$ be a closed interval.

Definition 1.1: A function $u: J\times J\to R$ is called a solution of equation (1.1) if $u\in C^2(J\times J)$ and satisfies the equation (1.1).

2. Preliminaries

Definition 2.1: The equation (1.1) is said to be HUR stable if the following holds:

Letφ: $J\times J\to (0, \infty)$ be a continuous function.Then there exists a continuous function $\Psi: J \times J \to (0, \infty)$, which depends on φ such that whenever $u: J \times J \to \mathbb{R}$ is a continuous function with

$$|p(x,t)u_{xx}(x,t)+p_x(x,t)u_x(x,t) + q(x,t)u_x(x,t)+ q_x(x,t)u(x,t)-g(x,t,u(x,t))|\leq\varphi(x,t),$$

There exists a solution $u_0: J\times J\to\mathbb{R}$of (1.1) such that

$$|u(x,t)-u_0(x,t)|\leq\Psi(x,t), \qquad \forall (x,t)\in J\times J.$$

We need the following.

Banach Contraction Principle:
Let(Y,d) be a complete metric space, then each contraction map $T: Y\to Y$ has a unique fixed point, that is, there exists $b\in Y$ such that $Tb=b$. Moreover,

$$d(b,w) \leq \frac{1}{(1-\alpha)} d(w,Tw), \ \forall\, w \in Y \quad \text{and } 0 \leq \alpha < 1.$$

Following the results from Gordji et al. [4], we establish the following result.

3. Main Result

In this section we prove the HUR stability of first order partial differential equation (1.1).

Theorem 3.1: Let $c\in J$. Let p, q and g be a sin (1.1) with additional conditions:

(i) $p(x,t)\geq 1, \forall x,t\in J$.
(ii) $\varphi: J\times J\to(0,\infty)$ be a continuous function and $M: J\times J\to[1,\infty)$ be an integrable function.
(iii) Assume that there exists α, $0<\alpha<1$ such that

$$\int_c^x M(\tau,t)\varphi(\tau,t)d\tau \leq \alpha\, \varphi(x,t). \quad (3.1)$$

and

$$K(x,t,u(x,t)) = \{p(x,t)\}^{-1}\left[p(c,t)u_x(c,t) - q(x,t)u(x,t) + q(c,t)u(c,t) + \int_c^x g\big(\tau,t,u(\tau,t)\big)d\tau\right] \quad (3.2)$$

Suppose that the following holds:
C1: $|K(\tau,t,l(\tau,t))-K(\tau,t,m(\tau,t))|\leq M(\tau,t)|l(\tau,t)-m(\tau,t)|, \forall \tau,t\in J$ and l, $m\in C\ (J\times J)$.
C2: $u: J\times J\rightarrow R$ be a function satisfying the inequality (2.1).

Then there exists a unique solution $u_0: J\times J\rightarrow R$ of the equation (1.1) of the form

$$u_0(x,t) = u(c,t) + \int_c^x K(\tau,t,u_0(xt))d\tau$$

Such that

$$|u(x,t) - u_0(x,t)| \leq \frac{\alpha}{(1-\alpha)}\varphi(x,t), \ \forall\, x,t \in J.$$

Proof: Consider
$|p(x,t)u_{xx}(x,t)+p_x(x,t)u_x(x,t) + q(x,t)u_x(x,t)+ q_x(x,t)u(x,t)- g(x,t,u(x,t))|$
$=| \{p(x,t)u_x(x,t)\}_x+\{q(x,t)u(x,t)\}_x - g(x,t,u(x,t))|$

From the inequality (2.1), we get
$|\{p(x,t)u_x(x,t)\}_x+\{q(x,t)u(x,t)\}_x - g(x,t,u(x,t))|\leq\varphi(x,t)$.
$\Rightarrow -\varphi(x,t)\leq\{p(x,t)u_x(x,t)\}_x+\{q(x,t)u(x,t)\}_x - g(x,t,u(x,t)\leq\varphi(x,t)$.
$\Rightarrow\{p(x,t)u_x(x,t)\}_x+\{q(x,t)u(x,t)\}_x - g(x,t,u(x,t\leq\varphi(x,t)$.

Integrating from c to x we get,

$$p(x,t)u_x(x,t) - p(c,t)u_x(c,t) + q(x,t)u(x,t) - q(c,t)u(c,t) - \int_c^x g(\tau,t,u(\tau,t))\,d\tau \leq \int_c^x \varphi(\tau,t)d\tau.$$

$\Rightarrow p(x,t)\{u_x(x,t) - \{p(x,t)\}^{-1}[p(c,t)u_x(c,t) -$ $qx,tux,t+ qc,tuc,t+$ $cxg\tau,t,u\tau,td\tau \leq cx\varphi\tau,td\tau.$

$$\Rightarrow \Big\{u_x(x,t) - \{p(x,t)\}^{-1}\Big[p(c,t)u_x(c,t) - q(x,t)u(x,t) + q(c,t)u(c,t) + \int_c^x g(\tau,t,u(\tau,t))d\tau\Big]\Big\} \leq \{p(x,t)\}^{-1}\int_c^x \varphi(\tau,t)d\tau.$$

$$\Rightarrow \Big\{u_x(x,t) - \{p(x,t)\}^{-1}\Big[p(c,t)u_x(c,t) - q(x,t)u(x,t) + q(c,t)u(c,t) + \int_c^x g(\tau,t,u(\tau,t))d\tau\Big]\Big\} \leq \int_c^x \varphi(\tau,t)d\tau,$$

$(\because p(x,t) \geq 1)$.

$$\Rightarrow \{u_x(x,t) - K(x,t,u(xt))\} \leq \int_c^x \varphi(\tau,t)d\tau.$$

where $K(x,t,u(xt))$ is given by equation (3.2).

Since $M: J\times J\rightarrow[1,\infty)$ be an integrable function, we have

$$\Rightarrow \{u_x(x,t) - K(x,t,u(xt))\} \leq \int_c^x M(\tau,t)\varphi(\tau,t)d\tau.$$

Using inequality (3.1) we have,

$$\{u_x(x,t) - K(x,t,u(xt))\} \leq \int_c^x M(\tau,t)\varphi(\tau,t)d\tau \leq \alpha\varphi(x,t).$$

$$\{u_x(x,t) - K(x,t,u(xt))\} \leq \alpha\varphi(x,t).$$

$\{u_x(x,t) - K(x,t,u(xt))\} \leq \varphi(x,t)$. (3.4)

Again, integrating from c to x we get,

$$u(x,t) - u(c,t) - \int_c^x K(\tau,t,u(\tau,t))d\tau \leq \int_c^x \varphi(\tau,t)d\tau.$$

Since $M: J\times J\rightarrow[1,\infty)$ be an integrable function, we have

$$u(x,t) - u(c,t) - \int_c^x K(\tau,t,u(\tau,t))d\tau \leq \int_c^x M(\tau,t)\varphi(\tau,t)d\tau.$$

Using inequality (3.1) we have,
$u(x,t) - u(c,t) - \int_c^x K(\tau,t,u(\tau,t))d\tau \leq$ $\int_c^x M(\tau,t)\varphi(\tau,t)d\tau \leq \alpha\varphi(x,t)$.
$\Rightarrow u(x,t) - u(c,t) - \int_c^x K(\tau,t,u(\tau,t))d\tau \leq \alpha\varphi(x,t)$. (3.3)

$$\Rightarrow u(x,t) - \left[u(c,t) + \int_c^x K(\tau,t,u(\tau,t))d\tau\right] \leq \varphi(x,t)$$

, $(\because 0 < \alpha < 1)$. (3.5)
In a similar way, from the left inequality of (3.3), we obtain
$-\left[u(x,t) - \left[u(c,t) + \int_c^x K(\tau,t,u(\tau,t))d\tau\right]\right] \leq \varphi(x,t)$.
(3.6)

From the inequalities (3.5) and (3.6) we get,

$$\left|u(x,t) - \left[u(c,t) + \int_c^x K(\tau,t,u(xt))d\tau\right]\right| \leq \varphi(x,t). \quad (3.7)$$

Let Y be the set of all continuously differentiable functions γ: $J \times J \rightarrow$ R. We define a metric d and an operator T on Y as follows: For l, $m\in Y$

$$d(l,m)=sup_{x,t\in J}\left|\frac{l(x,t)-m(x,t)}{\varphi(x,t)}\right|$$

and the operator

$$(Tm)(x,t) = \left[u(c,t) + \int_c^x K(\tau,t,m(\tau,t))d\tau\right]. \quad (3.8)$$

Consider,

$$d(Tl,Tm)=sup_{x,t\in J}\left\{\frac{Tl(x,t)-Tm(x,t)}{\varphi(x,t)}\right\}.$$

$$=sup_{x,t\in J}\left\{\frac{\left(\int_c^x K(\tau,t,l(\tau,t))\,d\tau - \int_c^x K(\tau,t,m(\tau,t))\,d\tau\right)}{\varphi(x,t)}\right\}.$$

$$\leq sup_{x,t\in J}\left\{\frac{\left(\int_c^x K(\tau,t,l(\tau,t))\,d\tau - \int_c^x K(\tau,t,m(\tau,t))\,d\tau\right)}{\varphi(x,t)}\right\}.$$

$$\leq sup_{x,t\in J}\left\{\frac{\int_c^x |K(\tau,t,l(\tau,t)) - K(\tau,t,m(\tau,t)|d\tau}{\varphi(x,t)}\right\}.$$

By using condition C1 we get,

$$d(Tl,Tm) \leq sup_{x,t\in J}\left\{\frac{\int_c^x\{M(\tau,t)|l(\tau,t)-m(\tau,t)|\}d\tau}{\varphi(x,t)}\right\}.$$

$$= sup_{x,t\in J}\left\{\frac{\int_c^x\left\{M(\tau,t)\varphi(\tau,t)\left(\frac{|l(\tau,t)-m(\tau,t)|}{\varphi(\tau,t)}\right)\right\}d\tau}{\varphi(x,t)}\right\}.$$

$$\leq sup_{x,t\in J}\left\{\frac{\int_c^x\left\{M(\tau,t)\varphi(\tau,t)\times sup_{\tau,t\in J}\left(\frac{|l(\tau,t)-m(\tau,t)|}{\varphi(\tau,t)}\right)\right\}d\tau}{\varphi(x,t)}\right\}.$$

$$d(Tl,Tm) \leq d(l,m)\times sup_{x,t\in J}\left\{\frac{\int_c^x\{M(\tau,t)\varphi(\tau,t)\}d\tau}{\varphi(x,t)}\right\}.$$

By using inequality (3.1) we get,

$$d(Tl,Tm) \leq \alpha d(l,m).$$

By using Banach contraction principle, there exists a unique $u_0\in X$ such that $Tu_0=u_0$, that is

$$\left[u(c,t)+\int_c^x K(\tau,t,u_0(\tau,t))d\tau\right]=u_0(x,t),$$

(By using equation (3.8))

and

$$d(u_0,u) \leq \frac{1}{(1-\alpha)}d(u,Tu). \qquad (3.9)$$

Now by using in equality (3.7) we get,

$$|u(x,t)-(Tu)(x,t)| \leq \alpha\,\varphi(x,t).$$

$$\Rightarrow \frac{|u(x,t)-(Tu)(x,t)|}{\varphi(x,t)} \leq \alpha.$$

$$\Rightarrow sup_{x,t\in J}\frac{|u(x,t)-(Tu)(x,t)|}{\varphi(x,t)} \leq \alpha.$$

Thus

$$d(u,Tu) \leq \alpha. \qquad (3.10)$$

Again

$$d(u_0,u)=sup_{x,t\in J}\left|\frac{u_0(x,t)-u(x,t)}{\varphi(x,t)}\right|.$$

From equation (3.9) we get,

$$d(u_0,u) \leq \frac{1}{(1-\alpha)}d(u,Tu).$$

$$sup_{x,t\in J}\left|\frac{u_0(x,t)-u(x,t)}{\varphi(x,t)}\right| \leq \frac{1}{(1-\alpha)}d(u,Tu).$$

$$\left|\frac{u_0(x,t)-u(x,t)}{\varphi(x,t)}\right| \leq sup_{x,t\in J}\left|\frac{u_0(x,t)-u(x,t)}{\varphi(x,t)}\right| \leq \frac{1}{(1-\alpha)}d(u,Tu).$$

$$\left|\frac{u_0(x,t)-u(x,t)}{\varphi(x,t)}\right| \leq \frac{1}{(1-\alpha)}d(u,Tu).$$

From equation (3.10) we get,

$$\left|\frac{u_0(x,t)-u(x,t)}{\varphi(x,t)}\right| \leq \frac{1}{(1-\alpha)}\alpha.$$

$$|u_0(x,t)-u(x,t)| \leq \frac{\alpha}{(1-\alpha)}\varphi(x,t), \forall\, x,t \in J.$$

Hence the result

4. Conclusion

In this paper we have proved the HUR stability of the second order partial differential equation (1.1) by employing Banach's contraction aprinciple.

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