International Journal of Science and Research (IJSR)
ISSN: 2319-7064
SJIF (2022): 7.942

Ultimate Proof of Collatz Conjecture (2'nd Version)

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Abstract: ***The Collatz conjecture is one of the most famous unsolved problems in mathematics. The conjecture asks whether repeating two simple arithmetic operations will eventually transform every positive integer into 1.***

Please note: It is just my a progress to prove collatz conjecture .It is not guaranteed that it is the final proof . So let's begin

1. Proof of collatz conjecture

For the purpose of this document, we like to reformulate the Collatz conjecture as follows:

Take any positive integer n. n can be written as in formula (2). If β
> 0 then make β = 0.

If β = 0 (and at leastone $\alpha_i > 0$) then multiply n by 3 and add 1 to obtain 3n+1. The new 3n+1 will again have a β > 0. Repeat this process indefinitely. The conjecture is that no matter what number you start with, you will always eventually reach a situation where β > 0 and all $\alpha_i = 0$. In the next step, n will eventually become 1 when we make β
= 0.

We present the proof of the Collatz conjecture for many types of sets defined by the remainder theorem of arithmetic. These sets are defined in mods 6,12,24,36,48,60,72,84,96,108 and we took only odd positive remainders to work with. It is not difficult to prove that the same results are true for any mod 12m, for positive integers m.

2. Division Sequence and Complete Division Sequence

Definition 1.1: A division sequence is a sequence given by arranging the numbers of division by 2 in each operation when the Collatz operation is continuously performed with a positive odd number, *n*, as the initial value.

For example, in the case of 9, the arrangement of numbers given by continuously performing 3*x* + 1, and dividing by 2 provides 9, 28, 14, 7, 22, 11, 34, 17, 52, 26, 13, 40, 20, 10, 5, 16, 8, 4, 2, 1 (stops when 1 is reached).

Therefore, the division sequence of 9 is [2, 1, 1, 2, 3, 4].

The division sequence of 1 is an empty list []. Further, [6] is a division sequence of 21, but [6, 2] and [6, 2, 2]... that repeats the loop of 1 to 4 to 2 to 1 are not division sequences.

When the division sequence is finite, it is equivalent to reaching 1 in a series of Collatz operations.

When the division sequence is infinite, it does not reach 1 in a series of Collatz operations. It is equivalent to entering a loop other than 4-2-1 or increasing the Collatz value endlessly.

Definition 1.2: A complete division sequence is a division sequence of multiples of 3.

- 9[2, 1, 1, 2, 3, 4] is a complete division sequence of 9.
- 7[1, 1, 2, 3, 4] is a division sequence of 7.

Definition 1.3: Supposing that only one element exists in the division sequence of *n*, no Collatz operation can be applied to *n*.

Theorem 1.1. When the Collatz operation is applied to *x* in the complete division sequence of *x* (two or more elements), (some) *y* and its division sequence are obtained.

Proof: This follows the Collatz operation and definition of a division sequence. Theorem 1.2. When the Collatz operation is applied to *y* in the division sequence of *y* (two or more elements), (some) *y* and its division sequence are obtained.

Proof: It is self-evident from the Collatz operation and definition of a division sequence.

3. One Only Looks at Odd Numbers of Multiples of 3

There is no need to look at even numbers.

By continuing to divide all even numbers by 2, one of the odd numbers is achieved.

Therefore, it is only necessary to check "whether all odd numbers reach 1 by the Collatz operation".

One only needs to look at multiples of 3.

For a number *x* that is not divisible by 3, the Collatz inverse operation is defined as obtaining a positive integer by $(x \times 2^k - 1)/3$. Multiple numbers can be obtained using the Collatz reverse operation.

Here, we consider the Collatz reverse operation on *x*.

The remainder of dividing *x* by 9 is one of 1, 2, 4, 5, 7, 8, *i.e.*:

$1 \times 2^6 \equiv 1$

$2 \times 2^5 \equiv 1$

$4 \times 2^4 \equiv 1$

$5 \times 2^1 \equiv 1$
$7 \times 2^2 \equiv 1$
$8 \times 2^3 \equiv 1 \pmod 9$

This indicates that multiplying any number by 2 appropriate number of times provides an even number with a reminder of 1 when divided by 9.

By subtracting 1 from this and dividing by 3, we get an odd number that is a multiple of 3.

Performing the Collatz reverse operation once from x provides an odd number y that is a multiple of 3.

If y reaches 1, then x, which was once given by the Collatz operation of y, also reaches 1. Therefore, the following can be stated.

Theorem 1.3. One only needs to check "whether an odd number that is a multiple of 3 reaches 1 by the Collatz operation".

4. Star Conversion

A star conversion is defined for a complete division sequence.

A complete division sequence of length, n, is copied to a complete division sequence of length, n or $n + 1$.

The remainder, which is given by dividing the Collatz value$x$ by 9 is
$x \equiv 3 \bmod 9$

The conversion to copy a finite or infinite sequence [a_1, a_2, a_3...] to a sequence [6, a_1 − 4, a_2, a_3...] is described as A[6, −4].

The conversion to copy a finite or infinite sequence [a_1, a_2, a_3...] to a sequence [1, a_1 − 2, a_2, a_3...] is described as B[1, −2].
$x \equiv 6 \bmod 9$

The conversion to copy a finite or infinite sequence [a_1, a_2, a_3...] to a sequence [4, a_1 − 4, a_2, a_3...] is described as C[4, −4].

The conversion to copy a finite or infinite sequence [a_1, a_2, a_3...] to a sequence [3, a_1 − 2, a_2, a_3...] is described as D[3, −2].
$x \equiv 0 \bmod 9$

The conversion to copy a finite or infinite sequence [a_1, a_2, a_3...] to a sequence [2, a_1 − 4, a_2, a_3...] is described as E[2, −4].

The conversion to copy a finite or infinite sequence [a_1, a_2, a_3...] to a sequence [5, a_1 − 2, a_2, a_3...] is described as F[5, −2].

Furthermore, the conversion to copy a finite or infinite sequence [a_1, a_2, a_3...] to a sequence [a_1 + 6, a_2, a_3...] is described as G[+6].

If the original first term is negative, G[+6] is performed in advance.

Example
$117 \equiv 0 \pmod 9$, 117[5, 1, 2, 3, 4] can be converted to E[2, −4] → 9 [2, 5−4, 1, 2, 3, 4] and F[5, −2] → 309[5, 5−2, 1, 2, 3, 4].

Table 1 shows the functions corresponding to each star conversion. The function represents a change in the Collatz value.

Table 1: Star conversion in mod 9

When	star conversion 1	star conversion 2
$x \equiv 3 \bmod 9$	A[6, −4] $y = 4x/3 - 7$	B[1, −2] $y = x/6 - 1/2$
$x \equiv 6 \bmod 9$	C[4, −4] $y = x/3 - 2$	D[3, −2] $y = 2x/3 - 1$
$x \equiv 0 \bmod 9$	E[2, −4] $y = x/12 - 3/4$	F[5, −2] $y = 8x/3 - 3$
Always	G[+6] $y = 64x + 21$	

Any Complete Division Sequence Is Obtained by Performing a Star Conversion on Any Complete Division Sequence

In Tables 2-8, we consider how the Collatz value changes with each star conversion. Therefore, for 3 mod 9 described as A[6, −4], the first term of the division sequence 4 or less is excluded.

Table 2: A copies $21 + 288t$ to $[21 + 384t]$

[6, −4]	$Y = 4x/3 - 7$	**3 + 9*t***	Excluded because x is even when t is
	$3 + 18t$	$(3(3 + 18t) + 1)/2 = 5 + 27t$	Excluded when t is even
	$21 + 36t$	$(3(21 + 36t) + 1)/4 = 16 + 27t$	Excluded when t is odd
	$21 + 72t$	$(3(21 + 72t) + 1)/8 = 8 + 27t$	Excluded when t is odd
	$21 + 144t$	$(3(21 + 144t) + 1)/16 = 4 + 27t$	Excluded when t is odd
	$21 + 288t$	$4(21 + 288t)/3 - 7 = 21 + 384t$	Star conversion

The star conversion A for 21[6] replaces [6]-A->[6, 2] with [6]-A->[6]. The Collatz value is 21, and it does not change. As B[1, −2], sum of which the first term of the division sequence is 2 or less is excluded.

Table 3: B copies $21 + 72t$ to $[3 + 12t]$.

[1, −2]	$Y = x/6 - 1/2$	$3 + 9t$	Excluded because x is even when t is
	$3 + 18t$	$(3(3 + 18t) + 1)/2 = 5 + 27t$	Excluded when t is even
	$21 + 36t$	$(3(21 + 36t) + 1)/4 = 16 + 27t$	Excluded when t is odd
	$21 + 72t$	$(21 + 72t)/6 - 1/2 = 3 + 12t$	Star conversion

6 mod 9 described as C [4, −4], sum of which the first term of the division sequence is 4 or less is excluded.

Table 4: C copies 213 + 288t to [69 + 96t].

[4, −4]	$Y = x/3 – 2$	$6 + 9t$	Excluded because x is even when t is
	$15 + 18t$	$(3(15 + 18t) + 1)/2 = 23 + 27t$	Excluded when t is even
	$33 + 36t$	$(3(33 + 36t) + 1)/4 = 25 + 27t$	Excluded when t is even
	$69 + 72t$	$(3(69 + 72t) + 1)/8 = 26 + 27t$	Excluded when t is odd
	$69 + 144t$	$(3(69 + 144t) + 1)/16 = 13 + 27t$	Excluded when t is even
	$213 + 288t$	$(213 + 288t)/3 – 2 = 69 + 96t$	Star conversion

As D[3, −2], sum of which the first term of the division sequence is 2 or less is excluded.

Table 5: D copies $69 + 72t$ to $[45 + 48t]$

[3, −2]	$Y = 2x/3 – 1$	$6 + 9t$	Excluded because x is even when t is even
	$15 + 18t$	$(3(15 + 18t) + 1)/2 = 23 + 27t$	Excluded when t is even
	$33 + 36t$	$(3(33 + 36t) + 1)/4 = 25 + 27t$	Excluded when t is even
	$69 + 72t$	$2(69 + 72t)/3 – 1 = 45 + 48t$	Star conversion

0 mod 9 is described as E[2, −4], therefore sum of which the first term of the division sequence is 4 or less is excluded.

Table 6: E copies $117 + 288t$ to $[9 + 24t]$.

[2, −4]	$Y = x/12 – 3/4$	$9t$	Excluded because x is even when even
	$9 + 18t$	$(3(9 + 18t) + 1)/2 = 14 + 27t$	Excluded when t is odd
	$9 + 36t$	$(3(9 + 36t) + 1)/4 = 7 + 27t$	Excluded when t is even
	$45 + 72t$	$(3(45 + 72t) + 1)/8 = 17 + 27t$	Excluded when t is even
	$117 + 144t$	$(3(117 + 144t) + 1)/16 = 22 + 27t$	Excluded when t is odd
	$117 + 288t$	$(117 + 288t)/12 – 3/4 = 9 + 24t$	Star conversion

As F[5, −2], sum of which the first term of the division sequence is 2 or less is excluded.

Table 7: F copies 45 + 72t to [117 + 192t]

[5, −2]	$Y = 8x/3 – 3$	$9t$	Excluded because x is even when t is
	$9+18t$	$(3(9 + 18t) + 1)/2 = 14 + 27t$	Excluded when t is odd
	$9+36t$	$(3(9 + 36t) + 1)/4 = 7 + 27t$	Excluded when t is even
	$45 + 75t$	$8(45 + 72t)/3 – 3 = 117 + 192t$	Star conversion

Table 8: G copies $3 + 6t$ to $[213 + 384t]$.

G[+6]	Y = 64x + 21	3 + 6t
3 + 6t	64(3 + 6t) + 21 = 213 + 384t	Star conversion

It can be seen that any conversion provides copying from $3 + 6t$ to $3 + 6t'$.

Synthesis

Figure 1 shows the synthesis of each calculated star conversion. Combining all star conversions gives "$3 + 6t$".

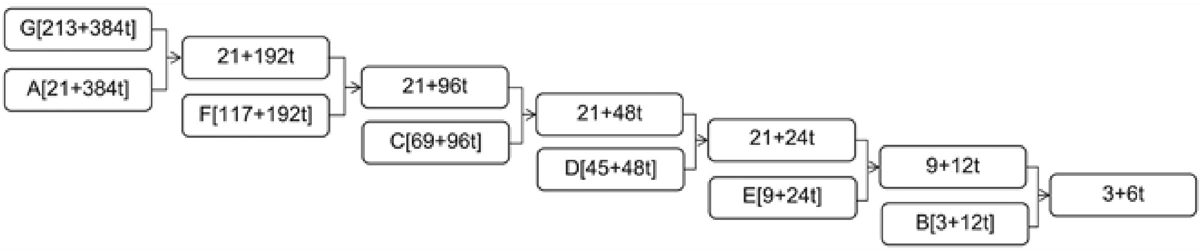

Figure 1: Star conversion synthesis

Theorem 2.1. Therefore, any complete division sequence is obtained by performing star conversion on any complete division sequence.

If all complete division sequences have finite lengths, then the Collatz conjecture is true.

5. Extended Star Conversion and Extended Complete Division Sequence

Definition 3.1. The extended star conversion is the conversion in which the star conversion is applied multiple times to the complete division sequence of x excluding the Collatz value x of 3, 9. Table 9 shows the extended star conversion.

Table 9: Extended star conversion

No.	When	Extended star conversion	After conversion
	0 mod 9		
1	9	None	
2	$72t + 45$	E[2, −4] $y = x/12 – 3/4$	$6t + 3$
3	$216t + 81$	DE[3, 0, −4] $y = x/18 – 3/2$	$12t + 3$
4	$216t + 153$	AE[6, −2, −4] $y = x/9 – 8$	$24t + 9$
5	$216t + 225$	FE[5, 0, −4] $y = 2x/9 – 5$	$48t + 45$
6	$108t + 27$	CF[4, 1, −2] $y = 8x/9 – 3$	$96t + 21$
7	$108t + 63$	BF[1, 3, −2] $y = 4x/9 – 1$	$48t + 27$
8	$108t + 99$	EF[2, 1, −2] $y = 2x/9 – 1$	$24t + 21$
	6 mod 9		
9	$18t + 15$	C[4, −4] $y = x/3 – 2$	$6t + 3$
	3 mod 9		
10	3	None	
11	$36t + 21$	B[1, −2] $y = x/6 – 1/2$	$6t + 3$
12	$108t + 39$	DB[3, −1, −2] $y = x/9 – 4/3$	$12t + 3$

13	108t + 75	AB[6, −3, −2] $y = 2x/9 - 23/3$	24t + 9
14	108t + 111	FB[5, −1, −2] $y = 4x/9 - 13/3$	48t + 45

The extended star conversion copies the initial value from 6t + 3 to 6t' + 3.

Definition 3.2: The extended complete division sequence is the division sequence obtained by performing the extended star conversion.

Elements of the extended complete division sequence can contain 0 or negative values.

Definition 3.3: Set of Collatz value and its division sequence
We call copying from (n, [a, b, ...]) to ((3n + 1)/2a, [b, ...]) an extended Collatz operation.

Definition 3.4: An extended Collatz division sequence is the division sequence when the Collatz value reaches an (odd number that is not a multiple of 3)/2r (r ≠ 0) after the extended Collatz operation.

Theorem 3.1: Applying the extended star conversion to a complete division sequence of x other than 3, 9 gives the extended complete division sequence of x'.

Proof: Self-evident from the definition of extended star conversion.

Theorem 3.2: Applying the extended Collatz operation to the extended complete division sequence of x gives the extended division sequence of z or the division sequence of y.

Proof: Suppose the Collatz value after the operation is a natural number y, then the division sequence of y is obtained. Otherwise, the extended division sequence of z is obtained.

Theorem 3.3: Applying the extended Collatz operation to the extended sequence z gives the extended division sequence of z' or the division sequence of y. Proof: If the Collatz value after the operation is a natural number y, the division sequence of y is obtained. Otherwise, the extended division sequence of z' is obtained.

6. Proof of Final Theorem

Base make Limited DivSeq

Theorem 6.1: Machine

```
makeLimitedDivSeq : (n : Nat)
  → ((z : Nat) → ((FirstLimited. B.allDivSeq) z → (AllLimited.B.allDivSeq) z))
    → (FirstLimited.B.allDivSeq) n
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Proof: We prove this using the well-founded induction method and case division. For the number of each case division:

- The number is reduced using rs;
- The predicate is converted to AllLimited using firstToAll;
- Finally, the predicate and numbers are undone using IsFirstLimitedxx.

Table 10 shows the extended star conversion and the decrease in the Collatz value after conversion (6t + 3 in the table below is n in the source code).

When	Extended star conversion	After conversion	Got smaller
0 mod 9			
9	None because of the base case		
72t + 45	E[2, −4] $y = x/12 - 3/4$	6t + 3	72t + 45 > 6t + 3
216t + 81	DE[3, 0, −4] $y = x/18 - 3/2$	12t + 3	216t + 81 > 12t + 3
216t + 153	AE[6, −2, −4] $y = x/9 - 8$	24t + 9	216t + 153 > 24t + 9
216t + 225	FE[5, 0, −4] $y = 2x/9 - 5$	48t + 45	216t + 225 > 48t + 45
108t + 27	CF[4, 1, −2] $y = 8x/9 - 3$	96t + 21	108t + 27 > 96t + 21
108t + 63 108t + 99	BF[1, 3, −2] $y = 4x/9 - 1$ EF[2, 1, −2] $y = 2x/9 - 1$	48t + 27 24t + 21	108t + 63 > 48t + 27 108t + 99 > 24t + 21
6 mod 9			
18t + 15	C[4, −4] $y = x/3 - 2$	6t + 3	18t + 15 > 6t + 3
3 mod 9			
3	None because of the base case		
36t + 21	B[1, −2] $y = x/6 - 1/2$	6t + 3	36t + 21 > 6t + 3
108t + 39	DB[3, −1, −2] $y = x/9 - 4/3$	12t + 3	108t + 39 > 12t + 3
108t + 75	AB[6, −3, −2] $y = 2x/9 - 23/3$	24t + 9	108t + 75 > 24t + 9
108t + 111	FB[5, −1, −2] $y = 4x/9 - 13/3$	48t + 45	108t + 111 > 48t + 45

Sufficient Condition for First To All
Since First Limited (first) is given as an argument, decomposing/combining it gives All Limited. Figure 4 shows the decomposing/combining from "first" to "All Limited".

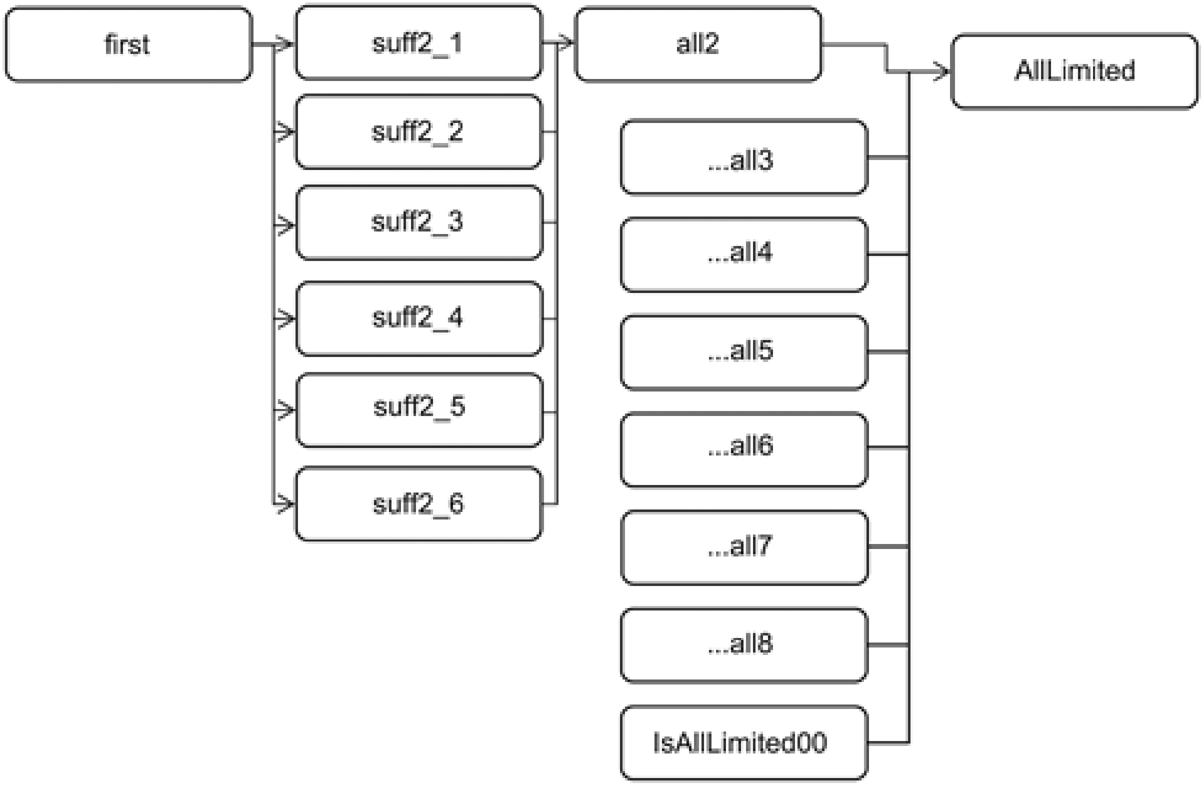

Figure 4: Decomposing/combining from "first" to "AllLimited". Decomposing/combining or creating "AllLimited" from "first" inside a function. The proved function firstToAll performs the following processing. "first" means "FirstLimited for all n". "suff2_1~6" means "FirstLimited in partial n". Combine suff2_1~6 to get all2. Combine all2~8, IsAllLimited00 to get All Limited.

Proof of Final Theorem limitedDivSeq
Theorem 6.2

limitedDivSeq: (n : Nat) → (FirstLimited. B.allDivSeq) n

Machine proof: Passing firstToAll to makeLimitedDivSeq gives limitedDivSeq.

7. Conclusion

It is just a case study or my opinion. I think this conjecture can be proved by these processes. Thanks to the readers