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Conjugate Graph and Some Properties of the $SL(2,3)$ Group

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Abstract: ***Conjugate graph is a graph in which the vertices are the non central elements of a group and two vertices are adjacent if they are conjugate. In this paper, the conjugate graph of $SL(2,3)$ is first calculated and then its adjacency matrix, Laplacian, spectrum, line graph and complement graphs are found. We also calculate the different graph properties like clique number, dominating number, chromatic number and independence number of this graph.***

Keywords: Conjugate graph, $SL(2,3)$ group, adjacency matrix, line graph, complement graph

1. Introduction

In this section, we give some preliminaries of graph theory and group theory that will be used throughout this paper:

$SL(2,3)$ group: [9] It is a group of order 24 with the presentation:

$$\{a, b, c \mid a^3 = b^3 = c^2 = abc\}$$

Conjugate of an element: [6] Suppose G is a finite group. Two elements a and b of G are called conjugate if there exists g in G with $g^{-1}ag = b$.

Conjugacy class: [6] The equivalence class that contains the element $a$in G is $a^G = \{gag^{-1} \mid g \in G\}$ and is called the conjugacy class of a.

Proper coloring: [1] A proper coloring of a graph X is a map from $V(X)$ into some finite set of colors such that no two adjacent vertices have the same color. If X can be properly colored with a set of k colors, then we say that X can be properly k-colored.

Chromatic number: [1] The least value of k for which X can be properly k-colored is the chromatic number of X and is denoted by $\chi(X)$.

Clique number: [1] A subset C of vertices of Γ is called a clique if the induced subgraph on C is a complete graph. The maximum size of a clique is called clique number of the graph G and is denoted by $\omega(G)$.

Independent set: [1] A subset X of the vertices of the graph G is called an independent set if the induced subgraph on X has no edges. The maximum size of an independent set in the graph G is called the independence number of the graph denoted by $\alpha(G)$.

Dominating number: [1] For a graph Γ and a subset S of vertices, denote by $N_\Gamma[S]$ the set of vertices in Γ which are in S or adjacent to a vertex in S. If $N_\Gamma[S] = V(\Gamma)$, then S is called a dominating set for Γ. The dominating number $\gamma(\Gamma)$ of Γ is the minimum size of a dominating set of the vertices of Γ.

Line graph: [4] The line graph of a graph X is the graph $L(X)$ with the edges of X as its vertices, and where two edges of X are adjacent in $L(X)$ if and only if they are incident in X.

Complement graph: [4] The complement $\overline{X}$ of a graph X has the same vertex set as X, where vertices x and y are adjacent in $\bar{X}$ if and only if they are not adjacent in X.

Conjugate graph: [1] A conjugate graph is a graph whose vertices are the non-central elements of a group G and two distinct vertices are adjacent if they are conjugate. It is denoted by Γ_G^c.

2. Results and Discussion

$SL(2,3)$ group can be defined as a group of 2×2 matrices with determinant '1' over the field of 3 elements.

Theorem 1: The conjugate graph of $SL(2,3)$ group is a union of five complete graphs, four of which is of order 4 and the last one is of order 6.

Proof: Number of conjugacy classes of $SL(2,3)$ group = 7

C.C (1) = $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

C.C (1) = $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$

C.C (1) = $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}$

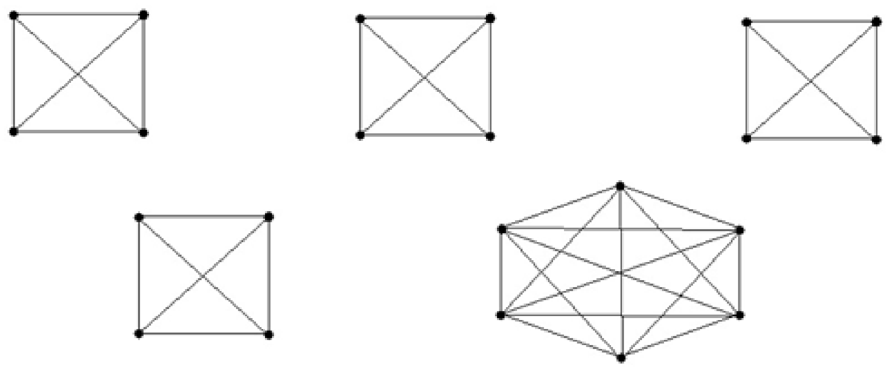
C.C (1) = $\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix}$

C.C (1) = $\begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix}$

C.C (1) = $\begin{bmatrix} -1 & -1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix}$

C.C (1) = $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} -1 & -1 \\ -1 & 1 \end{bmatrix}$

Hence the conjugate graph is given by:



Which is a disjoint union of five complete graphs, four of which are of order 4 and one is of order 6.

Proposition 1: The clique number and the chromatic number of the conjugate graph of the $SL(2,3)$ group is equal to 6.

Proof: Firstly, we know that the clique number and chromatic number of a conjugate graph are equal [1]. From theorem 1, from the figure, clearly, the chromatic number of the SL (2,3) group is equal to 6, since the largest complete component in the graph is of order 6. Hence,

clique number = chromatic number = 6

Proposition 2: The independence number and the dominating number of the conjugate graph of the $SL(2,3)$group is equal to 5.

Proof: We know that the independence number is equal to the dominating number for a conjugate graph. From theorem 1, from the figure, clearly, the dominating number is equal to 5 since there are 5 complete graphs. Hence,

independence number = dominating number = 5

Theorem 2: The adjacency matrix of the conjugate graph of the $SL(2,3)$ group is:

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$$\begin{bmatrix}
0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
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0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
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0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0
\end{bmatrix}$$

And the spectrum is = $\{(-1)^{(17)}, 3^4, 5\}$

Theorem 3: The Laplacian of the conjugate graph of the $SL(2,3)$ group is:

$$\begin{bmatrix}
3 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 3 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & -1 & 3 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & -1 & -1 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 3 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 3 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & -1 & 3 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & -1 & -1 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 3 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 3 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & -1 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 3 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
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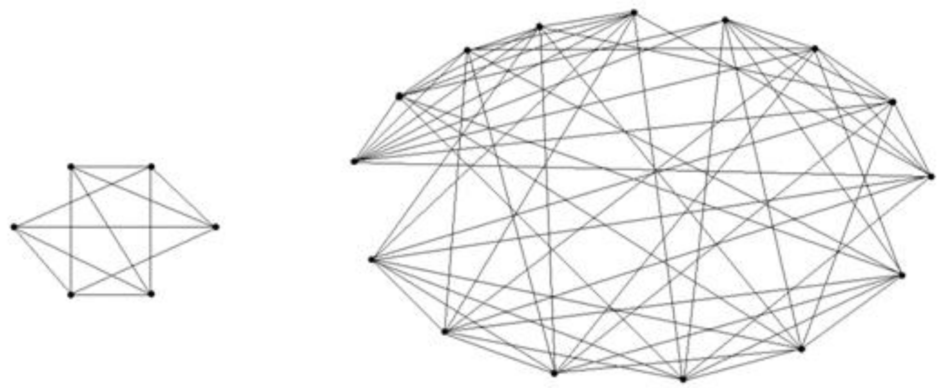
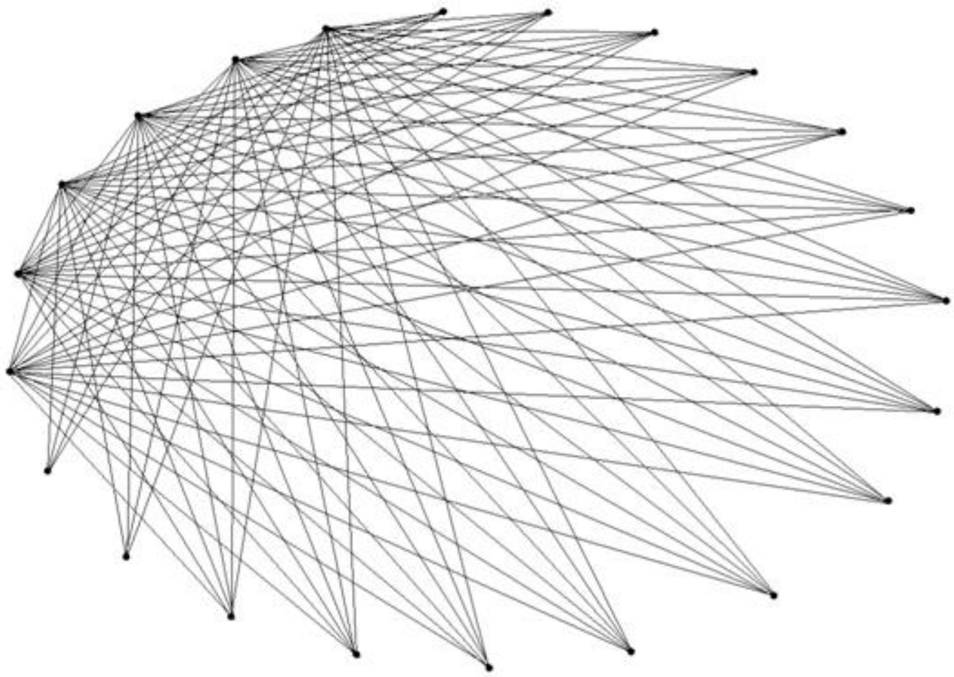
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & -1 & 3 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 5 & -1 & -1 & -1 & -1 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 5 & -1 & -1 & -1 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 5 & -1 & -1 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & -1 & 5 & -1 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & -1 & -1 & 5 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & -1 & -1 & -1 & 5
\end{bmatrix}$$

The spectrum of this is: $\{0^{(5)}, 4^{(12)}, 6^{(5)}\}$.

Proposition 3: The line graph of the conjugate graph of $SL(2,3)$ is a disjoint union of regular graphs, four of which are 4-regular graphs of order 6 and one is an 8 regular graph of order 15.



Proposition 4: The complement graph of the conjugate graph of $SL(2,3)$ is the complete multipartite graph $K_{4,4,4,4,6}$ of order 22.



3. Conclusion

In this research, we found that the conjugate graph of the $SL(2,3)$ group is a union of five complete graphs, four of which are of order 4 and the last one is of order 6. The independence number and dominating number are both found to be equal to 5, and the graph is also found to be a perfect graph. The line graph of this graph is found to be a disjoint union of regular graphs and the complement graph is found to be a complete multipartite graph.

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