A Retrial Queuing System with Two Types of Batch Arrivals and with Feedback to Orbit

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Abstract: ***A retrial queuing system with two types of batch arrivals is considered. The arrivals are called type I and type II customers. The type I customers arrive in batches of size k with probability c_k and type II customers arrive in batches of size k with probability d_k (k>0) according to two independent Poisson processes with rates $\lambda_1\ \bar{c} = \lambda_1 \sum_{k=1}^{\infty} k\, c_k$ and $\lambda_2\ \bar{d} = \lambda_2 \sum_{k=1}^{\infty} k\, d_k$ respectively. Service time distributions are identical, independent and are different for both type of customers. If the arriving customers are blocked due to server being busy, type I customers are queued in a priority queue of infinite capacity where as type II customers enter into a retrial group of infinite capacity in order to seek service again after a random amount of time. All the customers in the retrial group behave independent of each other. The retrial time is exponentially distributed with rate α. A type I or type II customer who has received service departs the system with probability (1-q) or returns to the retrial group with probability q. For this model the joint distribution of the number of customers in the priority queue and in the retrial group is obtained in closed form. Some particular models and operating characteristics are obtained. A real life example is given. Few numerical models are generated by assuming particular values to the parameters.***

Keywords: Batch arrival, Feedback, Retrial, Joint-distribution and Operating characteristics

AMS Subject classification number: 90B22, 60K25 and 60K30.

1. Introduction

In the field of queuing theory, retrial queues have been concentrated research topic for the last sixty years. Retrial queues are characterised by the fact that an arrival finds the server busy upon arrival is asked to leave or decides to leave the service area and joins a queue called orbit. After some random time the customer in the orbit can repeat their request for service. This request is independent of the rest of the customers in the orbit. For detailed survey one can see Yang and Templeton (1987), Falin (1992), Choi and Chang (1999), Artalejo and Gomez-Corral (2008) and Kim and Kim (2016).

Falin (1984) investigated a multichannel retrial queuing system. Choi and Park (1990) investigated an M/G/1 retrial queue with two types of customers in which the service time distribution for both type of customers are the same. Khalil et al. (1992) investigated the above model at Markovian level in detail. Falin et al. (1993) investigated a similar model, in which they assumed different service time distributions for both type of customers. In 1995, Choi et al., studied an M/G/1 retrial queue with two types of customers and finite capacity. Kalyanaraman and Srinivasan (2004), studied an M/G/1 retrial queue with geometric loss and with type I batch arrivals and type II single arrivals. Lee (2005) studied a non-Markovian retrial queue with two types of customers and with feedback of customers. In 2011, Thillaigovindan and Kalyanaraman have analyzed a feedback retrial queuing system with two types of arrivals. Kalyanaraman (2012) analysed a feedback retrial queue with two types of batch arrivals. Gao (2015) analyzed a retrial queue with two class of customers, in which the primary customers have pre-emptive priority over secondary customers. Toth and Sztrik (2021) studied the performance analysis of two-way communication retrial queuing systems with non-reliable server and impatient customers in the orbit.

This paper deal with a retrial queuing system with two types of batch arrivals and with feedback to orbit, in which both type of customers arrive in batches of variable size. In section 2, we give the descrition of the system. In section 3, we obtain the joint probability generating function for the number of customers in the priority queue and in the retrial group when the servers are busy as well as idle. The expressions for some particular models are deduced in section 4. Some operating characteristics are derived in section 5 and a real life related situation is given in Section 6. A numerical study is carried out in section 7. Finally, the last section ends with a conclusion.

2. The Model

A retrial queuing system with two types of customers is considered in this paper. The type I customers arrive in batches of size k with probability c_k and type II customers arrive in batches of size k with probability d_k (k>0) according to two independent Poisson processes with rates $\lambda_1\ \bar{c} = \lambda_1 \sum_{k=1}^{\infty} k\, c_k$ and $\lambda_2\ \bar{d} = \lambda_2 \sum_{k=1}^{\infty} k\, d_k$ respectively. The services are given singly by a server. If type II customers, upon arrival find the server busy, they enter in to an orbit of infinite capacity in order to seek service again after a random amount of time. All the customers in the retrial group behave independent of each other. The retrial time is exponentially distributed with mean $1/\alpha$. The type I customers are queued in a priority queue of infinite capacity after blocking, if the server is busy. As soon as the server is free, the customers in the priority queue are served using FCFS (First Come First Served) rule and the customers in the retrial group are served only if there are no customers in the priority queue. A type I or type II customer who has received service departs the system with probability (1-q) or returns to the retrial group with probability q, 0<q<1 for additional service, as feedback customer. The model description is given in Figure 2.1.

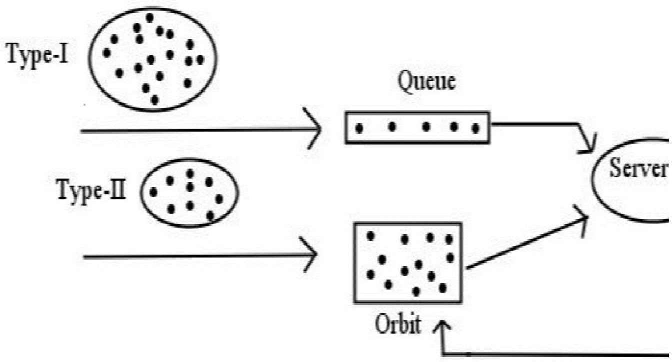

Figure: 2.1 System Model

The service time distributions for both type of customers are independently distributed random variables with different distributions. Supplementary variable technique is used for the analysis and the variable considered is the residual service time of a customer in service. The service time density functions are $b_k\ (x);\ k = 1,2$ and $B_k^{(*)}(s) = \int_0^\infty e^{-sx}\, b_k\ (x)dx, k = 1,2$ is the Laplace transformation of the distribution function $b_k\ (x)$.

The Stochastic process related to this model is $\{X(t)\} = \left\{\left(\xi(t), N_p(t), N_r(t), S_k(t)\right) : t \geq 0\right\}$ where

$N_p\ (t)$ = number of customers in the priority queue at time t

$N_r\ (t)$= number of customers in the retrial group at time t

$\xi\ (t)$= the server state at time t,

$$\xi(t) = \begin{cases} 0, & \textit{when the server is idle} \\ 1, & \textit{when the server is busy with a type I customer} \\ 2, & \textit{when the server is busy with a type II customer} \end{cases}$$

and

$S_k(t) =$ *the residual service time of a type k customer in service at time t,* $\quad k = 1,2.$

The stochastic process $\{X(t)\colon t \geq 0\}$ is Markov process with state space $\Omega = \{0,1,2\}\ X\ \{0,1,2,\ldots\}\ X\ \{0,1,2,\ldots\}\ X\ (0,\infty)$ and the corresponding stationary process is $\{(\xi,\ N_p,\ N_r,\ S_k)\}$.

The related probabilities are defined as

$q_j\,(t) = Pr\{\xi(t) = 0, N_r(t) = j\}$ and

$p\ (k,i,j;x,t)dx = Pr\{\xi\ (t) = k, N_p\ (t) = i, N_r\ (t) = j, S_k\ (t) \epsilon\ (x, x+dx)\}, k = 1,2.$

In steady state, the corresponding probabilities are, $q_j = \log_{t\to\infty} q_j\ (t)$ and $p(k,i,j;x) = \log_{t\to\infty} p(k,i,j;x,t)$ and the Laplace transformation of $p\ (k,i,j;x)$ is

$$p^{(*)}(k,i,j;s) = \int_0^\infty e^{-sx}\ p(k,i,j;x)dx,$$
$$k = 1,2,\colon i,j \geq\ 0.$$

It is clear that,

$p^{(*)}(k,i,j;0) = \int_0^\infty \mathrm{p(k,i,j;x)dx} = \mathrm{Pr} = \{\xi = \mathrm{k}, \mathrm{N_p} = \mathrm{i}, \mathrm{N_r} = \mathrm{j}\}$

is the steady state probability that there are i customers in the priority queue, j customers in the retrial group and the server services a k[th] type customer.

For $-1\ \leq\ Z_{1,}\,Z_2\ \leq 1,$ the following probability generating functions are defined for the analysis:

$$Q(Z_2) = \textstyle\sum_{j=0}^\infty q_j Z_2^j$$
$$C(Z_1) = \textstyle\sum_{j=0}^\infty c_j Z_1^j$$
$$D(Z_1) = \textstyle\sum_{j=0}^\infty d_j Z_2^j$$

$$P^{(*)}(k,i;s,Z_2) = \sum_{j=0}^\infty p^{(*)}\,(k,i,j;s)Z_2^j,\quad i = 0,1,2,\ldots;k = 1,2$$

$$P^{(*)}(k;s,Z_1,Z_2) = \sum_{j=0}^\infty P^{(*)}\,(k,i;s,Z_2)Z_1^j,\qquad k = 1,2$$

$$P(k,i;0,Z_2) = \sum_{j=0}^\infty p^{(*)}\,(k,i,j;0)Z_2^j,\quad i = 0,1,2,\ldots;k = 1,2$$

$$P(k;0,Z_1,Z_2) = \sum_{j=0}^\infty P\,(k,i;0,Z_2)Z_1^j,\qquad k = 1,2$$

3. The Analysis

Using the mean drift argument of Falin (1984), it can be shown that the system is stable if $\rho_1 + \rho_2 < 1$ where $\rho_1 = -\lambda_1\bar{c}\,B_1^{(*)'}(0), \rho_2 = -\lambda_2\bar{d}\,B_2^{(*)'}(0)$.

Now the mathematical equations that govern the system are obtained by employing the remaining service time as the supplementary variable. Relating the state of the system at time t and t+dt, the following partial differential difference equations are obtained.

For $j\ \geq 0, x\ \geq 0, i \geq 0$

$$(\lambda + j\,\alpha)\frac{d}{dt}q_j\ (t) = (1-q)p\ (1,0,j;0,t) + (1-q)p\ (2,0,j;0,t) + qp\ (2,0,j-1;0,t) \qquad \text{--------- (3.1)}$$

$$\frac{-\partial p(1,0,j;x,t)}{\partial x} + \frac{-\partial p(1,0,j;x,t)}{\partial t} = -\lambda p(1,0,j;x,t)\ +\ \lambda_1\, b_1(x)q_j\,(t) + (1-q)b_1(x)p(1,1,j;0,t) +\ \lambda_2 \textstyle\sum_{k=1}^j d_k\ p\ (1,0,j-k;x,t) \qquad \text{--------------------- (3.2)}$$

$$\frac{-\partial p(1,i,j;x,t)}{\partial x} + \frac{-\partial p(1,i,j;x,t)}{\partial t} = -\lambda p(1,i,j;x,t) + (1-q)b_1(x)p(1,i+1,j;0,t) +\ \lambda_1 \sum_{k=1}^i c_k\ p(1,i-k,j;x,t) + \lambda_2 \textstyle\sum_{k=1}^j d_k\ p(1,i,j-k;x,t) \qquad \text{------------- (3.3)}$$

$$\frac{-\partial p(2,0,j;x,t)}{\partial x} + \frac{-\partial p(2,0,j;x,t)}{\partial t} = -\lambda p(2,0,j;x,t) +\ \lambda_2\ b_2(x) \sum_{k=0}^j d_{k+1}\, q_{j-k}(t)$$

$+(j+1)\alpha b_2(x)q_{j+1}(t) + \lambda_2 \sum_{k=1}^{j} d_k\, p(2,0,j-k;x,t)$ -- ------------------------ (3.4)

$$\frac{-\partial p(2,i,j;x,t)}{\partial x} + \frac{-\partial p(2,i,j;x,t)}{\partial t} = -\lambda p(2,i,j;x,t) + \lambda_1 \sum_{k=1}^{i} c_k\, p(2,i-k,j;x,t) + \lambda_2 \sum_{k=1}^{j} d_k\, p(2,i,j-k;x,t)$$

------------------------ (3.5)

In steady state, the equations (3.1) to (3.5) become,

$(\lambda + j\,\alpha)q_j = (1-q)p(1,0,j;0) + (1-q)p(2,0,j;0) + qp(2,0,j-1;0)$ ------ (3.6)

$-p'(1,0,j;x) = -\lambda p(1,0,j;x) + \lambda_1 b_1(x)q_j + (1-q)b_1(x)p(1,1,j;0) + \lambda_2 \sum_{k=1}^{j} d_k\, p(1,0,j-k;x)$ ------------------------ (3.7)

$-p'(1,i,j;x) = -\lambda p(1,i,j;x) + (1-q)b_1(x)p(1,i+1,j;0) + \lambda_1 \sum_{k=1}^{i} c_k\, p(1,i-k,j;x)$
$\lambda_2 \sum_{k=1}^{j} d_k\, p(1,i,j-k;x)$ ---------------------- (3.8)

$-p'(2,0,j;x) = -\lambda p(2,0,j;x) + \lambda_2 b_2(x) \sum_{k=0}^{j} d_{k+1} q_{j-k}$
$+(j+1)\alpha b_2(x)q_{j+1} + \lambda_2 \sum_{k=1}^{j} d_k\, p(2,0,j-k;x)$
---- (3.9)

$$-p'(2,i,j;x) = -\lambda p(2,i,j;x) + \lambda_1 \sum_{k=1}^{i} c_k\, p(2,i-k,j;x) + \lambda_2 \sum_{k=1}^{j} d_k\, p(2,i,j-k;x)$$

--------------------- (3.10)

and the normalization condition is,

$\sum_{j=0}^{\infty}\sum_{i=0}^{\infty}\int_0^{\infty} p(1,i,j,x)dx + \sum_{j=0}^{\infty}\sum_{i=0}^{\infty}\int_0^{\infty} p(2,i,j,x)dx + \sum_{j=0}^{\infty} q_j = 1$ ------------ (3.11)
where $\lambda = \lambda_1 + \lambda_2.$

By taking Laplace transformation of equations (3.6) to (3.10) and multiplying by Z_2^j and then summing over j, from 0 to ∞ the following equations are obtained.

$$Q(Z_2) + \alpha Z_2\, Q'(Z_2) = (1-q)P(1,0;0,Z_2) + (1-q)P(2,0;0,Z_2) + qP(2,0;0,Z_2)$$

---------- (3.12)

$(s-\lambda+\lambda_2 D(Z_2))P^{(*)}(1,0;s,Z_2) = P(1,0;0,Z_2) - \lambda_1\ B_1^{(*)}(s)Q(Z_2) - (1-q)B_1^{(*)}(s)P(1,1;0,Z_2))$ --- (3.13)

$$(s-\lambda+\lambda_2 D(Z_2))P^{(*)}(1,i;s,Z_2) = P(1,i;0,Z_2) - \lambda_1 \sum_{k=1}^{i} c_k P^{(*)}(1,i-k;s,Z_2)$$
$- (1-q)B_1^{(*)}(s)P(1,i+1;0,Z_2))$
---- (3.14)

$(s-\lambda+\lambda_2 D(Z_2))P^{(*)}(2,0;s,Z_2) = P(2,0;0,Z_2) - \lambda_2\ B_2^{(*)}(s)\frac{D(Z_2)}{Z_2}Q(Z_2) - \alpha B_2^{(*)}(s)Q'(Z_2)$ ------ (3.15)

$(s-\lambda+\lambda_2 D(Z_2))P^{(*)}(2,i;s,Z_2) = -\lambda_1 \sum_{k=1}^{i} c_k P^{(*)}(2,i-k;s,Z_2)$ ----------- (3.16)

Multiplying equations (3.14) and (3.16) by Z_1^i and summing over i=1,2,... and using equations (3.13) and (3.15) we get,

$$(s-\lambda+\lambda_1 C(Z_1)+\lambda_2 D(Z_2))P^{(*)}(1;s,Z_1,Z_2) = P(1;0,Z_1,Z_2) - \lambda_1 B_1^{(*)}(s)Q(Z_2) + \frac{(1-q)B_1^{(*)}(s)}{Z_1}[P(1;0,0,Z_2) - P(1;0,Z_1,Z_2)]$$
----- (3.17)

$$(s-\lambda+\lambda_1 C(Z_1)+\lambda_2 D(Z_2))P^{(*)}(2;s,Z_1,Z_2) = P(2;0,0,Z_2) - \lambda_2 B_2^{(*)}(s)\frac{D(Z_2)}{Z_2}Q(Z_2)$$
$- \alpha B_2^{(*)}(s)Q'(Z_2)$
----------------- (3.18)

Substituting $s = \lambda - \lambda_1 C(Z_1) - \lambda_2 D(Z_2)$ in (3.17) and (3.18), we get

$P(1,0;0,Z_2) = \frac{\lambda_1 Z_1}{1-q}Q(Z_2) - \frac{Z_1-(1-q)B_1^{(*)}(l)}{(1-q)B_1^{(*)}(l)}P(1;0,Z_1,Z_2)$
------------- (3.19)

$P(2,0;0,Z_2) = B_2^{(*)}(l)\left[\lambda_2 \frac{D(Z_2)}{Z_2}Q(Z_2) + \alpha Q'(Z2)\right.$ --------------------- (3.20)

where $l = \lambda - \lambda_1 C(Z_1) - \lambda_2 D(Z_2)$

Using equations (3.19) and (3.20) in (3.12) and simplifying one can get the following equation,

$$\alpha[Z_2 - (1-q+q\,Z_2)B_2^{(*)}(l)]Q'(Z_2) + \left[\lambda - \lambda_1 Z_1 - (1-q+q\,Z_2)\lambda_2 B_2^{(*)}(l)\frac{D(Z_2)}{Z_2}\right]Q(Z_2) = \frac{(1-q)B_1^{(*)}(l)-Z_1}{B_1^{(*)}(l)}P(1;0,Z_1,Z_2)$$ ------------- (3.21)

Define,

$f(Z_1,Z_2) = \frac{(1-q)B_1^{(*)}(l)-Z_1}{B_1^{(*)}(l)}$ for each fixed Z_2, $-1 \le Z_2 \le 1$. By Rouche's theorem, there is a unique solution $Z_1 = h(Z_2)$ of the equation f (Z_1,Z_2)=0. Now (3.21) becomes

$$Q'(Z_2) = \frac{1}{\alpha}\frac{\lambda-\lambda_1 h(Z_2)-\lambda_2 U(Z_2)\frac{D(Z_2)}{Z_2}}{U(Z_2)-Z_2}Q(z_2)$$ ------- (3.22)

where $h(Z_2)$ is the root of the equation $Z_1 = B_1^{*}(l)$ and $U(Z_2) = (1-q+qZ_2)B_2^{(*)}(l)$

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Using (3.22) in equation (3.21), it can be seen that

$$P(1;0;Z_1,Z_2)=\frac{\left\{L\left[Z_2-(1-q+qZ_2)B_2^{(*)}(l)\right]+R[U(Z_2)-Z_2]\right\}B_1^{(*)}(l)}{\left[(1-q)B_1^{(*)}(l)-Z_1\right][U(Z_2)-Z_2]}Q\,(Z_2) \quad \text{------- (3.23)}$$

where $L=\lambda-\lambda_1 h(Z_2)-\lambda_2 U\,(Z_2)\frac{D\,(Z_2)}{Z_2}$ and

$$R=\lambda-\lambda_1 Z_1-\ (1-q+qZ_2)\lambda_2 B_2^{(*)}\ (l)\,\frac{D\,(Z_2)}{Z_2}$$

Using equation (3.23) in (3.19), we obtain

$$P(1;0{,}0,Z_2)=\frac{[\lambda_1 Z_1+R][U\,(Z_2)-Z_2]+L[Z_2-(1-q+qz_2)B_2^{(*)}\,(l)]}{(1-q)[U\,(Z_2)-Z_2]}Q(Z_2) \quad \text{-------- (3.24)}$$

Using equation (3.22) in (3.20), we get

$$P(2;0{,}0,Z_2)=\frac{\{\lambda_2\frac{D(Z_2)}{Z_2}[U\,(Z_2)-Z_2]+L\}B_2^{(*)}\,(l)}{[U\,(Z_2)-Z_2]}Q(Z_2) \quad \text{----- (3.25)}$$

The general solution of the differential equation (3.22) is

$$Q\,(Z_2)=Q\,(1)\exp\left\{\frac{-1}{\alpha}\int_{Z_2}^{1}\frac{\lambda-\lambda_1 h(x)-\lambda_2 U(x)\frac{D(x)}{x}}{U(x)-x}\,dx\right\} \quad \text{--- (3.26)}$$

Where Q (1) is a constant, which is the probability that the server is idle.

Putting $s=0$ in equation (3.13) and in equation (3.14) and summing over $i=0$ to ∞, we get

$$\lambda_2(D\,(Z_2)-1)\sum_{i=0}^{\infty}P^{(*)}(1,i;0,Z_2)=(1-q)\,P\,(1{,}0;0,Z_2)-\lambda_1\,Q\,(z_2) \quad \text{------------ (3.27)}$$

Putting $s=0$ in equation (3.15) and in equation (3.16) and summing over $i=0$ to ∞, we get

$$\lambda_2(D\,(Z_2)-1)\sum_{i=0}^{\infty}P^{(*)}(2,i;0,Z_2)=P(2;0{,}0,Z_2)-\lambda_2 Q(z_2)\frac{D(Z_2)}{Z_2}-\alpha Q'\,(Z_2) \quad \text{- (3.28)}$$

Adding equations (3.27) and (3.28) and using equation (3.12), leads to

$$\lambda_2\,(D(Z_2)-1)\sum_{i=0}^{\infty}\sum_{k=1}^{2}P^{(*)}(k,i;0,Z_2)=\lambda_2\left(1-\frac{D(Z_2)}{Z_2}-\right.$$

$q1-Z2B2\ (*)l$DZ2Z2Q$Z2$+ α1−$Z2$q$B2\ (*)l$−1$Q'Z2$ 3.29)

Evaluating at Z_2=1 and using normalization condition we get

$$Q'(1)=\frac{\lambda_2\bar{d}-\lambda_2\,(1-q)Q\,(1)}{\alpha\,(1-q)} \quad \text{----------- (3.30)}$$

Using equation (3.30) in (3.26) leads to

$$Q(1)=\frac{\bar{c}[(1-\rho_1)(1-q)-\rho_2\ ]}{(1-q)\ (\bar{c}+\rho_1-\rho_1\bar{c})} \quad \text{---------- (3.31)}$$

In steady state, the probability generating function for the number of customers in the orbit when the server is idle is obtained from equations (3.31) and (3.26).

Substituting $s=0$ in equation (3.17)

$$P^{(*)}(1{,}0,Z_1,Z_2)=\frac{A}{Z_1 l} \quad \text{------------ (3.32)}$$

Where

$A=[(1-q)-Z_1]P\,(1;0,Z_1,Z_2)+\lambda_1 Z_1 Q\,(Z_2)-(1-q)\,P\,(1;0{,}0,Z_2)$

(3.32) together with equations (3.23) and (3.19) yields the joint probability generating function for the number of customers in the priority queue and in the orbit when the server is busy with type I customer as

$$P^{(*)}(1{,}0,Z_1,Z_2)=\frac{B}{l[(1-q)B_1^{*}(l)-Z_1][U(Z_2)-Z_2]}Q(Z_2) \quad \text{---- (3.33)}$$

$$B=\ [1-B_1^{(*)}(l)]\{L[Z_2-(1-q+qZ_2)B_2^{(*)}(l)]+R[U(Z_2)-Z_2]\}$$

Putting s=0 in equations (3.18) and (3.22) we get

$$P^{(*)}(2{,}0,Z_1,Z_2)=\frac{1}{l}\left\{\left[\frac{L}{U(Z_2)-Z_2}+\lambda_2\frac{D(Z_2)}{Z_2}\right]Q(Z_2)-\right.$$

$P2,0,0,Z2$ ------------ (3.34)

(3.34) together with equations (3.22) and (3.20) yields the joint probability generating function for the number of customers in the priority queue and in the orbit when the server is busy with type II customer as

$$P^{(*)}(2{,}0,Z_1,Z_2)=\frac{[1-B_2^{(*)}(l)]}{l}\left\{\frac{\lambda_2 D(z_2)}{Z_2}+\frac{L}{U\,(Z_2-Z_2)}\right\}Q(Z_2) \quad (3.35)$$

Thus we have the following theorem:

Theorem 3.1: The stationary distribution of { (ξ,N_p, N_r,S_k} has the following generating functions

$$Q(Z_2)=\frac{\bar{c}[(1-\rho_1)(1-q)-\rho_2]}{(1-q)\ (\bar{c}+\rho_1-\rho_1\bar{c})}\exp\left\{\frac{1}{\alpha}\int_{1}^{Z_2}\frac{\lambda-\lambda_1 h(x)-\lambda_2 U(x)\frac{D(x)}{x}}{U(x)-x}\,dx\right\} \quad \text{--------- (3.36)}$$

$$P^{(*)}(1{,}0,Z_1,Z_2)=\frac{\left[1-B_1^{(*)}(l)\right]\left\{L\left[Z_2-(1-q+qZ_2)B_2^{(*)}(l)\right]+R[U(Z_2)-Z_2]\right\}}{l\left[(1-q)B_1^{(*)}(l)-Z_1\right][U(Z_2)-Z_2]}Q\,(Z_2) \quad \text{----------- (3.37)}$$

$$P^{(*)}(2{,}0,Z_1,Z_2)=\frac{[1-B_2^{(*)}(l)]}{l}\left\{\frac{\lambda_2 D(z_2)}{Z_2}+\frac{L}{U\,(Z_2-Z_2)}\right\}Q(Z_2) \quad \text{----------- (3.38)}$$

Corollary: 1

The probability that the server busy is

$$P_B=P^{(*)}(1;0{,}1{,}1)+P^{(*)}(2;0{,}1{,}1)=\frac{[\rho_1(1-q)+\bar{c}\rho_2]}{(1-q)\ (\bar{c}+\rho_1-\rho_1\bar{c})} \quad \text{----------- (3.39)}$$

Corollary: 2

The probability that the server idle is

$$P_I=Q(1)=\frac{\bar{c}[(1-\rho_1)(1-q)-\rho_2]}{(1-q)\ (\bar{c}+\rho_1-\rho_1\bar{c})} \quad \text{\_\_\_\_\_\_\_\_\_\_ (3.40)}$$

4. Particular Models

By taking particular values to some parameters of the queuing system analyzed in the paper, the following models can be obtained:

Theorem 4.1: For $d_k=c_k=0,k\neq 1,q=0$ and $B_1\,(x)=B_2\,(x)=B\,(x)$ the stationary distribution of $\{(\xi,\ N_p,\ N_r,\ S_k)\}$. has the following generating functions

$$Q\,(Z_2)=\ (1-\rho_1-\rho_2)\left\{\frac{1}{\alpha}\int_{1}^{z_2}\frac{\lambda-\lambda_1 h(x)-\lambda_2 U\,(x)}{U(x)-x}dx\right\}$$

$$P^{(*)}(1;0,Z_1,Z_2) = \frac{\left(1-B^{(*)}(l)\right)\{L[Z_2-B^{(*)}(l)]+R[U(Z_2)-Z_2]\}}{l[B^{(*)}(l)-Z_1][U(Z_2)-Z_2]}Q(Z_2)$$

$$P^{(*)}(2;0,Z_1,Z_2) = \frac{\left(1-B^{(*)}(l)\right)}{l}\left[\lambda_2 + \frac{L}{U(Z_2)-Z_2}\right]Q(Z_2)$$

where $\rho_1 = -\lambda_1 \bar{c}\, B^{(*)'}(0), \rho_2 = -\lambda_2 \bar{d}\, B^{(*)'}(0), l = \lambda - \lambda_1 Z_1 - \lambda_2 Z_2, L = \lambda - \lambda_1 h(Z_2) - \lambda_2 U(Z_2)$, $R = \lambda - \lambda_1 Z_1 - \lambda_2 B^{(*)}(l), U(Z_2) = B^{(*)}(l)$ and $h(Z_2)$ is the root of the equation $Z_1 = B^{(*)}(l)$ (Choi and Park (1990)).

Theorem 4.2. For $d_k = c_k = 0, k \neq 1, q = 0$, the stationary distribution of $\{(\xi,\ N_p,\ N_r,\ S_k)\}$. has the following generating functions

$$Q(Z_2) = (1-\rho_1-\rho_2)\left\{\frac{1}{\alpha}\int_1^{z_2}\frac{\lambda-\lambda_1 h(x)-\lambda_2 U(x)}{U(x)-x}dx\right\}$$

$$P^{(*)}(1;0,Z_1,Z_2) = \frac{\left(1-B_1{}^{(*)}(l)\right)\{L[Z_2-B_2{}^{(*)}(l)]+R[U(Z_2)-Z_2]\}}{l[B_1{}^{(*)}(l)-Z_1][U(Z_2)-Z_2]}Q(Z_2)$$

$$P^{(*)}(2;0,Z_1,Z_2) = \frac{\left(1-B_2{}^{(*)}(l)\right)}{l}\left[\lambda_2 + \frac{L}{U(Z_2)-Z_2}\right]Q(Z_2)$$

where $\rho_1 = -\lambda_1 B_1^{(*)'}(0), \rho_2 = -\lambda_2 B_2^{(*)'}(0), l = \lambda - \lambda_1 Z_1 - \lambda_2 Z_2, L = \lambda - \lambda_1 h(Z_2) - \lambda_2 U(Z_2)$, $R = \lambda - \lambda_1 Z_1 - \lambda_2 B_2{}^{(*)}(l), U(Z_2) = B_2{}^{(*)}(l)$ and $h(Z_2)$ is the root of the equation $Z_1 = B^{(*)}(l)$ (Falin et al. (1993)).

5. Operating Characteristics

Using straight forward calculations, the operating characteristics like the mean number of customers in the priority queue and the mean number of customers in the orbit are calculated. After putting $Z_2 = 1$ in equations (3.33) and (3.35), then differentiating with respect to Z_1 and taking the limit as $Z_1 \to 1$, we get

$\lim_{z_1\to 1} P^{(*)'}(1;0,Z_1,1) = (1 -$
$\rho 11-q-\rho 21-cA1+\lambda 1\lambda 2\beta 1\beta 2cd21-\rho 1-q1-q$
$-c2\rho\ 1[\ (1-\rho 1)1-q-\rho 2]\ 2c1-\rho 1-q1-q[\rho 1+c$
$(1-\rho 1)]$--------- (5.41)

where $A_1 = \frac{(1-q)[c_2\rho_1+\lambda_1^2\bar{c}^3\beta_1]+2q\bar{c}\rho_1^2}{2\bar{c}(1-\rho_1-q)^2(1-q)[\rho_1+\bar{c}(1-\rho_1)]}$

$\lim_{z_1\to 1} P^{(*)'}(2;0,Z_1,1) = \frac{\lambda_1\lambda_2\beta_2\bar{c}\bar{d}}{2(1-q)}$ --------- (5.42)

After putting $Z_1 = 1$ in equations (3.33) and (3.35) and then differentiating with respect to Z_2 and taking limit as $Z_2 \to 1$, we get

$$\lim_{Z_2\to 1} P^{(*)'}(1;0,1,Z_2) = \frac{D_1\left[\lambda_1\bar{c}^2(1-\rho_1)^2 D_2-(1-q-\rho_2)D_3\right]}{2\lambda_1\bar{c}^2\bar{d}^2(1-\rho_1)^2(q-1)[\rho_1+\bar{c}(1-\rho_1)]D_0} + \frac{(1-q-\rho_2)(D_4+\lambda_2\bar{c}D_5-D_6-D_7)}{2\lambda_1\lambda_2\bar{c}^2\bar{d}^2(1-\rho_1)^2(1-q)[\rho_1+\bar{c}(1-\rho_1)]} + \frac{D_8\bar{c}[(1-\rho_1)(1-q)-\rho_2]}{2\bar{d}^2(1-q^2)[\rho_1+\bar{c}(1-\rho_1)]} + \frac{\lambda_2\rho_{1D_1}(1-q-\rho_2-\rho_2\bar{c})}{\alpha\bar{c}(1-q)[\rho_1+\bar{c}(1-\rho_1)]D_0}$$ -------- (5.43)

Where

$D_0 = [(1-q)(1-\rho_1)-\rho_2]$

$D_1 = \rho_1\bar{d} + \rho_2\bar{c} + \bar{c}\,(1-\rho_1)\,(\bar{d}-1-q)$

$D_2 = D_0\left[2q\rho_2\bar{d} + \rho_2 d_2 + \lambda_2{}^2\bar{d}^3\beta_2\right]$

$D_3 = \lambda_1\bar{c}^2\rho_2(1-\rho_1)^2(2\bar{d}q+d_2) + \lambda_1\lambda_2{}^2\bar{c}^2\bar{d}^3\beta_2(1-\rho_1) + \lambda_2 c_2\bar{d}^2\rho_1{}^2\rho_2 + \lambda_2\lambda_1{}^2\bar{c}^3\bar{d}^2\beta_1\rho_2$

$D_4 = \lambda_1{}^2\lambda_2{}^2\bar{c}^3\bar{d}^3\beta_1 + \lambda_2{}^2\bar{d}^3\rho_1{}^3 c_2 + \lambda_1\lambda_2 d_2\bar{d}\rho_1\bar{c}^2(1-\rho_1)^2 + 2\lambda_1\lambda_2\bar{c}^3\bar{d}q\rho_2(1-\rho_1)^2$

$D_5 = \lambda_1\lambda_2{}^2\bar{c}^2\bar{d}^3\beta_2(1-\rho_1) + \lambda_2\bar{d}^2\rho_1{}^2\rho_2 c_2 + \lambda_1{}^2\lambda_2\bar{c}^3\bar{d}^2\beta_1\rho_2 + \lambda_1\bar{c}^2 d_2\rho_2(1-\rho_1)^2$

$D_6 = 2\lambda_1\lambda_2\bar{c}^3\bar{d}\,(\bar{d}-1)(1-\rho_1)^2[1-\rho_1-\rho_2-q(1-\rho_1)]$

$D_7 = \lambda_1\lambda_2\bar{c}^2 d_2(1-\rho_1)^2[\rho_1\bar{d} + \bar{c}[q(1-\rho_1)+\rho_2] - \bar{c}(1-\rho_1)]$

$D_8 = 2\bar{d}\,(1-\bar{d})\,(q+\rho_2-1) - \lambda_2{}^2\bar{d}^3$

$$\lim_{Z_2\to 1} P^{(*)'}(2;0,1,Z_2) = \frac{\lambda_2{}^2\bar{d}^2\beta_2}{2(1-q)} + \frac{\lambda_2\rho_2 D_1}{\alpha c(1-q)D_0} + \frac{\rho_2{}^2 q}{(1-q)[(1-\rho_1)(1-q)-\rho_2]} + \frac{\rho_2[(1-\rho_1-\rho_2)D_9+(\bar{c}+\rho_1-\rho_1\bar{c})D_5]}{2\lambda_1\bar{d}\bar{c}^2(1-\rho_1)^2 D_0\ (\bar{c}+\rho_1-\rho_1\bar{c})}$$ --------- (5.44)

Where

$D_9 = \lambda_1\lambda_2{}^2\bar{c}^2\bar{d}^3\beta_2(1-\rho_1) + \lambda_2\bar{d}^2\rho_1{}^2 c_2\rho_2 + \lambda_1{}^2\lambda_2\bar{c}^3\bar{d}^2\beta_1\rho_2 + \lambda_1\bar{c}^2 d_2\rho_2(1-\rho_1)^2$

From (3.30) and (3.31)

$$Q'(1) = \frac{\lambda_2}{\alpha}\left[\frac{\bar{d}}{1-q} - \frac{\bar{c}[(1-\rho_1)(1-q)-\rho_2]}{(1-q)\,(\bar{c}+\rho_1-\rho_1\bar{c})}\right]$$ --------------- (5.45)

(i) Mean number of customers in the priority queue is

$N_p = \lim_{Z_1\to 1} P^{(*)'}(1;0,Z_1,1) + \lim_{Z_2\to 1} P^{(*)'}(2;0,Z_1,1)$ -- (5.46)

Adding equations (5.41) and (5.42) we get (5.46)

(ii) Mean number of customers in the orbit as

$N_r = \lim_{Z_2\to 1} P^{(*)'}(1;0,1,Z_2) + \lim_{Z_2\to 1} P^{(*)'}(2;0,1,Z_2) + Q'(1)$ ---------- (5.47)

Adding equations (5.43), (5.44) and (5.45)) we get (5.47)

(iii) Mean busy period : Busy period T_b is the length of the time interval that keeps the server busy continuously and this continues till the instant the server becomes free again and let T_0 be the length of the idle period. For this model T_b and T_0 generates an alternating renewal process and therefore

$$\frac{E(T_b)}{E(T_0)} = \frac{Pr\{T_b\}}{1-Pr\{T_b\}} = \frac{P_B}{1-P_B}$$

But $\mathrm{E}(T_0) = \frac{1}{\lambda}, \mathrm{E}(T_b) = \frac{P_B}{\lambda\,(1-P_B)}$

Using equation (3.39) in the above equation, we get

$$\mathrm{E}(T_b) = \frac{[\rho_1(1-q)+\bar{c}\rho_2]}{\lambda\bar{c}[(1-\rho_1)(1-q)-\rho_2]} \quad \text{-------- (5.48)}$$

6. Numerical Models

In this section, some numerical examples related to the model analyzed in this paper are given. By varying type I arrival rate, type II arrival rate and the retrial rate, the probability that the server busy, the probability that the server idle, the mean number of customers in the priority queue and the mean number of customers in the orbit are calculated. For the analysis the parameters $q, \bar{c}, C''(1), \bar{d}, D''(1), \beta_1, \beta_2,$ $B_1^{(*)'}(0)$ and $B_2^{(*)'}(0)$ fixed. The results are shown in graphs and tables.

6.1. Graphs

The mean number of customers and the mean busy periods are given as graphs in the Figures 6.1-6.6. In Figures 6.1 and 6.2, the retrial rate is taken as 0.6 and type II arrival rate is taken as 0.7 and 0.9 respectively, the graphs of the mean number of customers in the priority queue and the mean number of customers in the orbit are drawn by varying the value of type I arrival rate. Whereas in Figures 6.3 and 6.4, the retrial rate has been fixed as 0.9 but the type I arrival rates are 0.7 and 0.9 respectively. The graphs of the mean number of customers in the priority queue and the mean number of customers in the orbit are drawn against varying the value of type II arrival rate.

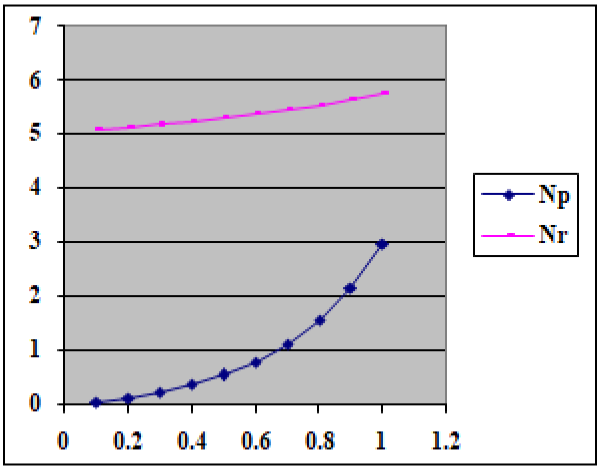

Figure 6.1: Type I arrival rate versus the mean number of customers

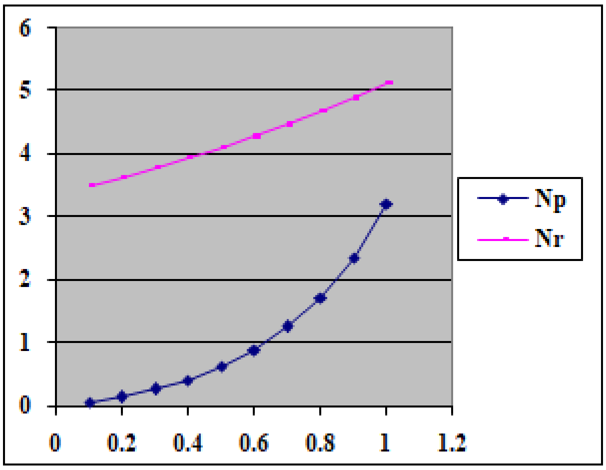

Figure 6.2: Type I arrival rate versus the mean number of customers

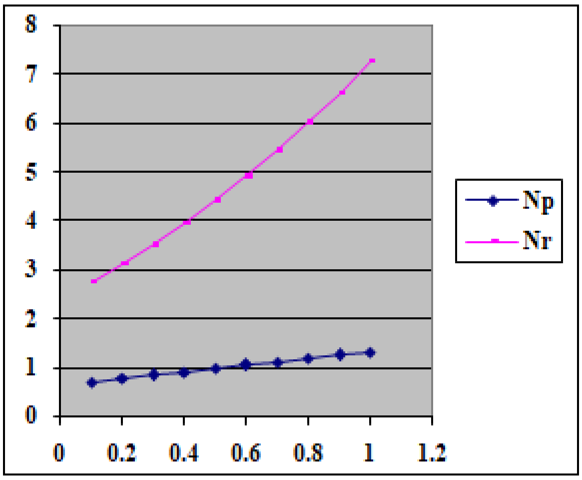

Figure 6.3: Type II arrival rate versus the mean number of customers

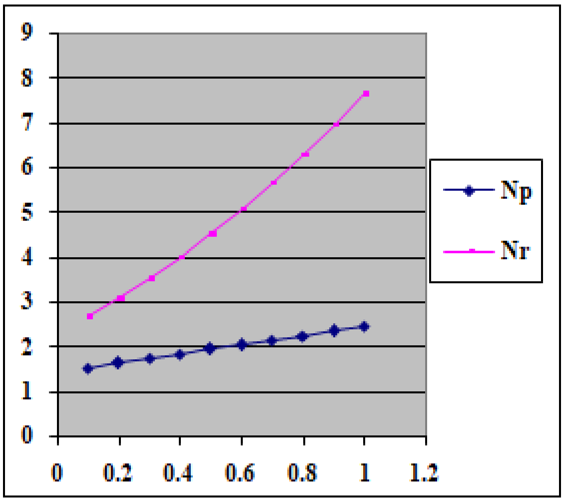

Figure 6.4: Type II arrival rate versus the mean number of customers

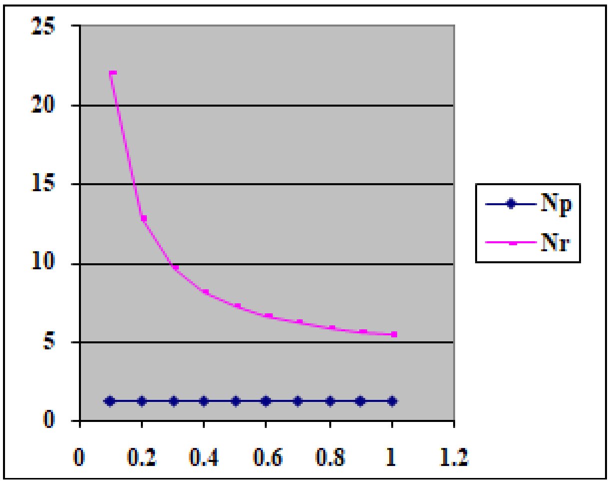

Figure 6.5: Retrial rate versus the mean number of customers

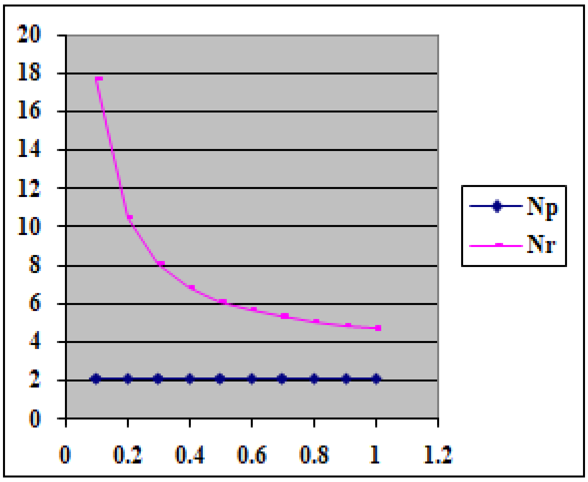

Figure 6.6: Retrial rate versus the mean number of customers

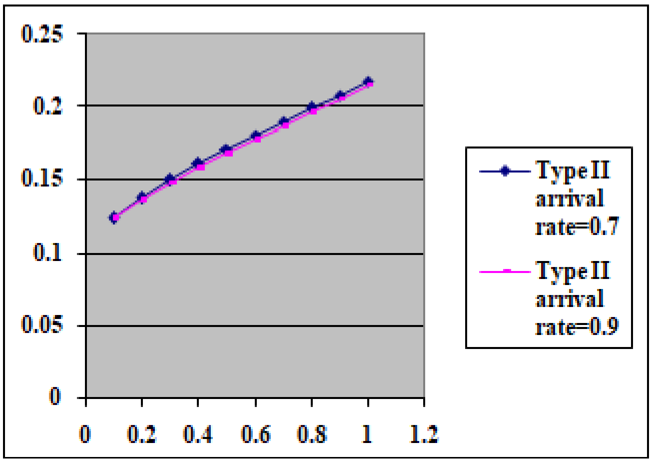

Figure 6.7: Type I arrival rate versus the mean busy period

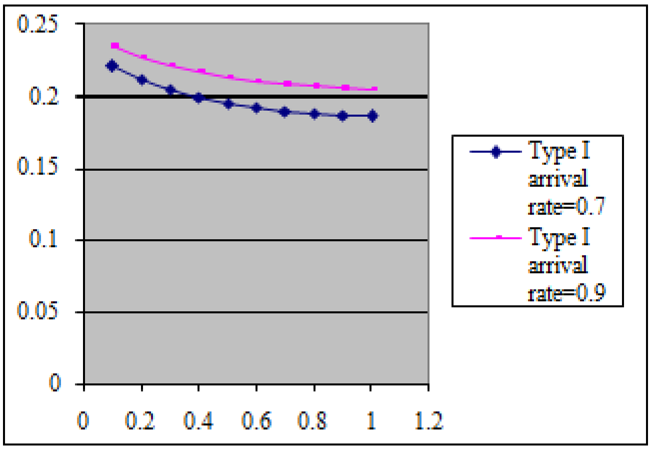

Figure 6.8: Type II arrival rate versus the mean busy period

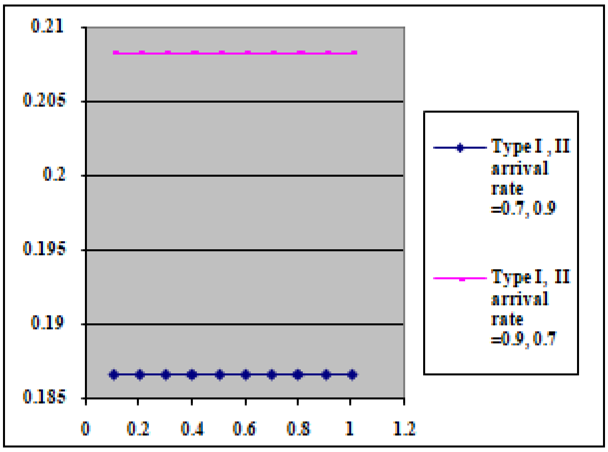

Figure 6.9: Retrial rate versus the mean busy period

In Figures 6.5 and 6.6, the same graphs with respect to varying retrial rate are drawn for fixed values of type I and type II arrival rates ($\lambda_1 = 0.7, \lambda_2 = 0.9$ and $\lambda_1 = 0.9, \lambda_2 = 0.7$). From the graphs it is clear that as type I arrival rate (type II arrival rate) increases the mean number of customers in the priority queue and the mean number of customers in the orbit also increase whereas as the retrial rate increases the mean number of customers in the orbit decreases and the mean number of customers in the priority queue remains a constant. In Figures 6.7,6.8 the graphs of mean busy period are drawn against varying values of type I arrival rate and type II arrival rate respectively. From the Figure 6.7, it is clear that as type I arrival rate increases the mean busy period also increases whereas from the Figure 6.8, it is clear that as type II arrival rate increases the mean busy period decreases. Figure 6.9 corresponds to the graph of mean busy period with respect to retrial rate and is like the graph of constant function.

6.2. Tables

The values of idle probabilities and busy probabilities are given in the Tables.

Tables 6.1, 6.2 presents the idle probability and busy probability. For various values of λ_1 for $\lambda_2 = 0.7,0.9$ the two probabilities are calculated and are given in Table 6.1 and Table 6.2. The results shows that the idle probability decreases for increasing value of λ_1. The same type of behaviour is attained if we interchange λ_1 and λ_2 and the results are shown in Table 6.2.

Table 6.1: The Probabilities P_I and P_B

λ_1	$\alpha = 0.6, \lambda_2 = 0.7$		$\alpha = 0.6, \lambda_2 = 0.9$	
	P_I	P_B	P_I	P_B
0.1	0.91	0.09	0.89	0.11
0.2	0.89	0.11	0.87	0.13
0.3	0.87	0.13	0.85	0.15
0.4	0.85	0.15	0.83	0.17
0.5	0.83	0.17	0.81	0.19
0.6	0.81	0.19	0.79	0.21
0.7	0.79	0.21	0.77	0.23
0.8	0.77	0.23	0.75	0.25
0.9	0.75	0.25	0.73	0.27
1.0	0.73	0.27	0.71	0.29

Table 6.2: The Probabilities P_I and P_B

λ_2	$\alpha = 0.6, \lambda_1 = 0.7$		$\alpha = 0.6, \lambda_1 = 0.9$	
	P_I	P_B	P_I	P_B
0.1	0.85	0.15	0.81	0.19
0.2	0.84	0.16	0.80	0.20
0.3	0.83	0.17	0.79	0.21
0.4	0.82	0.18	0.78	0.22
0.5	0.81	0.19	0.77	0.23
0.6	0.80	0.20	0.76	0.24
0.7	0.79	0.21	0.75	0.25
0.8	0.78	0.22	0.74	0.26
0.9	0.77	0.23	0.73	0.27
1.0	0.76	0.24	0.72	0.28

7. Real life example

In a wireless sensor system, the sensor transmits information to the central processing unit after getting information's. Assume that there are two types of information's, called primary important information and secondary non-important routine information. The information may have a one line information or cluster of information's. The information are processed one by one and a decision is taken by the central processing unit. The system always gives importance to primary information compare to routine information. Sometimes the primary information may be again processed after some times, but now it becomes secondary information. We assume that the primary information are pooled in a place, the system always gives first preference to this place. The secondary information are pooled in a place, this part is processed only if no primary information are available (Fraden (1997)). We can model this situation using our model analysed in this article.

8. Conclusion

In the foregoing analysis, an M/G/1 queue with retrial customers and system with two types of batch arrivals has considered. In addition, unsatisfied customers enter into the orbit as a feedback customer for additional service. The queue length distribution and mean queue length are derived. Extensive numerical works are carried out to observe the trends of the operating characters of the system.

The model can be generalised by taking retrial time as general distribution. Also the model is extended by incorporating the concept of MAP, BMAP, etc.

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