International Journal of Science and Research (IJSR)
ISSN: 2319-7064
SJIF (2022): 7.942

Generalization of the Group Classification with Application of Elliptic Partial Differential Equations

Mani Sharma[1], Dr. Chitra Singh[2]

Research Scholar, Rabindranath Tagore University Bhopal (M.P), India
mssharmamani108[at]gmail.com

Associate Professor Mathematics, Rabindranath Tagore University Bhopal (M.P), India
chitra.singh[at]aisectuniversity.ac.in

Abstract: ***This research work enhance the generalizing previous result on a one-dimensional nonlinear elliptic partial differential equation. By the use of new techniques of a partial differential equation with the equivalence of group ,class of group and groupoid, we extend the algebraic method of classification of nonlinear partial differential equations. The solution includes the preliminary group, sub algebras, and infinite-dimensional equivalence algebra. This result is examined by the example of Elliptic partial differential equation.***

Subject Classification Mathematics 2020: 35K10, 35L05, 05C25

Keywords: Partial differential Equation, Group, Elliptic partial differential equations, Subalgebra.

1. Introduction

In this research paper, with concerned ofgroup classification we finding an equivalent equation of a class of differential equation at containing the one or more arbitrary elements. These work are correlated with theoretical physics, which include the maximal number of symmetries from a given class yield and there describe the scientific model of real world problem.

In mathematics, the group classification problem of partial differential equation had been investigated by Sophus Lie's classifications of second-order ordinary differential equations and of second-order linear partial differential equations with two independent variables.

Recently a various novel techniques of group classification have been introduced, which include the algebraic method [5, 6,13, 8, 9] and advanced modification of the direct method which is called furcate splitting [7, 9, 4]. The algebraic method of group classification applied to the classes of differential equations with the arbitrary elements of functions of several arguments.

The class considered in the literature on group classification, is the most prominent ones are classes of (1+1)-dimensional evolution equations, [1, 2, 3, 4, 8, 14, 16, 15,]. There is no coincidence in the field of invariant discretization, numerical schemes for differential equations possessing the same symmetries as the original, undiscretized equations, mostly evolutionary equations [10].

Definition 1.Definition 1. Admissible transformations $T^1 = (\theta^1, \emptyset^1, \widetilde{\theta^1})$ and $T^2 = (\theta^2, \emptyset^2, \widetilde{\theta^2})$ in the class of $\mathscr{L}|s$ are called conjugate with respect to the equivalence group $G^{\sim}$of this class if there exist $\mathcal{T} \in G^{\sim}$ such that $\theta^2 = \mathcal{T}_*\theta^1$ and $\emptyset^2 = (w_*, \mathcal{T})\mathrm{o}\emptyset^1\mathrm{o}(w_*, \mathcal{T})^{-1}.w_*$denote the projection along with the group coordinate

Definition 2: The class $\mathscr{L}|s$ is called normalised if $G^{G^{\sim}} = G^{\sim}$. It is called semi normalised if $G^f * G^{G^{\sim}}$=$G^{\sim}$.

2. Main Result

Classification of admissible transformations and group in Elliptic Partial differential equation

Definition: The linear partial differential equation (1) of the second degree defined in some region D in the xy plane is said to be

Hyperbolic of a point (xy) in D,
If $\Delta = B^2 - AC > 0$

Parabolic, if $\Delta = B^2 - AC = 0$

Elliptic, if $\Delta = B^2 - AC < 0$ at the point. It is also satisfy the postulates of group
Closure, Associative, commutative and existence of identity, existence of inverse.

* **Solution** The given P.D.E can be written as

$$F^i(u_t, u_x) = r + x^2 t \tag{1}$$

By the **closure axiom** of group

$$a, b \in G \Rightarrow a + b \in G$$

Then $r, t \in G$

$$F^i(u_t, u_x) = r + x^2 t \in G$$

By **Associative axiom**$r, s, t \in G$

$$(r + s) + \mathrm{t} = \mathrm{r} + (\mathrm{s} + \mathrm{t}) \in G$$

Existence of identity

$$r.1 = r, \ \forall \ r \in G$$

Existence of inverse

$$r.r^{-1} = 1\ ,\ \forall\ r \in G$$

Commutative Axiom

$(r + s) = (s + r),\ \forall\ r, s \in G$

Comparing (1) with $Ar + 2Bs + Ct + f(x, y, u, z, p, q) = 0$

We get $A = 1,\ B = 0,\ C = x^2$

$$B^2 = 0,\ AC = 1.x^2 = x^2$$

$\therefore B^2 - AC < 0$ at all points where $x \neq 0$.

Hence the given equation is elliptic at all points except on the axis.

Also, the λ- quadratic $A\lambda^2 + 2B\lambda + C = 0$ gives $\lambda^2 + x^2 = 0$so that $\lambda = \pm ix$.

Hence $\frac{dy}{dx} + \lambda_1 = 0$ and $\frac{dy}{dx} + \lambda_2 = 0$ becomes

$$\frac{dy}{dx} + ix = 0\ \ and\ \ \frac{dy}{dx} - ix = 0.$$

Solution Thegiven P.D.E can be written as

$$F^i(u_t, u_x) = r + x^2 t \qquad (1)$$

Comparing (1) with $Ar + 2Bs + Ct + f(x, y, u, z, p, q) = 0$

$$\frac{dy}{dx} + ix = 0\ \ and\ \ \frac{dy}{dx} - ix = 0.$$

Solving for α, β, we get

$$\alpha = y\ \ and\ \beta = \frac{1}{2}x^2$$

$$p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial \alpha}.\frac{\partial \alpha}{\partial x} + \frac{\partial z}{\partial \beta}.\frac{\partial \beta}{\partial x} = x\frac{\partial z}{\partial \beta}$$

$$q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial \alpha}.\frac{\partial \alpha}{\partial y} + \frac{\partial z}{\partial \beta}\frac{\partial \beta}{\partial y} = \frac{\partial z}{\partial \alpha} so\ \frac{\partial}{\partial y} = \frac{\partial}{\partial \alpha}$$

$$r = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial x}\left(x\frac{\partial z}{\partial \beta}\right)$$

$$= 1.\frac{\partial z}{\partial \beta} + x\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial \beta}\right)$$

$$= \frac{\partial z}{\partial \beta} + x\left[\frac{\partial}{\partial \alpha}\left(\frac{\partial z}{\partial \beta}\right)\frac{\partial \alpha}{\partial x} + \frac{\partial}{\partial \beta}\left(\frac{\partial z}{\partial \beta}\right)\frac{\partial \beta}{\partial x}\right] = \frac{\partial z}{\partial \beta} + x^2\frac{\partial^2 z}{\partial \beta^2}$$

$$t = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial \alpha}\left(\frac{\partial z}{\partial \alpha}\right) = \frac{\partial^2 z}{\partial \alpha^2}.$$

Putting these values of $r\ and\ t$ in (1), we get

$$\frac{\partial z}{\partial \beta} + x^2\frac{\partial^2 z}{\partial \beta^2} + x^2\frac{\partial^2 z}{\partial \alpha^2} = 0$$

or

$$\frac{\partial^2 z}{\partial \alpha^2} + \frac{\partial^2 z}{\partial \beta^2} = \frac{-1}{2\beta}\frac{\partial z}{\partial \alpha}$$

canonical form of Elliptic Partial differential Equation $T^1 = (p^1, q^1, r^1, t^1)andT^2 = (p^2, q^2, r^1, t^1)$ is this class of $\mathscr{L}|s$ are called conjugate with respect to the equivalence group $G^{\sim}$ if there exist $T \in G^{\sim}$ such that $p^2 = \mathcal{T}_* P^1$ and $q^2 = (w_*, T)oQ^1o(w_*, \mathcal{T})^{-1}.w_*$ denote the projection along with the coordinate

It satisfies the condition of admissible transformation of partial differential equations.

Result: The Elliptic partial differential equation is satisfying the all postulates of group Axiom.

References

[1] Akhatov I.S., Gazizov R.K. and IbragimovN.Kh., Nonlocal symmetries.A heuristic approach, in Current Problems in Mathematics. Newest Results, Vol. 34, Akad. Nauk SSSR, Vsesoyuz. Inst. Nauchn.iTekhn .Inform., Moscow, 1989, pp. 3–83. (in Russian); translated in J. Soviet Math. 55 (1991), 1401–1450.

[2] Ames W.F., Anderson R.L., Dorodnitsyn V.A., Ferapontov E.V., Gazizov R.K., Ibragimov N.H. and Svirshchevski˘ı S.R., CRC Handbook of Lie group analysis of differential equations. Vol. 1. Symmetries, exact solutions and conservation laws, CRC Press, Boca Raton, FL, 1994.

[3] Basarab-Horwath P., Lahno V. and Zhdanov R., The structure of Lie algebras and the classification problem for partial differential equations, Acta Appl. Math. 69 (2001), 43–94, arxiv:math-ph/0005013.

[4] Basarab-Horwath P., Lahno V. and Zhdanov R., Classifying evolution equations, in Proceedings of the Third World Congress of Nonlinear Analysts, Part 8 (Catania, 2000), Nonlinear Anal. 47 (2001), 5135–5144.

[5] Bihlo A., Dos Santos Cardoso-Bihlo E. and Popovych R.O., Complete group classification of a class of nonlinear wave equations, J. Math. Phys. 53 (2012), 123515, arXiv:1106.4801.

[6] Bihlo A., Dos Santos Cardoso-Bihlo E. and Popovych R.O., Algebraic method for finding equivalence groups,J. Phys.: Conf. Ser. 621 (2015), 012001, arXiv:1503.06487.

[7] Bihlo A., Poltavets N. and Popovych R.O., Lie symmetries of two-dimensional shallow water equations with variable bottom topography, arXiv:1911.02097.

[8] Bihlo A. and Popovych R.O., Group classification of linear evolution equations, J. Math. Anal. Appl. 448 (2017), 982–2015, arXiv:1605.09251.

[9] Bihlo A. and Popovych R.O., Zeroth-order conservation laws of two-dimensional shallow water equations with variable bottom topography, arXiv:1912.11468.

[10] Bihlo A. and Valiquette F., Symmetry-preserving numerical schemes, in Symmetries and integrability of difference equations, CRM Ser. Math. Phys., Springer, Cham, 2017, pp. 261–324.

[11] Bluman G.W., Cheviakov A.F. and Anco S.C., Application of symmetry methods to partial differential equations, Springer, New York, 2010.

[12] Bluman G., Cheviakov A.F. and Ivanova N.M., Framework for nonlocally related partial differential equation systems and nonlocal symmetries: Extension, simplification, and examples, J. Math. Phys. 47 (2006), 113505.

[13] Bluman G. and Kumei S., On invariance properties of the wave equation, J. Math. Phys. 28 (1987), 307–318.

[14] Bluman G. and Kumei S., Symmetries and differential equations, Springer, New York, 1989.

[15] Dorodnitsyn V.A., On invariant solutions of the equation of non-linear heat conduction with a source, USSR Comput. Math.Math. Phys. 22 (1982), 115–122.

[16] Dos Santos Cardoso-Bihlo E., Bihlo A. and Popovych R.O., Enhanced preliminary group classification