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Total Graph of Z_n and It's Adjacency Matrix

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Abstract: ***Let G be a simple undirected graph. For a commutative ring R, $Z(R)$be the set of zero divisors of R. Total graph of R is denoted by $T(\mathbb{\Gamma}(R))$, all elements of R are taken as vertices of the graph.Two distinct vertices a, b are adjacent if and only if $a+b\in Z(R)$. Here we study total graph $T(\mathbb{\Gamma}(R))$ for $R=Z_n$. Also discuss degree, planarity, Eulerian property for different n. We also study adjacency matrix of the graph.***

Keyword: Total Graph, Planarity, Adjacency matrix

1. Introduction

In [1] D.F. Anderson and A. Badawi introduced the total graph of a commutative ring R. All elements of R considered as vertices of the graph, two distinct vertices connected by a line if and only if $a+b\in Z(R)$,Where $Z(R)$ is set of all zero divisors of the ring R. In this paper we consider the ring $R=Z_n$ where, Z_n is ring of integer modulo n.T.T. Chelvam and T.Asir [3] worked on total graph of Z_n, they discuss connectedness of the graph and discuss planarity for some values of n. In this paper we discuss about degree for different values of n and also their planarity.

The adjacency matrix of a graph G is denoted by A(G), whose (i,j)- entry $A_{ij}$is given by

$$A_{ij}=\begin{cases}1 & if\ vertices\ i,j\ are\ adjacent\\ & 0\end{cases}$$

Adjacency matrix A(G) is a symmetric matrix of order $n\times n$. Here we find adjacency matrix of $\mathrm{T}(\mathbb{\Gamma}(R))$, where, $R=Z_p$, $R=\mathrm{T}(\mathbb{\Gamma}(Z_{p^2}))$ and$R=\mathrm{T}(\mathbb{\Gamma}(Z_{2^k p}))$for any prime p and $k\in\mathbb{N}$. Using eigen values of adjacency matrix we can introduce different properties of graph.

2. Main Results

2.1 Degree, Planarity, Eulerian property of $\mathbf{T}(\mathbb{\Gamma}(Z_p))$, $\mathbf{T}(\mathbb{\Gamma}(Z_{p^2}))$ and $\mathbf{T}(\mathbb{\Gamma}(Z_n))$ for $n=2^n p$.

Theorem 2.1: In $\mathrm{T}(\mathbb{\Gamma}(Z_p))$degree of any unit vertex is 1 and only zero-divisor $\bar{0}$ is isolated vertex.

Proof: In $\mathrm{T}(\mathbb{\Gamma}(Z_p))$ unit vertices are all non-zero elementsZ_p and only $\bar{0}$ is zero divisor of $\mathrm{Z_p}$. For any unit element$\bar{a}$,
$\bar{a}+(\bar{p}-\bar{a})=\bar{0}$; So, $\bar{a}$ is adjacent only with $(\bar{p}-\bar{a})$. Therefore, degree of any unit element is 1.
Let if possible$\bar{0}$ is adjacent with $\bar{b}$
Then,$\bar{0}+\bar{b}=\bar{0}$ (Only zero-divisor in Z_p)
$\Rightarrow\bar{b}=\bar{0}$
But the graph is simple. Therefore, $\bar{0}$ is not adjacent with any vertex of the graph.
Hence, $\bar{0}$ is isolated vertex.

Theorem 2.2: In $\mathrm{T}(\mathbb{\Gamma}(Z_{p^2}))$ degree of any unit vertex is p.

Proof: In $\mathrm{T}(\mathbb{\Gamma}(Z_{p^2}))$ unit vertices are unit elements of Z_{p^2}. Let $\bar{u}$ be any unit element of Z_{p^2}. In Z_{p^2}zero divisors are $\bar{0}$, $\bar{p}$, $\overline{2p}$, $\overline{3p}$,$\cdots$ $\overline{p(p-1)}$. There are p number of zero divisor.
If $\bar{u}\sim\bar{a}$ then $\bar{u}+\bar{a}=\overline{pk}$ (zero divisor) where $k\in\mathbb{N}$.
So, $\bar{a}=\overline{pk}-\bar{u}$
Since, there are p number of $\overline{pk}$ in Z_{p^2}
Therefore, there are p number of $\overline{pk}-\bar{u}$ in Z_{p^2}.
Hence, degree of any unit vertex is p.

Theorem 2.3: In $\mathrm{T}(\mathbb{\Gamma}(Z_{p^2}))$degree of any zero divisor is $p-1$.

Proof: In $\mathrm{T}(\mathbb{\Gamma}(Z_{p^2}))$ every element of $Z_{p^2}$are taken as vertices.In Z_{p^2}zero divisors are $\bar{0}$, $\bar{p}$,$\overline{2p}$,$\overline{3p}$,$\cdots$ $\overline{p(p-1)}$. There are p number of zero divisor. Let $\bar{z}$ be any zero divisor of Z_{p^2}.
$\bar{z}.\bar{a}\equiv\bar{0}(mod\ p^2)$ if $\bar{a}$ is multiple of p
So, $\bar{a}$ is of the form$\overline{pk}$ which is zero divisor.
Therefore, any two distinct (graph is simple) zero divisors are adjacent.
There are p number of zero divisor.
Hence degree of any zero divisor is $p-1$.

Theorem 2.4: The graph $\mathrm{T}(\mathbb{\Gamma}(Z_{p^2}))$ is planar if and only if $p<5$.

Proof: In $\mathrm{T}(\mathbb{\Gamma}(Z_{p^2}))$ every element of $Z_{p^2}$are taken as vertices.In Z_{p^2}zero divisors are $\bar{0}$, $\bar{p}$,$\overline{2p}$,$\overline{3p}$,$\cdots$ $\overline{p(p-1)}$. There are p number of zero divisor. Any two distinct zero divisors are adjacent. So, zero divisors of Z_{p^2} form a complete graph. If $p=5$ there will be five zero-divisors and they form a complete graph K_5 which is not planar. For $p>5$, in the graph $\mathrm{T}(\mathbb{\Gamma}(Z_{p^2}))$ always a subgraph K_5. We know that a graph is planar if the graph has a subgraph which is homeomorphic to K_5 or $K_{3,3}$. Therefore, the graph $\mathrm{T}(\mathbb{\Gamma}(Z_{p^2}))$ is not planar for $p\geq 5$.

Theorem 2.5: The graph $\mathrm{T}(\mathbb{\Gamma}(Z_{p^2}))$ is not Eulerian for any prime p.

Proof: For any odd prime p degree of unit vertex in the graph $\mathrm{T}(\Gamma(Z_{p^2}))$ is p which odd. Therefore, the graph is not Eulerian. Degree of zero divisor is $p-1$ which is odd for $p=2$. Hence the graph is not Eulerian.

Theorem 2.6: If $n=2^n p$, then degree of any element of the graph $\mathrm{T}(\Gamma(Z_n))$ is $|Z(R)|-1$.

Proof: In this graph zero divisors are multiples of 2 and multiples of p. let a be any vertex of the graph. For vertex a there are $a+b \in Z(R)$, if there are $|Z(R)|$ number of $b$such that $\bar{a}+\bar{b} \in Z(R)$. Also, $\bar{a}+\bar{a}$ is multiple of 2 which is zero divisor, therefore number of vertex adjacent with $\bar{a}$ is $|Z(R)|-1$. Since $\bar{a}$ is an arbitrary Hence, degree of any element of the graph $\mathrm{T}(\Gamma(Z_n))$ is $|Z(R)|-1$.

Corollary: If $n=2^n p$, the graph $\mathrm{T}(\Gamma(\boldsymbol{Z_n}))$ is r regular; where, $r=|Z(R)|-1$

2.2 Adjacency matrix of $\mathbf{T}(\Gamma(\boldsymbol{Z_p}))$ and$\mathbf{T}(\Gamma(\boldsymbol{Z_{p^2}}))$

Theorem 3.1: Adjacency matrix of the graph $G=\mathrm{T}\left(\Gamma(Z_p)\right)$ is $A(G)$, whose (i,j)- entry is

$$A_{ij}=\begin{cases}1 & if\ v_i=\bar{a}\ and\ v_j=\bar{p}-\bar{a} \\ & or \\ & v_i=\bar{p}-\bar{a}\bar{a}\ and\ v_j=\bar{a} \\ 0 & otherwise\end{cases}$$

Proof: In the graph $\mathrm{T}(\Gamma(Z_p))$zero vertex is isolated. And any non-zero vertex $\bar{a}$ is adjacent only with $\bar{p}-\bar{a}$.

Because, $\bar{a}+(\bar{p}-\bar{a})=\bar{0}$ which is zero-divisor. Then by definition of adjacency matrix

$$A_{ij}=\begin{cases}1 & if\ v_i=\bar{a}\ and\ v_j=\bar{p}-\bar{a} \\ & or \\ & v_i=\bar{p}-\bar{a}\bar{a}\ and\ v_j=\bar{a} \\ 0 & otherwise\end{cases}$$

Theorem 3.2: Adjacency matrix of the graph $G=\mathrm{T}(\Gamma(Z_{p^2}))$ is $A(G)$, whose (i,j)- entry is

$$A_{ij}=\begin{cases}1 & if\ v_i=\bar{a}\ and\ v_j=\overline{p^2}-\bar{a} \\ & or \\ & v_i=\overline{p^2}-\bar{a}\ and\ v_j=\bar{a} \\ & or \\ & v_i=\bar{a}\ and\ v_j=\overline{pk}-\bar{a} \\ & or \\ & v_i=\overline{pk}-\bar{a}\ and\ v_j=\bar{a} \\ & or \\ & v_i\ and\ v_j\ both\ multiple\ of\ p \\ 0 & otherwise\end{cases}$$

Where,p is any prime and $k \in \mathbb{N}$

Proof: In the graph $\mathrm{T}(\Gamma(Z_p))$ zero divisors are multiple of p. Set of zero-divisors form a complete graph. So, by definition of adjacency matrix, if v_i and $v_j$are zero divisors (i.e multiple of p) then (i,j)- entry is 1. For any unit element $\bar{a}$,

$$\bar{a}+\overline{p^2}-\bar{a} \equiv \bar{0}\ (mod\ p^2)$$
$$\therefore \bar{a} \sim (\overline{p^2}-\bar{a})$$

And $\bar{a}+\left(\overline{pk}-\bar{a}\right)=\overline{pk}$ is zero divisor

So, $\bar{a} \sim \left(\overline{pk}-\bar{a}\right)$

Therefore, by definition of adjacency matrix $A(G)$, whose (i,j)- entry is

$$A_{ij}=\begin{cases}1 & if\ v_i=\bar{a}\ and\ v_j=\overline{p^2}-\bar{a} \\ & or \\ & v_i=\overline{p^2}-\bar{a}\ and\ v_j=\bar{a} \\ & or \\ & v_i=\bar{a}\ and\ v_j=\overline{pk}-\bar{a} \\ & or \\ & v_i=\overline{pk}-\bar{a}\ and\ v_j=\bar{a} \\ & or \\ & v_i\ and\ v_j\ both\ multiple\ of\ p \\ 0 & otherwise\end{cases}$$

Where, p is any prime and $k \in \mathbb{N}$.

Theorem 3.3: Adjacency matrix of the graph $G=\mathrm{T}(\Gamma(Z_n))$ for $n=2^k p$ is $A(G)$, whose (i,j)- entry is

$$A_{ij}=\begin{cases}1 & if\ v_i is\ unit\ element\ and\ v_j\ is\ any\ odd\ element\ (v_i \neq v_j) \\ & or \\ & if\ v_i+v_j\ is\ multiple\ of\ p\ where, v_i\ is\ unit \\ & or \\ & if\ v_i\ is\ an\ odd\ zero\ divisor\ and\ v_j is\ any\ odd\ element \\ & or \\ & if\ v_i\ is\ an\ odd\ zero\ divisor\ and\ v_i+v_j\ is\ multiple\ of\ p \\ & or \\ & if\ v_i\ is\ an\ even\ zero\ divisor\ and\ v_j is\ any\ even\ element \\ & or \\ & if\ v_i\ is\ an\ even\ and\ v_i+v_j\ is\ multiple\ of\ p \\ 0 & otherwise\end{cases}$$

Where p is prime and $k \in \mathbb{N}$.

Proof: In the graph $\mathrm{T}(\Gamma(Z_{2^k p}))$every even element and multiple of p are zero divisor of $Z_{2^k p}$. Sum of two unit element is even because, unit elements are prime to $n=2^k p$. Where, $n=2^k p$ is even. So, units are odd. Sum of two unit element is even which is zero divisor.

For any unit element v_i which is adjacent with any odd element because sum of v_i and any odd element is even. Also, for any unit element v_i, if sum of v_i and v_j is multiple of p, which is zero divisor then $v_i \sim v_j$.

If v_i is any odd zero divisor, then sum of v_i and any odd element is even, which is zero divisor. Therefore, $v_i \sim v_j$ if v_i is any odd zero divisor and v_j is any odd element of $Z_{2^k p}$.

If v_i is any even zero divisor, then sum of v_i and any even element is even, which is zero divisor. Therefore, $v_i \sim v_j$ if v_i is any even zero divisor and v_j is any even element of $Z_{2^k p}$.

For an even vertex $v_i$if $v_i + v_j$ is multiple of $p$then $v_i + v_j$ will be zero divisor. Therefore, $v_i \sim v_j$ if $v_i + v_j$ is multiple of p. Hence by definition of adjacency matrix

$$A_{ij} = \begin{cases} 1 & \begin{array}{c} if\ v_i is\ unit\ element\ and\ \ v_j\ is\ any\ odd\ element\ (v_i \neq v_j) \\ or \\ if\ v_i + v_j\ is\ multiple\ of\ p\ where, v_i\ is\ unit \\ or \\ if\ v_i\ is\ an\ odd\ zero\ divisor\ \ and\ v_j is\ any\ odd\ element \\ or \\ if\ v_i\ is\ an\ odd\ zero\ divisor\ \ and\ v_i + v_j\ is\ multiple\ of\ p \\ or \\ if\ v_i\ is\ an\ even\ zero\ divisor\ \ and\ v_j is\ any\ even\ element \\ or \\ if\ v_i\ is\ an\ even\ and\ v_i + v_j\ is\ multiple\ of\ p \end{array} \\ 0 & otherwise \end{cases}$$

Where p is prime and $k \in \mathbb{N}$.

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