A short review on lacunary and $\lambda-$statistical convergence

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Abstract

In this article the aim is to review some basic terms related to the generalized convergence known as lacunary statistical convergence and $\lambda-$statistical convergence. Some points related to these notions are given to show these convergence methods are more generalized than the usual convergence.

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1 Introduction

The generalization of the usual convergence has been done in many ways as the concept of convergence of sequence plays a vital role in the elementary theory of mathematics. Different authors discussed the concept of convergence of sequences in summability theory, probability theory, approximation theory and classical measure theory. One of the most recent and important convergence criteria is statistical convergence introduced by Fast [7] in 1951. The theory of summability is a fundamental tool to solve the problem related to convergence of sequence and series. The main purpose of this paper is to review the elementary results in the field of lacunary and $\lambda-$statistical convergence. Firstly, the basic definitions related to the different convergence criteria are given.

Definition 1.1 [15] A sequence of numbers $y = (y_m)$ is said to be statistically convergent to a number M provided that, for every $\epsilon > 0$, following condition holds

$$\delta\left(\{m \in \mathbb{N} : |y_m - M| \geq \epsilon\}\right) = 0.$$

In this case, we write $S - \lim_{m\to\infty} y_m = M$.

Definition 1.2 [8] Let $y = (y_m)$ be a sequence then $y = (y_m)$ will be statistically cauchy to M if for all $\epsilon > 0$, there exist p such that

$$\delta\left(\{m \in \mathbb{N} : |y_m - y_p| \geq \epsilon\}\right) = 0.$$

Definition 1.3 : [4] A sequence $y = (y_m) : m \in \mathbb{N}$ is called statistically bounded if there exist a number B_S such that

$$\delta\left(\{m \leq n : |y_m| > B_S\}\right) = 0.$$

Definition 1.4 Let $y = (y_m)$ be the real sequence and S_p is the partial sum of (y_m) i.e. $S_p = y_1 + y_2 + ...y_p$ then $y = (y_m)$ is known as Cesaro summable whenever

$$\lim_{m\to\infty} \frac{1}{m} \sum_{p=1}^{m} S_p = A$$

The resulting value of the limit is known as Cesaro sum of the series $\sum_{m=1}^{\infty} y_m$. This series is Cesaro summable if it is convergent.

Example The sequence $y_m = (-1)^m$ is Cesaro summable.

Definition 1.5 [5] Let f be the mapping such that f : N $\times$ N $\to$ R then the function is known as Double sequence if $(f(p, q)) = a_{pq}$ that is a sequence of elements denoted by two suffixes pq where $p, q = 1, 2, 3, 4, ...$
For example

$$a_{pq} = \frac{p}{p+q}$$

2 Lacunary Statistical Convergence

This paper discuss some well-known results on lacunary statistical convergence, λ-statistical convergence and may be of some interest for specialists in this domain.

In this section, the basic hypothesis of lacunary statistical convergence is given as follows ;
In 1993, the term lacunary statistical convergence defined by Fridy and Orhan [9] in which the authors take the non-decreasing sequence $\theta = \{h_s\}$. The authors

discuss that how N_θ (lacunary summability series) and Cesaro summability are related and also proves the connection between N_θ and S_θ convergence.

Definition 2.1 Let $y = (y_m)$ is a real sequence then $y = (y_m)$ is known as lacunary statistical convergent if for all $\epsilon > 0$ such that

$$\lim_s \frac{1}{h_s} (\{m \in I_s : |y_m - M| \geq \epsilon\}) = 0.$$

So, we can write $S_\theta - lim\ y = M$.

This is called θ-Density and $h_s = (m_s - m_{s-1}) \to \infty$ whenever s $\to \infty$ where (h_s) is a non-decreasing sequence of non-negative terms such as $m_0 = 1$ where $I_s = (m_{s-1}, m_s]$ and $\theta = h_s$.

Definition 2.2 Let $y = (y_m)$ be a real sequence then $y = (y_m)$ is known as lacunary summable to M if the following condition holds

$$[y\ : for\ some\ M,\ \lim_s [\frac{1}{h_s} \sum_{(m)\in I_s} |y_m - M| > \epsilon\ ] = 0].$$

In 1995, Ccakalli [3] have introduced the term lacunary statistical convergence in topological groups by using neighbourhoods in which limit is always unique. He proves some basic theorms related to statistically convergent sequences and lacunary statistical convergent sequences.

Definition 2.3 Let X is a topological group and let $y = (y_m)$ be the real sequence then $y = (y_m)$ is known as lacunary statistical convergence in topological space if for each neighbourhood R of 0

$$\lim_s \frac{1}{h_s} (\{m \in I_s : |y_m - M| \notin R\}) = 0.$$

where $\theta = m_s$ is the non-decreasing sequence and $m_0 = 0$ and $h_s = m_s - m_{s-1}$.

In 2006, Savas and Pattersonb [16] defined the lacunary statistical convergence of multiple sequence. The authors also discuss about the lacunary summability series of double sequence and has given the denition of P-statistical convergent.

Definition 2.4 Let $\theta_{a,b} = (p_a, q_b)$ be the double sequence then the double sequence $\theta_{a,b}$ is said to be double lacunary if

$$p_0, h_a = p_a - p_{a-1} \to \infty\ as\ a \to \infty.$$

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$$q_0, h_b = q_b - q_{b-1} \to \infty \ as \ b \to \infty.$$

where h_a and h_b is the non-decreasing sequence of positive numbers.

In 2009, with the help of continuous t-norm, continuous t-conorm and fuzzy sets Mursaleen and Mohiuddine [14] presented lacunary statistical convergence with respect to the intuitionistic fuzzy normed space.

In 2012, Tripathy, Hazarika and Choudhary [17] defined the term lacunary $\mathbb{I}$-convergent sequences and also defined lacunary $\mathbb{I}$-cauchy sequences that are more generalized than the ideal convergence.

Definition 2.5 [12] A non-null class $\mathbb{I}$ be a subset of power set of a vector space V will be an ideal in V if,

1. $\emptyset \in \mathbb{I}$
2. $P, Q \in \mathbb{I}$ implies $P \cup Q \in \mathbb{I}$ and
3. $P \in \mathbb{I}$ and $Q \subset P$ implies $Q \in \mathbb{I}$.

if $\mathbb{I} \neq 2^V$, then it is non-trivial ideal.

Definition 2.6 A non-null class $\mathbb{F}$ be a subset of power set of a vector space V will be a filter in V if,

1. $\emptyset \notin \mathbb{F}$.
2. $P, Q \in \mathbb{F}$ implies $P \cap Q \in \mathbb{F}$ and
3. $P \in F$ and $Q \supset P$ implies $Q \in \mathbb{F}$.

Definition 2.7 A non-trivial ideal $\mathbb{I} \subset 2^{\mathbb{V}}$ which have all the singletons $i.e.\{\{s\} : s \in V\}$ is termed as admissible ideal in V.

Definition 2.8 : A sequence $y = \{y_m\} : m \in \mathbb{N}$ of real numbers will $\mathbb{I}$-converge to a number M provided that, for any $\epsilon > 0$,

$$\{m \in \mathbb{N} : |y_m - M| \geq \epsilon\} \in \mathbb{I}$$

Definition 2.9 Let $\theta = k_s$ be a lacunary sequence and $y = (y_m)$ the sequence of real numbers then the sequence $y = (y_m)$ is called lacunary $\mathbb{I}$-convergent if for any $\epsilon > 0$,

$$\left\{s \in \mathbb{N} : \frac{1}{h_s} \sum_{m \in J_s} |y_m - M| \geq \epsilon\right\} \in \mathbb{I}$$

We write $\mathbb{I}_\theta - lim\ y_m = M$

Definition 2.10 Let $\theta = K_s$ be a lacunary sequence and $y = (y_m)$ the sequence of real numbers then the sequence $y = (y_m)$ is called lacunary $\mathbb{I}$-cauchy if a subsequence (y_{m_s}) of y_m is exist such that $k(s) \in J_s$ and for every s, $lim_{s\to\infty}(y_{m_s}) = M$ and for any $\epsilon > 0$,

$$\left\{ s \in \mathbb{N} : \tfrac{1}{h_s} \textstyle\sum_{m\in J_s} |y_m - y_{m_s}| \geq \epsilon \right\} \in \mathbb{I}$$

In 2014, lacunary Δ-statistical convergence introduced by Selma and Esra [2] with respect to intuitionistic fuzzy n-normed space.

In 2015, in probabilistic normed space Ayhan and Savas [6] introduced Lacunary statistically convergent triple sequences.

Definition 2.11 Let $\theta_{a,b,c} = (\{p_a, q_b, r_c\})$ be the triple sequence, then the triple sequence $\theta_{a,b,c}$ is said to be triple lacunary if

$$p_0, h_a = p_a - p_{a-1} \to \infty\ as\ a \to \infty.$$

$$q_0, h_b = q_b - q_{b-1} \to \infty\ as\ b \to \infty.$$

$$r_0, h_c = r_c - p_{c-1} \to \infty\ as\ c \to \infty.$$

3 λ-Statistical Convergence

In this section, first of all the basic hypothesis of λ-statistical convergence is given.

In 2000, Mursaleen [13] introduced the theory of λ-statistical convergence. He worked on (V;λ)-summability that is more generalized than classical convergence and statistical convergence and this mechanism is known as λ-statistical convergence. He also discuss about the connection between (V;λ)-summability and (C,1)-summability.

Definition 3.1 Let $y = (y_m)$ be the real sequence then $y = (y_m)$ is known as λ-statistical convergent to M if for $\beta > 0$ there is a non-negative number l such that

$$\lim_l \frac{1}{\lambda_l} |\{m \in I_l : |y_m - M| \geq \beta\}| = 0.$$

It is known as λ-density where λ_n is a non-decreasing sequence of non-negative numbers such as $\lambda_0 = 1, (\lambda_l - \lambda_{l-1}) \to \infty$ as $l \to \infty$ where $I_l = [\lambda_{l-1}, \lambda_l]$.

Definition 3.2 Let $y = (y_m)$ be the sequence of real numbers then the sequence $y = (y_m)$ is known as (V;λ)-summable if

$$\{y = y_m : \exists\ M \in \mathbb{R}, \lim_l \frac{1}{\lambda_l} \sum_{m \in I_l} |y_m - M| = 0\}.$$

Definition 3.3 Let $y = (y_m)$ be the sequence of real numbers then the sequence $y = (y_m)$ is known as (C, 1)-summable if

$$\{y = y_m : \exists\ M \in \mathbb{R}, \lim_l \frac{1}{l} \sum_{m=1}^{l} |y_m - M| = 0\}.$$

In 2012, in probabilistic normed space Savas and Mohiuddine presented the hypothesis of double $\bar{\lambda}$-statistically convergent sequence and double $\bar{\lambda}$-statisically cauchy sequences.

Definition 3.4 In probabilistic normed space let $y = (y_{k,l})$ be the real double sequence then $y = (y_{k,l})$ is known as double $\bar{\lambda}$-statistically convergent if for all $\beta > 0$,

$$\delta_{\bar{\lambda}}\{k \in I_n, l \in r_m; v_{y_{k,l}} - M(\beta) \leq 1 - t\}$$

Definition 3.5 In probabilistic normed space let $y = (y_{k,l})$ be the real double sequence then $y = (y_{k,l)}$ is known as double $\bar{\lambda}$-statistically cauchy if for all $\beta > 0$,

$$\delta_{\bar{\lambda}}\{k \in I_n, l \in r_m; v_{y_{k,l}} - v_{yp,q}(\beta) \leq 1 - t\}$$

In 2013, Hazarika and Savas [10] give the definition of λ-statistical convergence in n-normed space. The authors discuss the relation between (C,1)-summability and (V;λ)-summability in n-normed space.

Definition 3.6 In n-normed space suppose $y = (y_m)$ is a real sequence then $y = (y_m)$ is known as λ-statistically convergent if for any $\beta > 0$

$$\lim_l \frac{1}{\lambda_l} |\{m \in I_l : ||y_m - t, t_1, t_2, ...t_{l-1}|| \geq \beta\}| = 0.$$

In 2013, Mohammed and Mursaleen [1] have introduced the term λ-statistical convergence in para-normed space. The authors have given the definitions of λ-statistically cauchy sequence and also define the strongly λ_p-summability in para-normed space.

Definition 3.7 Suppose (V,f) is a paranormed space and $y = (y_m)$ be the sequence of real numbers, then the sequence $y = (y_m)$ is known as λ-Statistically Convergent to M if for any given $\beta > 0$

$$\lim_{l} \frac{1}{\lambda_l} |\{m \in I_l : |f(y_m - M)| \geq \beta\}| = 0.$$

In 2019, A new term was proposed in λ-statistical convergence by Omar kisi [11] named as $I_2 - \bar{\lambda}$-statistical convergence of double sequence in fuzzy normed linear space in which the concept of two non-decreasing sequence $\lambda = \lambda_r$ and $\mu = \mu_s$ is used. He developed the relation between $I_2 - [V; \lambda]$-summability and $I_2 - [C; 1]$ -summability.

Definition 3.8 Let $y = (y_{a,b})$ is a double sequence then $y = (y_{a,b})$ is known as $I_2 - \lambda$-statistical convergence to M in fuzzy normed space if for all $\epsilon, \delta > 0$, the following condition holds

$$\{(k,l) \in \mathbb{N} \times \mathbb{N} : \frac{1}{\bar{\lambda}_{k,l}} |\{(a,b) \in I_{k,l} : D(||y_{a,b} - M||, \bar{0}) \geq \epsilon\} \geq \delta\}$$

4 Observation

The hypothesis of lacunary statistical convergence is more generalized than Statistical convergence as θ-density is the extended form of natural density. Similarly λ-statistical convergence is more generalized than statistical convergence as λ-density is the extended form of natural density.

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