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Solving Linear Optimization Problem with Max-Archimedean Interval-Valued Fuzzy Relation Equations as Constraints

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Abstract

This paper introduces a linear optimization problem subject to max-Archimedean interval-valued fuzzy relation equations. According to the literature, three types of solution sets, namely; tolerable solution set, united solution set and controllable solution set can be identified with interval-valued fuzzy relation equations. Since the tolerable solutions are very useful in fuzzy control problems, thus optimization with such type of fuzzy relation equations is an important topic of research. The structure and the properties of the tolerable solution set are studied. The tolerable solution set can be characterized by one maximum solution and finitely many minimal solutions. Generally, determining all minimal solutions is a computationally difficult task, thus an efficient algorithm based on the rules of reduction is proposed which directly computes the tolerable optimal solution of the problem without finding the set of all minimal solutions. The concept of reduction is efficient for large size problems in terms of computation. The proposed method is illustrated with some examples.

Keywords: Interval-valued fuzzy relation equations, Archimedean t-norm, Linear optimization problem.

1 Introduction

The system of fuzzy relation equations (FRE) was first investigated by Sanchez [13], considering max-min composition. Since the system of FRE is useful in modelling many systems such as medical diagnosis, image processing, neural networks etc., therefore it is intensively studied by many researchers taking different type of compositions. A general representation of the system of FRE with sup–$\otimes$ is:

$$x \circ A = b \tag{1}$$

where $A = [a_{ij}], 0 \le a_{ij} \le 1$ is an $m \times n$ dimensional matrix, $b = [b_j], 0 \le b_j \le 1$, is an n–dimensional vector and $\circ$ denotes the sup–$\otimes$ composition of x and A, $\otimes$ being a continuous t-norm. The resolution problem of FRE (1) is to determine an m–dimensional vector $x = [x_i], 0 \le x_i \le 1$, such that (1) holds.

The general method for solving max-min FRE is given by Higashi and Klir [4]. Di Nola et al. [1] worked on the system of FRE with sup–$\otimes$ and proved that the solution set can be completely described by a unique maximum solution and finitely many minimal solutions. The system of FRE with max-product composition is studied by Markovskii [11]. It was shown that solving such type of system is closely related with the covering problem, which is an NP-hard problem. Lin [6] considered the generalization of this problem by taking sup–t FRE, t being any Archimedean t-norm and established a one to one correspondence between the minimal solutions and the irredundant coverings.

Fang and Li [2] first introduced a linear optimization problem subject to the system of FRE with max-min composition. They studied the nature of solution set, converted the system to an equivalent 0-1 integer programming problem and solved it using branch and bound technique. Loetamonphong and Fang [8] studied a linear optimization problem with max-product FRE. They derived special characteristics of the feasible domain, optimal solutions and proposed some methods to reduce the original problem. More work in this regard can be found in [12, 18]. Thapar et al. [14] considered a linear objective function subject to max–t FRE as constraints, where t is an Archimedean t-norm. They converted the problem to an equivalent 0-1 integer programming problem using the concept of covering problem and proposed a binary coded genetic algorithm to obtain the optimal solution.

A non-linear optimization problem subject to the system of max-min FRE was first considered by Lu and Fang [10]. They designed a domain specific genetic algorithm by taking advantage of the structure of the solution set of FRE. The individuals from the initial population were chosen from the feasible solution set and were kept within the feasible region during the mutation and crossover operations. Lin et al. [7] applied genetic algorithm for solving non-linear optimization model subject to max–t FRE as constraints.

Loetamonphong et al. [9] studied class of optimization problems with multiple objective functions subject to max-min FRE. They proposed a genetic algorithm to find the Pareto optimal solutions. Guu et al. [3] solved multi-objective linear functions subject to max-t FRE as constraint using two phase method to obtain a better solution. Thapar et al. [15] discussed satisficing solutions for multiobjective optimization problems with max-product FRE as constraints using genetic algorithm.

The study of interval-valued FRE has been an area of interest among many researchers. Being an extension of constant valued FRE, where each element of the relational matrix belongs to the unit interval [0,1], the interval-valued FRE are more flexible in terms of handling impreciseness and uncertainties. Each element of interval-valued fuzzy relational matrix are sub-intervals in the unit interval [0,1]. This nature enhances the flexibility of applications of interval-valued FRE in real life problems. The interval-valued FRE has important application in fuzzy control and medical diagnosis. Wang and Chang [16] first gave the method for resolving the composite interval-valued FRE. They explored the properties of interval-valued FRE with max-min composition and proposed an algorithm to resolve it.

Li and Fang [5] analyzed the properties of the solution set of interval-valued max-min FRE and converted the problem into system of fuzzy relational inequalities. They suggested more efficient method in terms of computational work for finding the complete solution set. Wang et al. [17] discussed three types of solution sets namely: tolerable solution sets, united solution set and controllable solution set for interval-valued max–t FRE. They studied the properties of each solution set and gave relationship between them.

According to Wang et al. [17], the nonempty solution set of max–t interval-valued FRE, t being a continuous t–norm, can be characterized by one maximum and finitely many minimal solutions. Being motivated by the study of linear optimization problem with FRE, in this paper we are focusing on optimization of a linear objective function subject to max– Archimedean interval-valued FRE. An efficient algorithm is proposed to find the optimal solution of the problem without finding the set of all minimal solutions. In Section 2, basic concepts related to interval-valued FRE are recalled. In Section 3, a linear optimization problem with max-Archimedean interval-valued FRE is introduced and some results are proved that characterise the components of the optimal solution. Based on these results, some rules are proposed in Section 4 that determines each component of the optimal solution to optimize the objective function. The proposed method is illustrated using some examples in Section 5.

2 Preliminary properties

Let $I = \{1, 2, \ldots, m\}$ and $J = \{1, 2, \ldots, n\}$ be the index sets, $A^- = [a_{ij}^-]$ and $A^+ = [a_{ij}^+]$ with $0 \le a_{ij}^- \le a_{ij}^+ \le 1$, are $m \times n$ dimensional matrices; $b^- = [b_j^-]$ and $b^+ = [b_j^+]$ with $0 \le b_j^- \le b_j^+ \le 1$, are n–dimensional vectors. Consider an $m \times n$ interval-valued matrix $A^I = [[a_{ij}^-, a_{ij}^+]]$, formed by A^- and A^+ and an n–dimensional interval-valued vector $b^I = [[b_j^-, b_j^+]]$, formed by b^- and b^+, then

$$x \circ A^I = b^I \tag{2}$$

represents an interval-valued FRE, where $\circ$ denotes the max–$\otimes$ composition of x and A^I, $\otimes$ being an Archimedean t–norm. A continuous t–norm that is subidempotent, i.e. $a \otimes a < a; \forall a \in [0, 1]$, is called an Archimedean t–norm. In [17], three types of solution sets for (2) are discussed, one of them is the tolerable solution set. Let $X(A^I, b^I)$ denotes the set of tolerable solutions of (2), then determining $X(A^I, b^I)$ for interval-valued FRE (2), means to find the set of all vectors $x = [x_i], 0 \leq x_i \leq 1, \forall i \in I$, such that

$$b_j^- \leq \max_{i \in I}(x_i \otimes a_{ij}) \leq b_j^+, \forall a_{ij} \in [a_{ij}^-, a_{ij}^+], \forall j \in J$$

Moreover, if we denote $X = \{x = [x_i]_{i\in I} | 0 \leq x_i \leq 1\}$, $A_M^I = \{A | A^- \leq A \leq A^+\}$ and $b_M^I = \{b | b^- \leq b \leq b^+\}$, then the tolerable solution set can be defined as $X(A^I, b^I) = \{x \in X|$ for each $A \in A_M^I$, there exists $b \in b_M^I$ such that $x \circ A = b\}$. If $X(A^I, b^I) \neq \emptyset$, then it is said to be consistent and it contains one maximum solution and finitely many minimal solutions, otherwise it is said to be inconsistent.

Theorem 1 *The problem of finding the tolerable solution set $X(A^I, b^I)$ of the system of interval-valued FRE $x \circ A^I = b^I$ is equivalent to the problem of finding all solutions of the following system of fuzzy relational inequalities:*

$$\begin{cases} x \circ A^+ \leq b^+ \\ x \circ A^- \geq b^- \end{cases} \tag{3}$$

i.e. x is a solution of (2) iff x is a solution of (3).

Proof *Refer [17], for the proof.*

If A^I and b^I are given for (2), then according to [17], define

$$\hat{x} = [\hat{x}_i]_{i\in I} = [\min_{j\in J}(a_{ij}^+ \rightarrow_s b_j^+)]_{i\in I} \tag{4}$$

where $\rightarrow_s$ denotes the fuzzy implication operator defined as $a \rightarrow_s b = sup\{x \in [0, 1] | x \otimes a \leq b\}$, given $a, b \in [0, 1]$.

Theorem 2 *If $X(A^I, b^I) \neq \emptyset$ then $\hat{x}$ defined by (4) is the maximum solution of $X(A^I, b^I)$.*

Proof *Refer [17], for the proof.*

Lemma 1 *$X(A^I, b^I) \neq \emptyset$ if and only if $\hat{x} \circ A^- \geq b^-$.*

Proof *Refer [17], for the proof.*

Lemma 2 *If $x \in X(A^I, b^I)$, then for every equation $j \in J$, $\max_{i\in I}(x_i \otimes a_{ij}^+) \leq b_j^+$, and there exists $i_0 \in I$ such that $(x_{i_0} \otimes a_{i_0 j}^-) \geq b_j^-$.*

Proof *For a solution $x \in X(A^I, b^I)$, since $x \circ A^+ \leq b^+$, thus $\max_{i\in I}(x_i \otimes a_{ij}^+) \leq b_j^+$, $\forall j \in J$. Also $x \circ A^- \geq b^-$ implies that $\max_{i\in I}(x_i \otimes a_{ij}^-) \geq b_j^-$, for $j \in J$. Thus, for each $j \in J$, there exists $i_0 \in I$ such that $(x_{i_0} \otimes a_{i_0 j}^-) \geq b_j^-$.*

Lemma 3 *If for any equation $j \in J$, $a_{ij}^+ < b_j^-, \forall i \in I$, then $X(A^I, b^I) = \emptyset$.*

Proof *If for any equation $j \in J$, $a_{ij}^+ < b_j^-, \forall i \in I$, this implies $a_{ij}^- < b_j^-, \forall i \in I$. Thus, $x_i \otimes a_{ij}^- \leq 1 \otimes a_{ij}^- = a_{ij}^- < b_j^-, \forall i \in I$, i.e. $x_i \otimes a_{ij}^- < b_j^-, \forall i \in I$, i.e. $\max_{i\in I}(x_i \otimes a_{ij}^-) < b_j^-$. Hence $X(A^I, b^I) = \emptyset$.*

Definition 1 *For $x = [x_i]_{i\in I} \in X(A^I, b^I)$, x_i is called a binding variable if $x_i \otimes a_{ij}^- \geq b_j^-$ holds for some equation $j \in J$ and a constraint $j \in J$ is said to be a binding constraint if $x_i \otimes a_{ij}^- \geq b_j^-$ holds for some $i \in I$.*

Let $X(A^I, b^I) \neq \emptyset$, then define $I_j = \{i \in I | \hat{x}_i \otimes a_{ij}^- \geq b_j^-\}, \forall j \in J$ and $J_i = \{j \in J | \hat{x}_i \otimes a_{ij}^- \geq b_j^-\}, \forall i \in I$. I_j denotes the set of binding variables for the j^{th} equation and J_i denotes the set of binding constraints for the i^{th} variable. Thus for $x = [x_i]_{i \in I} \in X(A^I, b^I)$, x_i is a binding variable iff $J_i \neq \emptyset$.

Let $\rightarrow_i$ denotes the fuzzy implication operator defined as $a \rightarrow_i b = inf\{x \in [0,1] | x \otimes a \geq b\}$, given $a, b \in [0,1]$ and $a \geq b$. From Definition 1, for a binding variable x_i, $x_i \otimes a_{ij}^- \geq b_j^-$ for some $j \in J$ and for a binding constraint $x_i \otimes a_{ij}^- \geq b_j^-$ for some $i \in I$. This implies that $x_i \geq a_{ij}^- \rightarrow_i b_j^- \ \forall j \in J_i$. By using property of t-norms $a \otimes b \leq min(a, b)$, we get $x_i \otimes a_{ij}^- \leq min(x_i, a_{ij}^-)$, therefore $a_{ij}^- \geq b_j^-$. Thus, $a_{ij}^- \rightarrow_i b_j^-$ is clearly defined.

Theorem 3 *Let $x = [x_i]_{i \in I} \in X(A^I, b^I)$. For any i^{th} component of the solution,*

(i) if $J_i \neq \emptyset$, then $\max_{j \in J_i}(a_{ij}^- \rightarrow_i b_j^-) \leq x_i \leq \hat{x}_i$,

(ii) if $J_i = \emptyset$, then $0 \leq x_i \leq \hat{x}_i$.

Proof *(i) For $x = [x_i]_{i \in I} \in X(A^I, b^I)$, if $J_i \neq \emptyset$, then x_i is a binding variable. Thus $x_i \otimes a_{ij}^- \geq b_j^-, \forall j \in J_i$, i.e. $x_i \geq a_{ij}^- \rightarrow_i b_j^-, \forall j \in J_i$. Thus $x_i \geq \max_{j \in J_i}(a_{ij}^- \rightarrow_i b_j^-)$. Since $\hat{x} = [\hat{x}_i]_{i \in I}$ is the maximum solution of (2), hence $\max_{j \in J_i}(a_{ij}^- \rightarrow_i b_j^-) \leq x_i \leq \hat{x}_i$.*

(ii) For $x = [x_i]_{i \in I} \in X(A^I, b^I)$, if $J_i = \emptyset$, then x_i is a nonbinding variable. Thus $x_i \otimes a_{ij}^- < b_j^-, \forall j \in J$. Since $x_i \in [0,1]$ and $\hat{x} = [\hat{x}_i]_{i \in I}$ is the maximum solution, therefore $0 \leq x_i \leq \hat{x}_i$.

From Theorem 3, for $x = [x_i]_{i \in I} \in X(A^I, b^I)$, if x_i is a binding variable, then $\max_{j \in J_i}(a_{ij}^- \rightarrow_i b_j^-)$ serves as the lower bound of x_i and $\hat{x}_i$ serves as the upper bound of x_i. And if x_i is a nonbinding variable, then 0 serves as the lower bound of x_i and $\hat{x}_i$ serves as the upper bound of x_i.

For binding variable x_i, we denote lower bound of x_i as $\check{x}_i$, i.e. $\check{x}_i = \max_{j \in J_i}(a_{ij}^- \rightarrow_i b_j^-)$.

3 The Problem

According to Theorem 3, the value of each component x_i, for $x = [x_i]_{i \in I} \in X(A^I, b^I)$ either lies within $\check{x}_i$ and $\hat{x}_i$, or lies within 0 and $\hat{x}_i$, i.e. $\check{x}_i \leq x_i \leq \hat{x}_i$ or $0 \leq x_i \leq \hat{x}_i$ and $X(A^I, b^I)$ contains one maximum solution and finitely many minimal solutions, thus optimization problem can be associated to it. We are interested in solving the following optimization problem:

$$Min\ Z = \sum_{i \in I} c_i x_i \tag{5}$$

$$s.t.\ b_j^- \leq \max_{i \in I}(x_i \otimes a_{ij}) \leq b_j^+, \forall a_{ij} \in [a_{ij}^-, a_{ij}^+], \forall j \in J \tag{6}$$

$$0 \leq x_i \leq 1, \forall i \in I$$

where $c_i \in R$ is the cost associated with the variable x_i and the other symbols have their usual meaning as defined in Section 2.

Finding the optimal solution of the problem (5)–(6) is to find one or more tolerable solutions from $X(A^I, b^I)$ that minimize (5). Since the objective function is linear, thus any component x_i^* of the optimal solution $x^* = [x_i^*]_{i \in I} \in X(A^I, b^I)$ will be equal to its lower bound or its upper bound. Our aim is to determine each component of the optimal solution, so as to optimize the considered objective function. Each component x_i^* of the optimal solution $x^* = [x_i^*]_{i \in I} \in X(A^I, b^I)$ can be found by considering the cost c_i associated with x_i in the objective function as well as the nature of x_i, i.e. whether it is a binding variable or a nonbinding variable.

Theorem 4 *Let $x^* = [x_i^*]_{i\in I} \in X(A^I, b^I)$ be the optimal solution of the problem (5)–(6). Then for $k \in I$, the following holds:*

(i) If $c_k < 0$, then $x_k^* = \hat{x}_k$,

(ii) If $c_k > 0$, then $x_k^* = \check{x}_k$, if x_k is a binding variable and $x_k^* = 0$, if x_k is a nonbinding variable.

(iii) If $c_k = 0$, x_k^* can take any value within its lower and upper bound.

Proof *(i) For the optimal solution $x^* \in X(A^I, b^I)$, if $c_k < 0, k \in I$, let $x_k^* \neq \hat{x}_k$. Then $\sum_{i\in I} c_i x_i^* = \sum_{i\in I, i\neq k} c_i x_i^* + c_k x_k^* > \sum_{i\in I, i\neq k} c_i x_i^* + c_k \hat{x}_k$. It contradicts the fact that x^* is the optimal solution. Hence, $x_k^* = \hat{x}_k$.*

(ii) For the optimal solution $x^* \in X(A^I, b^I)$, if $c_k > 0, k \in I$, let x_k^* be a binding variable and $x_k^* \neq \check{x}_k$. Then using Theorem 3, $\check{x}_k \leq x_k^* \leq \hat{x}_k$. Since $c_k > 0$, therefore $\sum_{i\in I} c_i x_i^* = \sum_{i\in I, i\neq k} c_i x_i^* + c_k x_k^* > \sum_{i\in I, i\neq k} c_i x_i^* + c_k \check{x}_k$. It contradicts the fact that x^* is the optimal solution. Hence, $x_k^* = \check{x}_k$.

Similarly, let x_k be a nonbinding variable and $x_k^* \neq 0$, then using Theorem 3, $0 \leq x_k^* \leq \hat{x}_k$. Since $c_k > 0$, therefore $\sum_{i\in I} c_i x_i^* = \sum_{i\in I, i\neq k} c_i x_i^* + c_k x_k^* > \sum_{i\in I, i\neq k} c_i x_i^* + c_k \cdot 0$. It contradicts the fact that x^* is the optimal solution. Hence, $x_k^* = 0$.

(iii) For the optimal solution $x^* \in X(A^I, b^I)$, if $c_k = 0$ for some $k \in I$, there will be no effect on objective function with the value of x_k^*. Thus, x_k^* can take any value with in its lower and upper bound.

4 Rules for reducing the problem

On the basis of the properties discussed in Section 2 and Section 3, in this section, we employ a value based matrix method to reduce the original problem. Moreover using Theorem 4, for $c_i < 0, i \in I$, the optimal value x_i^* of x_i is always fixed to $\hat{x}_i$. Thus, we can limit our search to find the optimal values of the components $x_i, i \in I$ with $c_i > 0$. In this case a component x_i^* of the optimal solution is equal to $\check{x}_i$, if x_i is binding and equal to 0, if x_i is nonbinding. Hence, binding variables can provide useful information in searching an optimal solution of the problem (5)–(6). Using these properties a value matrix $M = [m_{ij}]$ is defined as

$$m_{ij} = \begin{cases} c_i \check{x}_i, & \text{if } j \in J_i \\ \infty, & \text{otherwise} \end{cases}, \forall i \in I, \forall j \in J$$

Define $\bar{I}_j = \{i \in I | m_{ij} = c_i \check{x}_i\}, \forall j \in J$ and $\bar{J}_i = \{j \in J | m_{ij} = c_i \check{x}_i\}, \forall i \in I$. Some rules are proposed to optimize the problem by employing value matrix.

Rule 1 *If for some $i \in I$, $m_{ij} < 0$ for some $j \in J$, then assign $x_i^* = \hat{x}_i$ in the optimal solution x^*.*

Proof *If for some $i \in I$, $m_{ij} < 0$ for some $j \in J$, this implies that $c_i < 0$. Thus, from Theorem 4, $x_i^* = \hat{x}_i$.*

Rule 2 *If for some $j \in J$, $\bar{I}_j = \{i\}$ is a singleton set and $c_i > 0$, then assign $x_i^* = \check{x}_i$ in the optimal solution x^*.*

Proof *If for some $j \in J$, $\bar{I}_j = \{i\}$ is a singleton set, then this implies that the j^{th} equation can only be satisfied by the variable x_i, i.e. x_i is the only binding variable for the j^{th} equation. Also, since $c_i > 0$, thus by Theorem 4, $x_i^* = \check{x}_i$.*

Rule 3 *If $\bar{I}_s \subseteq \bar{I}_t$ for some s, t in the value matrix, then t^{th} column of the value matrix M can be deleted.*

Proof *If $\bar{I}_s \subseteq \bar{I}_t$ for some $s, t \in J$ in the value matrix, then all the variables that are binding for the s^{th} equation are also binding for the t^{th} equation. Thus, column corresponding to the t^{th} equation can be deleted from consideration.*

Rule 4 *If $\emptyset \neq \bar{J}_p \subseteq \bar{J}_q$ for some $p, q \in I$ and $0 < c_q \check{x}_q < c_p \check{x}_p$, then assign $x_p^* = 0$ in the optimal solution x^*.*

Proof *Let $x^* = [x_i^*]_{i \in I}$ be any optimal solution. Since $\bar{J}_p \subseteq \bar{J}_q$, this implies that $J_p \subseteq J_q$. Also since $0 < c_q \check{x}_q < c_p \check{x}_p$, thus $c_p > 0$. This implies that $x_p^* = 0$ or $x_p^* = \check{x}_p$, using Theorem 4. If $x_p^* = 0$, then the proof is done. If $x_p^* = \check{x}_p > 0$, then if $x_q^* = 0$, then we can construct a solution vector x, equal to x^*, except for $x_p = 0$ and $x_q = \check{x}_q$. Since $J_p \subseteq J_q$, the constraints satisfied by $x_p^* = \check{x}_p$ are also satisfied by $x_q = \check{x}_q$. Thus, x is a solution of the problem. Also $Z(x^*) - Z(x) = \sum\limits_{i \in I} c_i x_i^* - \sum\limits_{i \in I} c_i x_i = c_p x_p^* - c_q x_q = c_p \check{x}_p - c_q \check{x}_q > 0$. This contradicts the assumption of $x^* = [x_i^*]_{i \in I}$ being the optimal solution. Therefore, if $0 < c_q \check{x}_q < c_p \check{x}_p$, then for the optimal solution $x^* = [x_i^*]_{i \in I}$, $x_p^* = 0$.*

If $x_p^* = \check{x}_p > 0$, then if $x_q^* = \check{x}_q$, then we can construct a solution vector x, equal to x^*, except for $x_p = 0$. Since $J_p \subseteq J_q$, the constraints satisfied by $x_p^* = \check{x}_p$ are also satisfied by $x_q = \check{x}_q$. Thus, x is a solution of the problem. Also $Z(x^*) - Z(x) = \sum\limits_{i \in I} c_i x_i^* - \sum\limits_{i \in I} c_i x_i = c_p x_p^* = c_p \check{x}_p > 0$. This contradicts the assumption of $x^* = [x_i^*]_{i \in I}$ being the optimal solution. Therefore, if $0 < c_q \check{x}_q < c_p \check{x}_p$, then for the optimal solution $x^* = [x_i^*]_{i \in I}$, $x_p^* = 0$.

Rule 5 *Let $\hat{I} \subseteq I$ and $\bigcup\limits_{i \in \hat{I}} \bar{J}_i = \hat{J}$. If $p \in I$, $p \notin \hat{I}$, $\bar{J}_p \subseteq \hat{J}$, and $\sum\limits_{i \in \hat{I}} c_i \check{x}_i < c_p \check{x}_p$, then assign $x_p^* = 0$ in the optimal solution x^*.*

Proof *Similar to Rule 4.*

Rule 6 *If for some $p \in I$, $c_p > 0$ and $\bar{J}_p = \emptyset$, then assign $x_p^* = 0$ in the optimal solution x^*.*

Proof *If for some $p \in I$, $\bar{J}_p = \emptyset$, then x_p is a nonbinding variable. Since $c_p > 0$, thus by Theorem 4, p^{th} component of the optimal solution vector can be assigned value 0.*

Algorithm for obtaining the optimal solution of the problem (5)–(6)

Step 1 *Compute the maximum solution $\hat{x}$, according to (4).*

Step 2 *Check feasibility. If $\hat{x} \circ A^- \geq b^-$, continue to the next step. Otherwise stop, the problem is inconsistent, i.e. $X(A^I, b^I) = \emptyset$.*

Step 3 *Find index sets $J_i, \forall i \in I$.*

Step 4 *For $i \in I$, if $J_i \neq \emptyset$, compute $\check{x}_i$ using $\check{x}_i = \max\limits_{j \in J_i}(a_{ij}^- \rightarrow_i b_j^-)$. Hence, compute the value of $c_i \check{x}_i$.*

Step 5 *Obtain the value matrix M.*

Step 6 *Find index sets $\bar{J}_i, \forall i \in I$ and $\bar{I}_j, \forall j \in J$ for the value matrix M.*

Step 7 *Apply Rule 1- Rule 6 to determine the values of as many decision variables as possible. If all the components of the optimal solution x^* are determined, generate the optimal solution and hence the optimal value of the objective function, else go to next step.*

Step 8 *Apply branch and bound method as discussed in [18], to determine the remaining undecided decision variables and hence the optimal solution x^* and the optimal value of the objective function.*

5 Numerical illustrations

Example 1 *Consider the following linear optimization problem subject to the system of interval-valued FRE with max-product composition defined as* $x \otimes a = x \cdot a$,

$$Min\ Z = 3x_1 + 2x_2 + x_3 + 0.5x_4$$

$$s.t.\ x \otimes A^I = b^I$$

$$0 \leq x_i \leq 1, \forall i \in I$$

where

$$A^I = \begin{bmatrix} [0.51, 0.80] & [0.34, 0.72] & [0.17, 0.39] \\ [0.42, 0.58] & [0.60, 0.96] & [0.30, 0.80] \\ [0.60, 0.65] & [0.12, 0.52] & [0.20, 0.70] \\ [0.35, 0.82] & [0.25, 0.91] & [0.36, 0.60] \end{bmatrix}$$

$$b^I = \begin{bmatrix} [0.30, 0.60] & [0.20, 0.60] & [0.10, 0.40] \end{bmatrix}$$

In this problem, we have

$$A^- = \begin{bmatrix} 0.51 & 0.34 & 0.17 \\ 0.42 & 0.60 & 0.30 \\ 0.60 & 0.12 & 0.20 \\ 0.35 & 0.25 & 0.36 \end{bmatrix}$$

$$A^+ = \begin{bmatrix} 0.80 & 0.72 & 0.39 \\ 0.58 & 0.96 & 0.80 \\ 0.65 & 0.52 & 0.70 \\ 0.82 & 0.91 & 0.60 \end{bmatrix}$$

$$b^- = \begin{bmatrix} 0.30 & 0.20 & 0.10 \end{bmatrix}$$

$$b^+ = \begin{bmatrix} 0.60 & 0.60 & 0.40 \end{bmatrix}$$

Step 1 *Compute the maximum solution* $\hat{x}$*, according to (4). We get* $\hat{x}_1 = 0.75, \hat{x}_2 = 0.50, \hat{x}_3 = 0.57, \hat{x}_4 = 0.66$*. Thus,* $\hat{x} = \begin{bmatrix} 0.75 & 0.50 & 0.57 & 0.66 \end{bmatrix}$ *is the maximum solution.*

Step 2 *Since* $\hat{x} \otimes A^- \geq b^-$*, the given system of interval-valued FRE is consistent.*

Step 3 *Find index sets* J_i*,* $\forall i \in I$*.*
$J_1 = \{1, 2, 3\}, J_2 = \{2, 3\}, J_3 = \{1, 3\}, J_4 = \{3\}$.

Step 4 *Since* $J_i \neq \emptyset, \forall i \in I$*, thus we have* $\check{x}_1 = 0.59, \check{x}_2 = 0.33, \check{x}_3 = 0.50, \check{x}_4 = 0.28$ *and* $c_1\check{x}_1 = 1.77, c_2\check{x}_2 = 0.66, c_3\check{x}_3 = 0.50, c_4\check{x}_4 = 0.14$.

Step 5 *Obtain the value matrix* M.

$$M = \begin{array}{c c} & \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{array}{c} x_1 \\ x_2 \\ x_3 \\ x_4 \end{array} & \begin{bmatrix} 1.77 & 1.77 & 1.77 \\ \infty & 0.66 & 0.66 \\ 0.50 & \infty & 0.50 \\ \infty & \infty & 0.14 \end{bmatrix} \end{array}$$

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Step 6 *Find index sets* $\bar{J}_i, \forall i \in I$ *and* $\bar{I}_j, \forall j \in J$ *corresponding to the matrix* M. $\bar{J}_1 = \{1,2,3\}, \bar{J}_2 = \{2,3\}, \bar{J}_3 = \{1,3\}, \bar{J}_4 = \{3\}$ *and* $\bar{I}_1 = \{1,3\}, \bar{I}_2 = \{1,2\}, \bar{I}_3 = \{1,2,3,4\}$.

Step 7 *In the matrix* M, $\bar{I}_1 \subseteq \bar{I}_3$, *thus using Rule 3, delete* 3^{rd} *column of the value matrix* M. *After reduction, the updated matrix* M *is obtained as*

$$M = \begin{array}{c c} & \begin{array}{cc} 1 & 2 \end{array} \\ \begin{array}{c} x_1 \\ x_2 \\ x_3 \\ x_4 \end{array} & \begin{bmatrix} 1.77 & 1.77 \\ \infty & 0.66 \\ 0.50 & \infty \\ \infty & \infty \end{bmatrix} \end{array}$$

In the updated matrix M, $\bar{J}_4 = \emptyset$, therefore using Rule 6, assign $x_4^* = 0$. Delete the row corresponding to the variable x_4 and obtain the updated matrix M.

$$M = \begin{array}{c c} & \begin{array}{cc} 1 & 2 \end{array} \\ \begin{array}{c} x_1 \\ x_2 \\ x_3 \end{array} & \begin{bmatrix} 1.77 & 1.77 \\ \infty & 0.66 \\ 0.50 & \infty \end{bmatrix} \end{array}$$

Now in the updated matrix M, we have $\bar{J}_1 = \{1,2\} \subseteq \bigcup_{i \in \{2,3\}} \bar{J}_i = \{1,2\}$, and also $c_2\check{x}_2 + c_3\check{x}_3 = 1.16, c_1\check{x}_1 = 1.77$, i.e. $c_2\check{x}_2 + c_3\check{x}_3 < c_1\check{x}_1$. Thus, using Rule 5, assign $x_1^* = 0$. Delete the row corresponding to the variable x_1 to obtain the updated matrix M.

$$M = \begin{array}{c c} & \begin{array}{cc} 1 & 2 \end{array} \\ \begin{array}{c} x_2 \\ x_3 \end{array} & \begin{bmatrix} \infty & 0.66 \\ 0.50 & \infty \end{bmatrix} \end{array}$$

In the updated matrix M, $\bar{I}_1 = \{3\}$ and $\bar{I}_2 = \{2\}$ are singleton sets, thus using Rule 2, assign $x_2^* = \check{x}_2 = 0.33$ and $x_3^* = \check{x}_3 = 0.50$. Since all the components of the optimal solution are determined, thus the optimal solution is $x^* = \begin{bmatrix} 0 & 0.33 & 0.50 & 0 \end{bmatrix}$ and the corresponding optimal value of the objective function is $Z^* = 1.16$.

Example 2 *Consider the following linear optimization problem subject to the system of interval-valued FRE with max-Lukasiewicz composition defined as* $x \otimes a = max(0, x + a - 1)$,

$$Min\ Z = 2x_1 + 5x_2 + 1.4x_3 + 0.5x_4 - 3x_5 + 0.7x_6 + 2x_7 + x_8$$

$$s.t.\ x \otimes A^I = b^I$$

$$0 \leq x_i \leq 1, \forall i \in I$$

where

$$A^I = \begin{bmatrix} [0.2,0.6] & [0.3,0.6] & [0.5,1.0] & [0.7,0.7] & [0.2,0.5] & [0.3,0.7] & [0.4,0.8] & [0.2,0.7] & [0.2,0.6] & [0.1,0.3] \\ [0.1,0.5] & [0.6,0.9] & [0.2,0.7] & [0.5,0.6] & [0.6,0.9] & [0.7,0.8] & [0.3,0.9] & [0.3,0.4] & [0.7,0.9] & [0.7,0.7] \\ [0.3,0.7] & [0.5,0.8] & [0.7,0.8] & [0.1,0.8] & [0.3,0.4] & [0.5,0.7] & [1.0,1.0] & [0.5,0.8] & [0.8,0.9] & [0.2,0.8] \\ [0.5,0.8] & [0.6,0.9] & [0.3,0.7] & [0.1,0.5] & [0.1,0.6] & [0.1,0.3] & [0.9,0.9] & [0.6,0.7] & [0.4,0.5] & [0.4,0.6] \\ [0.3,0.4] & [0.2,0.7] & [0.6,0.8] & [0.4,0.9] & [0.5,0.7] & [0.4,0.8] & [0.4,0.6] & [0.9,1.0] & [0.7,0.8] & [0.3,0.6] \\ [0.2,0.5] & [0.3,0.7] & [0.8,0.9] & [0.3,0.6] & [0.3,0.8] & [0.2,0.6] & [0.3,0.5] & [0.8,0.9] & [0.2,1.0] & [0.5,0.8] \\ [0.6,0.7] & [0.4,0.8] & [0.5,0.6] & [0.5,0.8] & [0.7,0.8] & [0.6,0.7] & [0.5,0.7] & [0.7,0.8] & [0.5,0.7] & [0.4,0.8] \\ [0.1,0.3] & [0.2,0.9] & [0.2,0.8] & [0.4,0.5] & [0.8,0.8] & [0.5,0.8] & [0.2,0.9] & [0.2,0.7] & [0.2,0.7] & [0.5,0.5] \end{bmatrix}$$

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$$b^I = \begin{bmatrix}[0.2, 0.7] & [0.3, 0.9] & [0.3, 0.8] & [0.5, 0.5] & [0.7, 0.8] & [0.4, 0.8] & [0.6, 0.9] & [0.3, 0.7] & [0.4, 0.8] & [0.4, 0.6]\end{bmatrix}$$

In this problem, we have

$$A^- = \begin{bmatrix} 0.2 & 0.3 & 0.5 & 0.7 & 0.2 & 0.3 & 0.4 & 0.2 & 0.2 & 0.1 \\ 0.1 & 0.6 & 0.2 & 0.5 & 0.6 & 0.7 & 0.3 & 0.3 & 0.7 & 0.7 \\ 0.3 & 0.5 & 0.7 & 0.1 & 0.3 & 0.5 & 1.0 & 0.5 & 0.8 & 0.2 \\ 0.5 & 0.6 & 0.3 & 0.1 & 0.1 & 0.1 & 0.9 & 0.6 & 0.4 & 0.4 \\ 0.3 & 0.2 & 0.6 & 0.4 & 0.5 & 0.4 & 0.4 & 0.9 & 0.7 & 0.3 \\ 0.2 & 0.3 & 0.8 & 0.3 & 0.3 & 0.2 & 0.3 & 0.8 & 0.2 & 0.5 \\ 0.6 & 0.4 & 0.5 & 0.5 & 0.7 & 0.6 & 0.5 & 0.7 & 0.5 & 0.4 \\ 0.1 & 0.2 & 0.2 & 0.4 & 0.8 & 0.5 & 0.2 & 0.2 & 0.2 & 0.5 \end{bmatrix}$$

$$A^+ = \begin{bmatrix} 0.6 & 0.6 & 1.0 & 0.7 & 0.5 & 0.7 & 0.8 & 0.7 & 0.6 & 0.3 \\ 0.5 & 0.9 & 0.7 & 0.6 & 0.9 & 0.8 & 0.9 & 0.4 & 0.9 & 0.7 \\ 0.7 & 0.8 & 0.8 & 0.8 & 0.4 & 0.7 & 1.0 & 0.8 & 0.9 & 0.8 \\ 0.8 & 0.9 & 0.7 & 0.5 & 0.6 & 0.3 & 0.9 & 0.7 & 0.5 & 0.6 \\ 0.4 & 0.7 & 0.8 & 0.9 & 0.7 & 0.8 & 0.6 & 1.0 & 0.8 & 0.6 \\ 0.5 & 0.7 & 0.9 & 0.6 & 0.8 & 0.6 & 0.5 & 0.9 & 1.0 & 0.8 \\ 0.7 & 0.8 & 0.6 & 0.8 & 0.8 & 0.7 & 0.7 & 0.8 & 0.7 & 0.8 \\ 0.3 & 0.9 & 0.8 & 0.5 & 0.8 & 0.8 & 0.9 & 0.7 & 0.7 & 0.5 \end{bmatrix}$$

$$b^- = \begin{bmatrix} 0.2 & 0.3 & 0.3 & 0.5 & 0.7 & 0.4 & 0.6 & 0.3 & 0.4 & 0.4 \end{bmatrix}$$

$$b^+ = \begin{bmatrix} 0.7 & 0.9 & 0.8 & 0.5 & 0.8 & 0.8 & 0.9 & 0.7 & 0.8 & 0.6 \end{bmatrix}$$

Step 1 *Compute the maximum solution* $\hat{x}$*, according to (4). We get* $\hat{x}_1 = 0.8, \hat{x}_2 = 0.9, \hat{x}_3 = 0.7, \hat{x}_4 = 0.9, \hat{x}_5 = 0.6, \hat{x}_6 = 0.8, \hat{x}_7 = 0.7, \hat{x}_8 = 1.0$*. Thus,* $\hat{x} = \begin{bmatrix} 0.8 & 0.9 & 0.7 & 0.9 & 0.6 & 0.8 & 0.7 & 1 \end{bmatrix}$ *is the maximum solution.*

Step 2 *Since* $\hat{x} \otimes A^- \geq b^-$*, the given system of interval-valued FRE is consistent.*

Step 3 *Find index sets* J_i*,* $\forall i \in I$*.*

$J_1 = \{3, 4\}, J_2 = \{2, 6, 9, 10\}, J_3 = \{3, 7, 9\}, J_4 = \{1, 2, 7, 8\}, J_5 = \{8\}, J_6 = \{3, 8\}, J_7 = \{1, 8\}, J_8 = \{5, 6, 10\}$.

Step 4 *Since* $J_i \neq \emptyset, \forall i \in I$*, thus we have* $\check{x}_1 = 0.80, \check{x}_2 = 0.70, \check{x}_3 = 0.60, \check{x}_4 = 0.70, \check{x}_5 = 0.40, \check{x}_6 = 0.50, \check{x}_7 = 0.60, \check{x}_8 = 0.90$ *and* $c_1\check{x}_1 = 1.60, c_2\check{x}_2 = 3.50, c_3\check{x}_3 = 0.84, c_4\check{x}_4 = 0.35, c_5\check{x}_5 = -0.12, c_6\check{x}_6 = 0.35, c_7\check{x}_7 = 1.20, c_8\check{x}_8 = 0.90$.

Step 5 *Obtain the value matrix* M*.*

$$M = \begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \\ x_8 \end{array} \begin{array}{c} \begin{array}{cccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \end{array} \\ \begin{bmatrix} \infty & \infty & 1.60 & 1.60 & \infty & \infty & \infty & \infty & \infty & \infty \\ \infty & 3.50 & \infty & \infty & \infty & 3.50 & \infty & \infty & 3.50 & 3.50 \\ \infty & \infty & 0.84 & \infty & \infty & \infty & 0.84 & \infty & 0.84 & \infty \\ 0.35 & 0.35 & \infty & \infty & \infty & \infty & 0.35 & 0.35 & \infty & \infty \\ \infty & \infty & \infty & \infty & \infty & \infty & \infty & -0.12 & \infty & \infty \\ \infty & \infty & 0.35 & \infty & \infty & \infty & \infty & 0.35 & \infty & \infty \\ 1.20 & \infty & \infty & \infty & \infty & \infty & \infty & 1.20 & \infty & \infty \\ \infty & \infty & \infty & \infty & 0.90 & 0.90 & \infty & \infty & \infty & 0.90 \end{bmatrix} \end{array}$$

Step 6 *Find index sets $\bar{J}_i, \forall i \in I$ and $\bar{I}_j, \forall j \in J$ corresponding to the matrix M.*

$\bar{J}_1 = \{3,4\}, \bar{J}_2 = \{2,6,9,10\}, \bar{J}_3 = \{3,7,9\}, \bar{J}_4 = \{1,2,7,8\}, \bar{J}_5 = \{8\}, \bar{J}_6 = \{3,8\}, \bar{J}_7 = \{1,8\}, \bar{J}_8 = \{5,6,10\}$ *and* $\bar{I}_1 = \{4,7\}, \bar{I}_2 = \{2,4\}, \bar{I}_3 = \{1,3,6\}, \bar{I}_4 = \{1\}, \bar{I}_5 = \{8\}, \bar{I}_6 = \{2,8\}, \bar{I}_7 = \{3,4\}, \bar{I}_8 = \{4,5,6,7\}, \bar{I}_9 = \{2,3\}, \bar{I}_{10} = \{2,8\}$.

Step 7 *In the matrix M, $m_{58} = -0.120 < 0$. Thus, using Rule 1, assign $x_5^* = \hat{x}_5 = 0.60$. Delete the row corresponding to the variable x_5 and the column corresponding to the 8^{th} equation. Obtain the updated matrix M.*

$$M = \begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_6 \\ x_7 \\ x_8 \end{array} \begin{array}{c} \begin{array}{ccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 9 & 10 \end{array} \\ \begin{bmatrix} \infty & \infty & 1.60 & 1.60 & \infty & \infty & \infty & \infty & \infty \\ \infty & 3.50 & \infty & \infty & \infty & 3.50 & \infty & 3.50 & 3.50 \\ \infty & \infty & 0.84 & \infty & \infty & \infty & 0.84 & 0.84 & \infty \\ 0.35 & 0.35 & \infty & \infty & \infty & \infty & 0.35 & \infty & \infty \\ \infty & \infty & 0.35 & \infty & \infty & \infty & \infty & \infty & \infty \\ 1.20 & \infty & \infty & \infty & \infty & \infty & \infty & \infty & \infty \\ \infty & \infty & \infty & \infty & 0.90 & 0.90 & \infty & \infty & 0.90 \end{bmatrix} \end{array}$$

In the updated matrix M, $\bar{I}_4 = \{1\} \subseteq \bar{I}_3 = \{1,3,6\}, \bar{I}_5 = \{8\} \subseteq \bar{I}_6 = \{2,8\}$ and $\bar{I}_5 = \{8\} \subseteq \bar{I}_{10} = \{2,8\}$. Therefore using Rule 3, the columns corresponding to the 3^{rd}, 6^{th} and 10^{th} equation can be deleted to obtain the updated matrix M.

$$M = \begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_6 \\ x_7 \\ x_8 \end{array} \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 4 & 5 & 7 & 9 \end{array} \\ \begin{bmatrix} \infty & \infty & 1.60 & \infty & \infty & \infty \\ \infty & 3.50 & \infty & \infty & \infty & 3.50 \\ \infty & \infty & \infty & \infty & 0.84 & 0.84 \\ 0.35 & 0.35 & \infty & \infty & 0.35 & \infty \\ \infty & \infty & \infty & \infty & \infty & \infty \\ 1.20 & \infty & \infty & \infty & \infty & \infty \\ \infty & \infty & \infty & 0.90 & \infty & \infty \end{bmatrix} \end{array}$$

Now, in the updated matrix M, $\bar{J}_6 = \emptyset$, thus, using Rule 6, assign $x_6^* = 0$ and delete the row corresponding to the variable x_6 to obtain the updated matrix M.

$$M = \begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_7 \\ x_8 \end{array} \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 4 & 5 & 7 & 9 \end{array} \\ \begin{bmatrix} \infty & \infty & 1.60 & \infty & \infty & \infty \\ \infty & 3.50 & \infty & \infty & \infty & 3.50 \\ \infty & \infty & \infty & \infty & 0.84 & 0.84 \\ 0.35 & 0.35 & \infty & \infty & 0.35 & \infty \\ 1.20 & \infty & \infty & \infty & \infty & \infty \\ \infty & \infty & \infty & 0.90 & \infty & \infty \end{bmatrix} \end{array}$$

In the updated matrix M, $\bar{I}_4 = \{1\}$ and $\bar{I}_5 = \{8\}$ are singleton sets, thus using Rule 2, assign $x_1^* = \check{x}_1 = 0.80$ and $x_8^* = \check{x}_8 = 0.90$. Delete the rows corresponding to the variables x_1 and x_8 and the columns corresponding to the 4^{th} and 5^{th} equation. Obtain the updated matrix M.

$$M = \begin{array}{c} \\ x_2 \\ x_3 \\ x_4 \\ x_7 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 7 & 9 \end{array} \\ \begin{bmatrix} \infty & 3.50 & \infty & 3.50 \\ \infty & \infty & 0.84 & 0.84 \\ 0.35 & 0.35 & 0.35 & \infty \\ 1.20 & \infty & \infty & \infty \end{bmatrix} \end{array}$$

In the updated matrix M, $\bar{J}_7 = \{1\} \subseteq \bar{J}_4 = \{1,2,7\}$ and also $c_4\check{x}_4 = 0.35 < c_7\check{x}_7 = 1.20$. Thus, using Rule 4, assign $x_7^* = 0$ and delete the row corresponding to the variable x_7 to obtain the updated matrix M.

$$M = \begin{array}{c} \\ x_2 \\ x_3 \\ x_4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 7 & 9 \end{array} \\ \begin{bmatrix} \infty & 3.50 & \infty & 3.50 \\ \infty & \infty & 0.84 & 0.84 \\ 0.35 & 0.35 & 0.35 & \infty \end{bmatrix} \end{array}$$

In the updated matrix M, $\bar{I}_1 = \{4\} \subseteq \bar{I}_2 = \{2,4\}$ and $\bar{I}_1 = \{4\} \subseteq \bar{I}_7 = \{3,4\}$. Thus, using Rule 3, the columns corresponding to the 2^{nd} and 7^{th} equation can be deleted to obtain the updated matrix M.

$$M = \begin{array}{c} \\ x_2 \\ x_3 \\ x_4 \end{array} \begin{array}{c} \begin{array}{cc} 1 & 9 \end{array} \\ \begin{bmatrix} \infty & 3.50 \\ \infty & 0.84 \\ 0.35 & \infty \end{bmatrix} \end{array}$$

In the updated matrix M, $\bar{I}_1 = \{4\}$ is a singleton set. Thus, using Rule 2, assign $x_4^* = \check{x}_4 = 0.70$. Delete the row corresponding to the variable x_4 and the column corresponding to the 1^{st} equation. Obtain the updated matrix M.

$$M = \begin{array}{c} \\ x_2 \\ x_3 \end{array} \begin{array}{c} 9 \\ \begin{bmatrix} 3.50 \\ 0.84 \end{bmatrix} \end{array}$$

In the updated matrix M, $\bar{J}_2 = \{9\} \subseteq \bar{J}_3 = \{9\}$ and also $c_3\check{x}_3 = 0.84 < c_2\check{x}_2 = 3.50$. Thus, using Rule 4, assign $x_2^* = 0$ and delete the row corresponding to the variable x_2 to obtain the updated matrix M.

$$M = \begin{array}{c} \\ x_3 \end{array} \begin{array}{c} 9 \\ \begin{bmatrix} 0.84 \end{bmatrix} \end{array}$$

In the updated matrix M, $\bar{I}_9 = \{3\}$ is a singleton set. Thus, using Rule 2, assign $x_3^* = \check{x}_3 = 0.60$. Since all the components of the optimal solution are determined, thus the optimal solution is

$x^* = \begin{bmatrix} 0.80 & 0 & 0.60 & 0.70 & 0.60 & 0 & 0 & 0.90 \end{bmatrix}$

and the corresponding optimal value of the objective function is $Z^* = 1.89$.

6 Conclusion

In this study, a linear optimization problem subject to interval-valued FRE with max-Archimedean composition is considered. The applications of tolerable solutions of interval-valued FRE can be found in fuzzy control and diagnosis problems. Therefore optimization with such type of FRE are useful. Basic properties related to interval-valued FRE are studied. Some necessary conditions are proved for the considered type of optimization problem, that deal with the value of each component in the optimal solution. Using these conditions, some rules are proposed that reduce the problem size and compute the components of the optimal solution efficiently.

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