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On Conjugate Graphs of Dihedral Group

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Abstract: ***The conjugate graph $\Gamma_C^G(G)$ of a non-abelian group G is a graph whose vertices are the elements of G and two vertices are adjacent if they belong to the same conjugacy class. In this paper, we study the line graphs and complement of conjugate graphs of D_n and explore graph properties like chromatic number, independence number, diameter, planarity, vertex and edge connectivity. Further, results related to the spectrum and energy of line graphs have also been discussed.***

Keywords: Conjugate graph, Dihedral Group, Line Graph, Energy

Mathematics Subject Classification (2020): 05C25, 05C50

1. Introduction

The structural properties of a group can be studied by associating it to a graph and is an interesting field of study in Algebraic Graph Theory where algebraic methods are applied to solve problem about graphs and vice-versa. Concept of associating a graph to a group or a ring have resulted in many fascinating results about their structures and have attracted several authors and research scholars to work on them and bring out their own observations and results. Initially, commutative groups were mostly studied as they are much simpler as compared to non-commutative ones. Since non-commutative groups constitute a major part of finite groups, they eventually gained importance and are now extensively studied.

The study of conjugacy classes of a non-abelian group is fundamental in determining the structure of the group. An element $a \in G$ of a group G is said to be conjugate to an element $b \in G$ if $\exists$ an element $c \in G$ such that $b = c^{-1}ac$. The set of all elements conjugate to a is called the conjugacy class of a denoted by $Cl(a)$. The conjugacy is an equivalence relation and partitions G into disjoint equivalence classes. In 2012, Erfanian and Tolue [1] introduced the conjugate graph $\Gamma_C^G(G)$of a non-abelian group G where the vertices are the elements of G and two vertices are adjacent if they belong to the same conjugacy class. Clearly, conjugate graphs are disconnected and union of complete graphs. Sarmin [7] obtained some graph properties of conjugate graphs of metacyclic 2-groups. Zamri and Sarmin [9] found out the conjugate graphs of metacyclic 3-groups and metacyclic 5-group. In 2017, Zulkarnain [2] discussed the conjugate graphs of finite p-groups.

In this paper, we study the structure of conjugate graphs and their complement graphs for dihedral group and explore some graph properties. The spectra and energy of these graphs and their line graphs have also been studied. The spectra of a graph is a representation of various structural properties of the graph. The concept of energy of a graph was first introduced by Gutman [6].

The energy $E(G)$ of a graph G is defined as :
Let G be a graph with n vertices and let $\lambda_1, \lambda_2, \dots\dots, \lambda_n$ be the spectrum of its adjacency matrix. Then,

$$E(G) = \sum_{i=1}^{n} |\lambda_i|$$

In 1978, Gutman [6] conjectured that among all graphs with n vertices, the complete graph K_n has the maximum energy. This was however disproved by Walikar et al [5] and was found out that there exists graphs of order n for which the energy is greater than that of complete graph K_n , ie, $E(G) > 2n - 2$.

2. Preliminaries

This sections includes some basic definitions and results which have been referred to for establishing our main results.

Definition 2.1: A graph G is said to be connected if there exists a path between any two distinct vertices in the graph; otherwise G is said to be disconnected.

Definition 2.2: The diameter of a connected graph is the maximum distance between any two vertices in the graph.

Definition 2.3: The Chromatic number of a graph G, denoted by $\chi(G)$, is the smallest number of colors required to color the vertices of G so that no two adjacent vertices share the same color.

Definition 2.4: A set of vertices that induces an empty subgraph is called an independent set, say X and the size of the largest independent set X is called the independence number, denoted by $\omega(X)$.

Definition 2.5: The edge connectivity of a graph G is defined as the minimum number of edges whose removal leaves G disconnected. The vertex connectivity of a graph G is defined as the minimum number of vertices whose removal from G leaves the remaining graph disconnected.

Definition 2.6: A graph G is said to be $k-$ regular if the degree of each vertex in G is k.

Definition 2.7: The Line graph $L(G)$ of a graph G is a graph whose vertices are the edges of G, with two vertices of $L(G)$ adjacent whenever the corresponding edges of G are adjacent.

Result 2.1: A graph is planar if and only if it contains no subgraph isomorphic to $K_{3,3}$ or K_5 .

Result 2.2: If a graph G with n vertices is k −regular, then its eigen values $d_1, d_2, \ldots, d_n$ are such that $d_1 = k$ and $|d_i| < k$ for $i = 2, \ldots, n$.

3. Main Results

This section is divided into four subsections. In the first section, we determine the structure of conjugate graph of D_n. In the second and third section, we discuss the structure, spectrum and energy of line graphs of $\Gamma_C^G(D_n)$. Lastly, graph properties of the complement graph of $\Gamma_C^G(D_n)$ have been investigated.

3.1 Conjugate graph of D_n

The dihedral group $D_n (n > 2)$ of order $2n$ is the group of symmetries of a regular n-sided polygon with the group presentation

$$D_n =< a, b : a^n = b^2 = e, bab = a^{-1} >$$

The structure of conjugate graphs are solely dependent on the structure of conjugacy classes for any group. So given the conjugacy class structure, we can easily find the conjugate graph of a group. We state an important lemma which characterises the structure of conjugacy classes of dihedral groups.

Lemma 3.1.1 : The number of conjugacy classes for $D_n(n\ odd)$ is $\frac{n+3}{2}$ and the conjugacy classes are given by

$$\{e = b^n\}, \{a, ab, ab^2, ab^3, \ldots, ab^{n-2}, ab^{n-1}\}, \{b, b^{n-1}\}, \{b^2, b^{n-2}\}, \ldots \ldots, \{b^{\frac{n-1}{2}}, b^{\frac{n+1}{2}}\}$$

And number of conjugacy classes for $D_n(n\ even)$ is $\frac{n+6}{2}$ and the conjugacy classes are given by

$$\{e = b^n\}, \{a, ab^2, ab^4, \ldots, ab^{n-2}\}, \{ab, ab^3, ab^5, \ldots, ab^{n-1}\}, \{b, b^{n-1}\}, \{b^2, b^{n-2}\}, \ldots \ldots, \{b^{\frac{n}{2}}\}$$

As an immediate consequence of **Lemma 3.1.1**, we state the following theorem.

Theorem 3.1.1 : For odd n,

$$\Gamma_C^G(D_n) = K_1 \cup (\frac{n-1}{2})K_2 \cup K_n$$

and

$$\Gamma_C^G(D_n) = 2K_1 \cup (\frac{n}{2} - 1)K_2 \cup 2K_{\frac{n}{2}}$$

for even values of n.

Proof : Obviously, $\Gamma_C^G(D_n)$ is disconnected. The number of components of $\Gamma_C^G(D_n)$ is the number of conjugacy classes of D_n and each component is complete with number of vertices same as the size of the corresponding conjugacy classes.

3.2. Line Graphs of $\Gamma_C^G(D_n)$: Chromatic and Independence number

The Line graph $L(G)$ of a graph G is a graph whose vertices are the edges of G, with two vertices of $L(G)$ adjacent whenever the corresponding edges of G are adjacent. Thus, line graph represents the adjacencies between the edges of the graph. We state the following results on the structure and graph properties of line graphs of $\Gamma_C^G(D_n)$

Before moving onto our main results, we state an important lemma.

Lemma 3.2.1 : The line graph $L(K_n) = G$ of the complete graph K_n has the following properties :

- G has $\binom{n}{2}$ vertices.
- G is regular of degree $2(n-2)$
- Every two non-adjacent points are mutually adjacent to exactly four points.
- Every two adjacent points are mutually adjacent to exactly $n-2$ points.

Theorem 3.2.1: For odd n, $L(\Gamma_C^G(D_n))$ is disconnected and consists of $\frac{n-1}{2}$ isolated vertices and a $2(n-2)$- regular graph of $\binom{n}{2}$ vertices.

***Proof*:** The conjugate graph $\Gamma_C^G(D_n)$ consists of $\frac{n+3}{2}$ components out of which $\frac{n-1}{2}$ are K_2 graphs, one K_n graph and a isolated vertex. So the line graph $L(\Gamma_C^G(D_n))$ will have $\frac{n-1}{2}$ isolated vertices corresponding to the $\frac{n-1}{2}$ K_2 graphs. The K_n component of the conjugate graph will correspond to a graph with $\binom{n}{2}$ vertices in $L(\Gamma_C^G(D_n))$ and shall be $2(n-2)$- regular.

Theorem 3.2.2: For even n, $L(\Gamma_C^G(D_n))$ is disconnected and consists of $\frac{n}{2} - 1$ isolated vertices and two $2(\frac{n}{2} - 2)$-regular graphs each with $\binom{n/2}{2}$ number of vertices.

***Proof*:** The conjugate graph $\Gamma_C^G(D_n)$ consists of $\frac{n+6}{2}$ components out of which $\frac{n}{2} - 1$ are K_2 graphs, two $K_{\frac{n}{2}}$ graphs and two isolated vertices. So the line graph $L(\Gamma_C^G(D_n))$ will have $\frac{n}{2} - 1$ isolated vertices corresponding to the $\frac{n}{2} - 1$ K_2 graphs. The $K_{\frac{n}{2}}$ components of the conjugate graph will correspond to two disconnected components with $\binom{n/2}{2}$ vertices in $L(\Gamma_C^G(D_n))$ and shall be $2(\frac{n}{2} - 2)$-regular.

Theorem 3.2.3: The Chromatic number χ, of $L(\Gamma_C^G(D_n))$ satisfies

$$3 \le \chi \le 2n - 3; \text{ for odd } n$$

$$3 \le \chi \le 2n-3; \text{ for even } n \text{ except for } D_4$$

***Proof*:** For odd values of n, $L(\Gamma_C^G(D_n))$ is regular of degree $2(n-2)$ excluding the isolated vertices. So maximum vertex degree is $2n-4$. Since, the chromatic number is atmost one greater than maximum degree, so

$$\chi \le 2n-3$$

Also, the line graph corresponding to K_n is such that every two adjacent points are mutually adjacent to exactly $n-2$ points. So the minimum number of colors needed to color the vertices so that none of the adjacent vertices receives the same color is atleast 3, ie,

$$3 \le \chi$$

Hence for odd n, we have,

$$3 \le \chi \le 2n-3$$

Similarly, for even n, the first part of the inequality holds [Lemma 3.2.1]

Since $L(\Gamma_C^G(D_n))$ is regular of degree $2(n-2) = n-4$, (excluding the isolated vertices), so

$$\chi \le 2n-3$$

Therefore,

$$3 \le \chi \le 2n-3$$

for even values of n.

Theorem 3.2.4: The independence number ω of $L(\Gamma_C^G(D_n))$ is given by ,

$$\omega\big(L(\Gamma_C^G(D_n))\big) = n-1, \textit{when } n \textit{ is odd and } \frac{n}{2} \textit{ is even}$$
$$= n-2, \textit{when } \frac{n}{2} \textit{ is odd}$$

Proof : **Case 1** For odd n, $L(\Gamma_C^G(D_n))$ is disconnected and consists of $\frac{n-1}{2}$ isolated vertices and a K_n corresponding line graph.

We know that the independence number of line graph of K_n is $\frac{n-1}{2}$, when n is odd. So the independence number of $L(\Gamma_C^G(D_n))$ is the sum total of the number of isolated vertices and the independence number of $L(K_n)$ which is $n-1$.

Case 2 For even n, $L(\Gamma_C^G(D_n))$ is disconnected and consists of $\frac{n}{2}-1$ isolated vertices and two $K_{n/2}$ corresponding line graph.

We know that the independence number of line graph of K_n is $\frac{n}{2}$ when n is even and $\frac{n-1}{2}$ when n is odd.

Now,

$$\omega\left(L\left(K_{n/2}\right)\right) = \frac{n}{4}, \textit{if } \frac{n}{2} \textit{ is even}$$
$$= \frac{\frac{n}{2}-1}{2}, \textit{if } \frac{n}{2} \textit{ is odd}$$

Therefore,

$$\omega\big(L(\Gamma_C^G(D_n))\big) = 2.\frac{n}{4} + \frac{n}{2} - 1 = n-1, \textit{when } \frac{n}{2} \textit{ is even}$$
$$= 2\left(\frac{\frac{n}{2}-1}{2}\right) + \frac{n}{2} - 1 = n-2, \textit{when } \frac{n}{2} \textit{ is odd}$$

3.3. Spectrum and Energy

The ordinary spectrum is the set of all the eigen values of the adjacency matrix along with their multiplicities. The Adjacency matrix $A(G)$ of a directed graph G is the integer matrix with rows and columns indexed by the vertices of G, such that the uv-entry of $A(G)$ is equal to the number of edges from u to v which is usually 0 or 1.

The Adjacency matrix of $\Gamma_C^G(D_n) = X$ can be put in the form

$$A(X) = \begin{pmatrix} A(G_1) & 0 & 0 & \cdots & 0 \\ 0 & A(G_2) & 0 & & 0 \\ & \vdots & & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & A(G_k) \end{pmatrix}$$

where k is the number of components and $G_1, G_2, \ldots, G_k$ are the components of $\Gamma_C^G(D_n)$.

$A(G_i)$ is the adjacency matrix corresponding to the component G_i. The eigen values of $A(X)$ are precisely the union of all eigen values of $A(G_i)$, for all i.

From the structure of $\Gamma_C^G(D_n)$, the spectrum can be easily computed as

$$\{0, 1^{\left(\frac{n-1}{2}\right)}, -1^{\left(\frac{3n-3}{2}\right)}, n-1\}$$

and

$$\{0^{(2)}, 1^{\left(\frac{n}{2}-1\right)}, -1^{\left(\frac{3n-6}{2}\right)}, \frac{n}{2}-1^{(2)}\}$$

for odd and even values of n respectively.

The following results follow consequently.

Theorem 3.3.1 : $\Gamma_C^G(D_n)$ is non-hyperenergetic.

Proof: From the spectrum, $E\big(\Gamma_C^G(D_n)\big) = 3(n-1)$, for odd n and,

$E\big(\Gamma_C^G(D_n)\big) = 3(n-2)$, for even n.

Clearly, $3(n-1) \ngtr 2(2n-1), \forall\, odd\ n \ge 3$

and $3(n-2) \ngtr 2(2n-1), \forall\, even\ n \ge 4$.

Theorem 3.3.2 : $\Gamma_C^G(D_n)$ is non-hypoenergetic except for $n = 4$.

Proof : Clearly, $3(n-1) \nless 2n, \forall\, odd\ n \ge 3$

and $3(n-2) \nless 2n, \forall\, even\ n > 4$.

Theorem 3.3.3 :For odd n, $\Gamma_C^G(D_n)$ and $\Gamma_C^G(D_{n+1})$ have the same energy.

***Proof* :** $E\big(\Gamma_C^G(D_n)\big) = 3(n-1)$
$= 3\{(n+1)-2\}$
$= E\big(\Gamma_C^G(D_{n+1})\big)$

Theorem 3.3.4 : $L(\Gamma_C^G(D_n))$ is non-hyperenergetic for $n = 3,4,6$.

Proof : From the structure of $\Gamma_C^G(D_n)$, it is clear that for odd n, $L(\Gamma_C^G(D_n))$ is disconnected and consists of $\frac{n-1}{2}$ isolated vertices and a $2(n-2)$-regular graph of $\binom{n}{2}$ vertices.

Since, $L(\Gamma_C^G(D_n))$ is $2(n-2)$-regular, so its eigen values $d_1 \geq d_2 \geq \cdots \geq d_k$ are such that $d_1 = 2(n-2)$ and $|d_i| < 2(n-2)$ for $i = 2, \ldots, k$.

Thus,

Maximum energy of $L(\Gamma_C^G(D_n)) = \frac{n(n-1)}{2}(2(n-2))$

$$= n(n-1)(n-2) \ngtr 2\left(\frac{n(n-1)}{2} + \frac{n-1}{2} - 1\right), for\ n = 3$$

For even n, $L(\Gamma_C^G(D_n))$ is disconnected and consists of $\frac{n}{2} - 1$ isolated vertices and two $2(\frac{n}{2} - 2)$- regular graphs each with $\binom{n/2}{2}$ number of vertices. Since, $L(\Gamma_C^G(D_n))$ consists of two $2(\frac{n}{2} - 2)$-regular graphs, so their eigen values eigen values $d_1 \geq d_2 \geq \cdots \geq d_k$ are such that $d_1 = 2(\frac{n}{2} - 2)$ and $|d_i| < 2(\frac{n}{2} - 2)$ for $i = 2, \ldots, k$.
Thus,

Maximum energy of $L(\Gamma_C^G(D_n)) = 2[\frac{n}{4}(\frac{n}{2} - 1)2(\frac{n}{2} - 2)]$

$$= n\left(\frac{n}{2} - 1\right)\left(\frac{n}{2} - 2\right)$$

$$\ngtr 2\left[2.\frac{n}{4}\left(\frac{n}{2} - 1\right) + \left(\frac{n}{2} - 1\right) - 1\right], for\ n = 4,6.$$

Hence, $L(\Gamma_C^G(D_n))$ is non-hyperenergetic for $n = 3,4,6$.

Theorem 3.3.5 : $L(\Gamma_C^G(D_4))$ is hypoenergetic.
Proof : From the above theorem,

Maximum energy of $L(\Gamma_C^G(D_n) = n(n-1)(n-2, for\ odd\ n$

$$= n\left(\frac{n}{2} - 1\right)\left(\frac{n}{2} - 2\right), for\ even\ n$$

The number of vertices of $L(\Gamma_C^G(D_n))$ for odd and even values of n being $\frac{(n+1)(n-1)}{2}$ and $(\frac{n}{2} - 1)(\frac{n}{2} + 1)$, it is clearly visible that $E(L(\Gamma_C^G(D_n))) \nless$ number of vertices of $L(\Gamma_C^G(D_n))$, except for $n = 4$.

3.4. The complement of $\Gamma_C^G(D_n)$

In this section, we have obtained and established some of the graph properties for the complement of conjugate graphs of dihedral groups. We will be representing the complement of $\Gamma_C^G(D_n)$ by $\overline{\Gamma_C^G(D_n)}$.

Theorem 3.4.1 The diameter of $\overline{\Gamma_C^G(D_n)}$ is 2.
Proof : By definition, diameter is the maximum distance between any two vertices in a graph, where distance is the shortest path between any two vertices. Since the identity element e is non-adjacent to all other vertices in the conjugate graph, so it is adjacent to every other vertex in the complement graph $\overline{\Gamma_C^G(D_n)}$. Thus, every pair of non-adjacent vertices of $\Gamma_C^G(D_n)$ meet at a common vertex e in $\overline{\Gamma_C^G(D_n)}$. Therefore the shortest path between each pair of vertices is 2 and hence diameter is 2.

Theorem 3.4.2: Chromatic number of $\overline{\Gamma_C^G(D_n)}$ is equal to the number of components of $\Gamma_C^G(D_n)$.

Proof: Let the number of components of $\Gamma_C^G(D_n)$ be k and the components be $G_1, G_2, \ldots, G_k$. Each component forms a point (or vertex) as a set of non-adjacent vertices in $\overline{\Gamma_C^G(D_n)}$. Each disconnected component in $\Gamma_C^G(D_n)$ form a complete graph of k vertices in $\overline{\Gamma_C^G(D_n)}$. Hence, k is the minimum number of colors required to color the vertices of $\overline{\Gamma_C^G(D_n)}$ and hence chromatic number = k = number of components of $\Gamma_C^G(D_n)$.

Theorem 3.4.3 : The independence number of $\overline{\Gamma_C^G(D_n)}$ is equal to n for odd n and $\frac{n}{2}$ for even n.

Proof : Clearly, the point (vertex) with the highest number of non-adjacent vertices in $\overline{\Gamma_C^G(D_n)}$ gives the independence number of $\overline{\Gamma_C^G(D_n)}$.

For odd n, the conjugate graph is a union of K_1, K_2 and K_n graphs. The n vertices of K_n in $\Gamma_C^G(D_n)$ forms the vertex with the highest number n of non-adjacent vertices in $\overline{\Gamma_C^G(D_n)}$. Hence, independence number for odd n is equal to n.

For even n, the $\Gamma_C^G(D_n)$ is a union of K_1, K_2 and $K_{\frac{n}{2}}$ graphs. The $\frac{n}{2}$ vertices in $K_{\frac{n}{2}}$ of the conjugate graph forms the point with maximum number of non-adjacent vertices in $\overline{\Gamma_C^G(D_n)}$. Hence, independence number is $\frac{n}{2}$.

Theorem 3.4.4: $\overline{\Gamma_C^G(D_n)}$ is non-planar for all n except for $n = 3,5$.
Proof : The number of components in $\Gamma_C^G(D_n)$ is 5 except for $n = 3,5$, so the least number of vertices in $\overline{\Gamma_C^G(D_n)}$ is 5. Since all the components are disconnected in the conjugate graph, they form a complete graph K_5 in $\overline{\Gamma_C^G(D_n)}$. Hence, $\overline{\Gamma_C^G(D_n)}$ is non-planar for all n except for $n = 3,5$.

Theorem 3.4.5: The edge connectivity of $\overline{\Gamma_C^G(D_n)}$ is n for odd n and $\frac{3n}{2}$ for all even values of n.
Proof: Since diameter of $\overline{\Gamma_C^G(D_n)}$ is 2, so each pair of vertices is connected to each other by a path of length at most 2. The only way to disconnect the graph into two components is to eliminate all the edges adjacent to the vertex with the smallest degree. Since the smallest degree of a vertex in $\overline{\Gamma_C^G(D_n)}$ is n (for odd n) and $\frac{3n}{2}$ (for even n), hence the result.

Theorem 3.4.6: Vertex connectivity of $\overline{\Gamma_C^G(D_n)}$ is n for odd n and $\frac{3n}{2}$ for all even values of n.
Proof: The vertices of $\overline{\Gamma_C^G(D_n)}$ can be partitioned into two categories - the one with vertices which forms the independent set and the other with remaining vertices.

Removal of any one or all of the former set of vertices has no effect on the connectivity of the graph. The latter set of vertices are such that they are either adjacent to each other or connects to each other by a path of at most length 2 and each vertex is adjacent to all the former set of vertices. The only way to disconnect the graph is to eliminate all the vertices from the latter set of connecting vertices.

Since the number of vertices in the independence set is $n,$ so vertex connectivity is $2n - n = n$ for odd n and independence number for even n being $\frac{n}{2}$ so vertex connectivity is $2n - \frac{n}{2} = \frac{3n}{2}$.

4. Conclusion

In this paper, the structure of line graphs of conjugate graph of Dihedral groups has been found out to be union of regular graphs and some isolated vertices. A bound for the chromatic number and the independence number have been found for the line graphs. The energies of conjugate graphs of Dihedral groups have been investigated and found out to be non-hyperenergetic for all values of n and non-hypoenergetic except for $n = 4$. The line graph of conjugate graphs is non-hyperenergetic for $n = 3,4,6$ and non-hypoenergetic except for $n = 4$. The diameter, chromatic number, independence number, vertex and edge connectivity of the complement graph of $\Gamma_C^G(D_n)$ are found out. Also, complement graph of $\Gamma_C^G(D_n)$is non-planar for all n except for $n = 3,5$.

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