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Composite Graph of a Group

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Abstract: ***Let G be a finite group. The Composite graph $\zeta(G)$ of G is a graph with vertices as the elements of G, and two vertices a and b are adjacent if either $o(a) = o(b)$ or $gcd(o(a), o(b))$ is a composite number. In this paper, we study the structure of $\zeta(G)$ graphs, for some abelian and non-abelian groups. Further, we see that if two groups G_1 and G_2 are such that $G_1 \cong G_2$, then $\zeta(G_1) \cong \zeta(G_2)$.***

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Mathematics Subject Classification (2010): 05C25

1. Introduction

The study of structural properties of a graph by associating it to a group is an interesting field of study in Algebraic Graph Theory. A graph can be associated to a group in many ways. Power graphs [5], Prime graphs [3] are result of such associations. The order of an element in defining a graph have previously been studied extensively by many researchers and authors throughout.

Based on the order of an element, we have introduced a new graph called Composite graph $\zeta(G)$ of a finite group G. We define a graph on G with vertices as the elements of G such that two vertices a and b are adjacent if either $o(a) = o(b)$ or $\gcd(o(a), o(b))$ is a composite number. In this paper, we study the structure of $\zeta(G)$, for some abelian and non-abelian groups. Moreover, we check that if G_1 and G_2 are two groups such that $o(G_1) = o(G_2)$, then is it necessary that $\zeta(G_1) \cong \zeta(G_2)$, and if not, under what circumstances are these graphs isomorphic.

From the definition of $\zeta(G)$ it is obvious that -

1) $\zeta(G)$ is connected
2) If $o(G) = n$, then $\deg(e) = n - 1$, where e is the identity element of G.
3) If $\zeta(G)$ is regular, then it is complete.

Throughout the paper, the *order* of G is the number of vertices of $\zeta(G)$. Also, G is finite and $\zeta(G)$ is undirected and simple.

2. Preliminaries

This section includes some results and definitions that we shall be using in the subsequent sections.

Definition 2.1: (Order of an element)
Let G be a group and $a \in G$ be any element. The *order* of a is the least positive integer n such that $a^n = e$, where $e \in G$ is the identity element .We denote the *order* of a as $|a|$ or $o(a)$. The order of the identity element is always 1.

Definition 2.2: (Cyclic group)
A group G is called a cyclic group if there exists an element $a \in G$ such that every element of G can be expressed as a power of a. In that case a is called the *generator* of G and we express it by writing $G =< a >$.

Definition 2.3: (Conjugate Element and Conjugacy Class)
Let G be a group and $a, b \in G$. We say that a is conjugate to b if there exists an element $c \in G$ such that $a = c^{-1}bc$. We denote it by $a{\sim}b$. The set of all conjugates of a is called the conjugacy class of a in G and we denote it by $Cl(a)$. Also, $G = \bigcup_{a \in G} Cl(a)$

Definition 2.4: (Euler's phi function)
Given any integer n, $\varphi(n)$ is the number of non-negative integers less than n that are relatively prime to n. If p is a prime, then $\varphi(p) = p - 1$.

Definition 2.5: (Isomorphic Graphs)
Two graphs G and H are said to be isomorphic if there exists a bijection between the vertex sets of G and H

$$f: V(G) \to V(H)$$

such that two vertices u and v are adjacent in G if and only if $f(u)$ and $f(v)$ are adjacent in H.

Result 2.6: (Cauchy's Theorem)
If G is a finite group and p is a prime dividing the order of G, then G contains an element of order p.

Result 2.7: The order of an element divides the order of the group.

Result 2.8: A group G of prime order is cyclic and every element of G other than identity can be taken as its generator.

Result 2.9: If G is a cyclic group of order n, there exists $\varphi(m)$ elements of order m for every divisor m of n.

3. Main Results

Theorem 1: If G is a group of order p, where p is a prime, then $\zeta(G)$ is a complete graph.

Proof: Let $|G| = p$.

Then, G is cyclic.

Since, order of a cyclic group is equal to the order of its generator, so order of every element of G is p.

Hence $\zeta(G)$ is a complete graph.

Theorem 2: Let G be a group such that every non-identity element of G has prime order. Then $\zeta(G)$ is a complete graph. Converse is true.

Proof : Clearly, if every element of G has prime order then G is complete.

Conversely, let $\zeta(G)$ be a complete graph.

Let $v_1, v_2, \dots, v_n$ be the vertices of G such that

$$o(v_i) = m_i \, , 1 \leq i \leq n$$

If possible, let m_i be not prime. Then $\exists$ a prime divisor p of m_i.

By Cauchy's Theorem, G has an element of order p.

So the vertices m_i and p are non-adjacent in $\zeta(G)$, which is a contradiction to the fact that $\zeta(G)$ is complete.

Hence, $o(v_i) = m_i$, a prime.

Theorem 3 : Let G be a cyclic group of order n such that distinct divisors m_i of n are prime. Then $\zeta(G)$, contains atmost one $K_{m_1+m_2+\dots+m_k+1-k}$ and one $K_{\varphi(n)+1}$ subgraph.

Proof : Let $m_1, m_2, \dots, m_k$ be all the divisors of n which are prime. There are

$\varphi(m_1)$ elements of order m_1

$\varphi(m_2)$ elements of order m_2
….
$\varphi(m_k)$ elements of order m_k

$\varphi(n)$ elements of order n

Since m_i's are mutually coprime and $\gcd(m_i, n) = m_i$, a prime, so $\zeta(G)$ consists of two complete graphs $K_{\varphi(m_1)+\varphi(m_2)+\dots+\varphi(m_k)+1} = K_{m_1+m_2+\dots+m_k+1-k}$ and $K_{\varphi(n)+1}$ and both connects at the vertex e. Hence the result.

Corollary : If G is a cyclic group of order pq $(p \neq q)$, then $\zeta(G)$ contains atmost one K_{p+q-1} and one $K_{(p-1)(q-1)+1}$ subgraph.

Theorem 4 : Composite graph of the Dihedral group D_p, where p is prime, is K_{2p}

Proof : Clearly, the identity element is adjacent to every element of D_p.

Since $|D_p| = 2p$, there are p elements of order 2 and $\varphi(p) = p - 1$ elements of order p in D_p

Since the orders $1, 2, p$ are mutually coprime to each other, so $\zeta(D_p) = K_{2p}$.

Theorem 5 : Let G be a finite group and $k(G)$ be the number of conjugacy classes of G. If $k(G) = 3$, then $\zeta(G)$ is a complete graph.

Proof : If G is finite and $k(G) = 3$, then G is either a cyclic group of order 3 or is the non-abelian group S_3 of order 6.

If G is cyclic of order 3, then by **Theorem 1**, $\zeta(G)$ is complete.

If $G \cong S_3$, then we know that the order of elements of S_3 are 1, 2 and 3. Since the orders are mutually coprime, so $\zeta(G)$ is complete.

Theorem 6 : Let G be a group which possesses an element of order 2. Then $\zeta(G)$ contains a complete subgraph with odd number of vertices.

Proof : Let $a \in G$ such that $o(a) = 2$.

Since $o(a) | o(G)$, so $o(G)$ must be even.

Also, $o(e) = 1$, so we are left with odd number of elements in G of order ≥ 2.

Let $x \in G$ such that $o(x) = 2$.

For every element $x \in G$, there exists a unique element $x^{-1} \in G$ such that $xx^{-1} = e$.

Also, $o(x) = 2 \Leftrightarrow x = x^{-1}$

Thus, elements of order 2 are their own inverse.

Let C be the set of all elements of order greater than 2.

Since each element of a group has unique inverse, so elements of C must occur in pairs. Thus, order of C must be even.

Therefore, there must exist odd number of elements of order 2 in G. These odd number of elements forms a complete graph.

Theorem 7 : Let G_1 and G_2 be two groups such that $G_1 \cong G_2$. Then $\zeta(G_1) \cong \zeta(G_2)$.

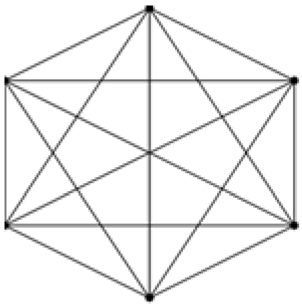
Proof : Let $G_1 \cong G_2$. Then $o(G_1) = o(G_2)$ and under an isomorphism, the order of any element is preserved. So there exists a bijection between the vertex set of G_1 and G_2

$$f : V(G_1) \rightarrow V(G_2)$$

such that two vertices u and v are adjacent in $\zeta(G_1)$ if and only if $f(u)$ and $f(v)$ are adjacent in $\zeta(G_2)$. Hence, $\zeta(G_1) \cong \zeta(G_2)$.

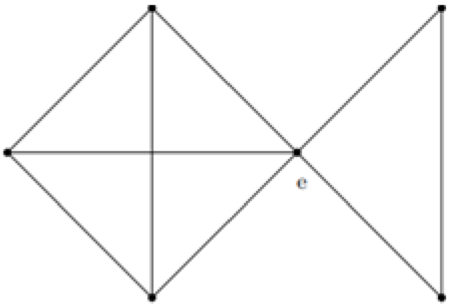
Remark : If G_1 and G_2 are group such that $o(G_1) = o(G_2)$, then it is not necessary that $\zeta(G_1) \cong \zeta(G_2)$.

For example - Consider the dihedral group D_3 and the cyclic group $(Z_6, +)$. Both are of order 6 but $\zeta(D_3)$ is the complete graph K_6 (Fig. 1(a)) while $\zeta(Z_6)$is not complete (Fig. 1(b)).



$\zeta(D_3)$

Fig. 1(a)



$\zeta(Z_6, +)$

Fig. 1(b)

Theorem 8: If G_1 and G_2 are cyclic groups of same order, then $\zeta(G_1) \cong \zeta(G_2)$.

Proof : We know that cyclic groups of same order are isomorphic.

If G_1 and G_2 are cyclic groups of same order, so $G_1 \cong G_2$ which implies $\zeta(G_1) \cong \zeta(G_2)$.

Corollary: If G is a cyclic group of order n, then $\zeta(G) \cong \zeta(Z_n)$, where Z_n is the additive group of residue classes modulo n.

Theorem 9: Composite graph can never be a complete bipartite graph $K_{p,q}$ where $p, q > 1$. Star graphs are the only bipartite cases possible for Composite graph of a group.

Proof : Let G be a group of order n and if possible let $\zeta(G)$ be a complete bipartite graph $K_{p,q}$ where $p, q > 1$.

Let V_1 and V_2 be the partition vertex sets of $\zeta(G)$ such that $|V_1| = p$, $|V_2| = q$ and $p + q = n$ with $p, q > 1$.

Then for every $x \in G$, $o(x) < n - 1$.

But from the definition of Composite graph of a group, $o(e) = n - 1$, a contradiction.

So either $V_1 = \{e\}$ or $V_2 = \{e\}$ so that $\zeta(G)$ is a star graph.

Therefore, $\zeta(G)$ can never be a complete bipartite graph $K_{p,q}$ where $p, q > 1$.

Corollary: Chromatic number of $\zeta(G) \geq 2$ for any group G.

4. Conclusion

In this paper, we defined a new graph called the Composite graph $\zeta(G)$ of a group G whose vertices are the elements of G and two vertices a and b are adjacent if either $o(a) = o(b)$ or $\gcd(o(a), o(b))$ is a composite number. The Composite graphs are complete for cyclic groups of prime order, groups whose non-identity element has prime order and Dihedral group D_p where p is prime. We also found out that if G is a group with order $n = p_1 p_2 \ldots . p_k$, then $\zeta(G)$ consists of complete subgraphs. Also, star graphs are the only possible bipartite cases for Composite graph of a group. Further, if two groups are isomorphic, then their Composite graphs are also isomorphic.

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