Complete Intuitionistic Fuzzy Graphs

S. Vijayalakshmi[1], S. Geetha[2], S. Sahaya Arockia Selvi[3]

Associate Professor, Department of Mathematics, St. Michael College of Engineering and Technology, Kalaiyarkovil, Sivagangai District - 630 551
EMail: [1]*vimalvishali[at]gmail.com*, [2]*geethae836[at]gmail.com, josemistika[at]gmail.com*

Abstract: ***In this paper, we provide three operations on Intuitionistic fuzzy graphs, namely direct product, semi strong product and strong product on Intuitionistic fuzzy graphs. We give sufficient condition for each one of them to be complete and we show that if any of these product is complete, then at least one factor is a complete intuitionistic fuzzy graph. Moreover, we introduced and study the notion and sufficient conditions for the preceding products of two Intuitionistic fuzzy balanced graphs to be balanced and we prove that any isomorphic Intuitionistic fuzzy graph to a balanced Intuitionistic fuzzy graph must be balanced.***

MSC 2010 No.: 47H10, 54H25

1. Introduction

The concept of fuzzy relation which has a widespread application in pattern recognition was introduced by Zadeh [8] in his landmark is Fuzzy sets in 1965. Fuzzy graph and fuzzy analogs of graph theoretic concepts were first introduced by Rosenfeld [6] in 1975. Mordeson and Peng [2] defined the concept of complete of fuzzy graph and studied some operations on Intuitionistic fuzzy graphs. In [7] the definition of complement of an Intuitionistic fuzzy graph was modified so that the complement of the complement is the original Intuitionistic fuzzy graph.

Moreover some properties of self-complementary Intuitionistic fuzzy graphs and the complement of the operations of union join and composition on Intuitionistic fuzzy graphs that were introduced in [2] we studied. Formally, a fuzzy subset of a non - empty set V is a mapping $\mu: V \rightarrow [0,1]$ and a fuzzy relation γ on fuzzy subset μ is a fuzzy subset $V \times V$. All throughout this paper, we assume that μ is reflexive, γ is symmetric and V is finite.

2. Preliminaries

Definition 2.1

An Intuitionistic fuzzy graph is of the form G= (V,E) where

(i) $V = \{v_1, v_2, \dots, v_n\}$ such that $\mu_1 : V \rightarrow [0,1]$ and $\gamma_1: v \rightarrow [0,1]$ denote the degree of membership and non - membership of the element $v_i \in V$ respectively and $0 \leq \mu_1(v_1) + \gamma_1(v_1) \leq 1, \forall v_i \in V\ (i = 1,2, \dots, n)$

(ii) $E \leq V \times V$ where $\mu_2: V \times V \rightarrow [0,1]$ and $\gamma_2: V \times V \rightarrow [0,1]$ and $\gamma_2: V \times V \rightarrow [0,1]$are such that

$$\mu_2(v_1, v_2) \leq \min[\mu_1(v_1) + \mu_2(v_2)]\ \gamma_2(v_1, v_2) \leq \max[\gamma_1(v_1) + \gamma_2(v_2)]$$

and$0 \leq \mu_2(v_1, v_2) + \gamma(v_1, v_2) \leq 1 \forall\ (v_1, v_2) \in E(1,2, \dots, n)$

Definition 2.2

An Intuitionistic fuzzy graph $G(V,E)$ is completeif $\mu_2(v_1, v_2) = \mu_1(v_1) \wedge \mu_1(v_2)$ and $\gamma_2(v_1, v_2) = \gamma_1(v_1) \wedge \gamma_1(v_2)$ for all $v_1, v_2 \in V$

Definition 2.3

Let $G: (V, E)$ be a self complementary Intuitionistic fuzzy graph, then $\sum_{v_1, v_2 \in V} \mu_2(v_1, v_2) = \frac{1}{2} \sum_{v_1, v_2 \in V} \mu_1(v_1) \wedge$ *μ2(v2)* and *v1,v2∈ Vγ2(v1,v2)=12v1,v2∈ Vγ1v1∧γ2(v2)*

Definition 2.4

Two Intuitionistic fuzzy graph $G_1: (\mu_1, \mu_2)$ and (γ_1, γ_2) with crisp graph $G_1^*: V_1, E_1$ and $G_2: (\mu_1^*, \mu_2^*)$ and (γ_1^*, γ_2^*) are isomorphic if there exists a bijeciton $h: V_1 \rightarrow V_2$ such that $\mu_1(u) = \mu_2(h(u)),\ \ \gamma_1(u) = \gamma_2(h(u))$ and $\mu_2(u, v) = \mu_2^*(h(u), h(v)), \gamma_2 = \gamma_2^*(h(u), h(v)) \forall\, u, v \in V$.

In this paper, we provide three new operations on Intuitionistic fuzzy graphs, namely direct product, semi - strong product and strong product, we give sufficient conditions for each one of them to be complete and if any one of these product of two Intuitionistic fuzzy graphs is complete, then at least one of the two intuitionistic fuzzy graphs must be complete. Moreover, we introduce and study the notion of balanced Intuitionistic fuzzy graph and show that this notion is weaker than complete and we give necessary and sufficient conditions for the direct product. Semi - strong product and strong product of two balanced intuitionistic fuzzy graphs to be balanced. Finally we prove that given a balanced Intuitionistic fuzzy graph to G must be balanced.

3. Complete Intuitionistic fuzzy graphs

Definition 3.1

The direct product of two Intuitionistic fuzzy graph $G_1: (\mu_1, \gamma_1)$ with crisp graph $G_1^*: (V_1, E_1)$ and $G_2: (\mu_2, \gamma_2)$ with crisp graph $G_2^*: (V_2, E_2)$ where we assume that $V_1 \cap V_2 = \emptyset$ is defined to be the Intuitionistic fuzzy graph $G_1 \Pi G_2: [(\mu_1 \Pi \backslash mu_1^*), (\gamma_1 \Pi \gamma_1^*), (\mu_2 \Pi \mu_2^*), (\gamma_2 \Pi \gamma_2^*)]$ with crisp graph $G^*: (V_1 \times V_2, E)$ where, $E = \{(u_1, v_1)(u_2, v_2): (u_1, u_2) \in E_1, (v_1, v_2) \in E_2\}$

$(\mu_1 \Pi \mu_1^*)(u, v) = \mu_1(u) \wedge \mu_1^*(v)\ \forall\ (u, v) \in V_1 \times V_2$

$(\gamma_1 \Pi \gamma_1^*)(u, v) = \gamma_1(u) \wedge \gamma_1^{*(v)} \forall\ (u, v) \in V_1 \times V_2$and

$(\mu_2 \Pi \mu_2^*)((u_1, v_1), (u_2, v_2)) = \mu_2(u_1, u_2) \wedge \mu_2^*(v_1, v_2)$

$(\gamma_2 \Pi \gamma_2^*)((u_1, v_1)(u_2, v_2)) = \gamma_2(u_1, u_2) \wedge \gamma_2^*(v_1, v_2)$

Definition 3.2

The semi - strong product of two Intuitionistic fuzzy graph G_1: $((\mu_1, \mu_2), (\gamma_1, \gamma_2))$ with crisp graph $G_1^*: (V_1, E_1)$ and $G_2: ((\mu_1^*, \mu_2^*)(\gamma_1^*, \gamma_2^*))$ with crisp graph $G_2^*: (V_2, E_2)$ where we assume that $V_1 \cap V_2 = \emptyset$ is defined to be the Intuitionistic fuzzy graph $G_1 \cdot G_2$: $[(\mu_1 \cdot \mu_1^*), (\gamma_1 \cdot$ *γ1∗ ,μ2· μ2∗,γ2· γ2∗* with crisp graph *G∗: (V1× V2, E)* where, $E = \{(u, v_1)(u, v_2): u \in V_1, (v_1, v_2) \in E_2\} \cup \{(u_1, v_1)(u_2, v_2): (u_1, u_2) \in E_1, (v_1, v_2) \in E_2\}$

$(\mu_1 \cdot \mu_1^*)(u, v) = \mu_1(u) \wedge \mu_1^*(v) \ \forall \ (u, v) \in V_1 \times V_2$

$(\gamma_1 \cdot \gamma_1^*)(u, v) = \gamma_1(u) \wedge \gamma_1^*(v) \ \forall \ (u, v) \in V_1 \times V_2$

$(\mu_2 \cdot \mu_2^*)((u, v_1), (u, v_2)) = \mu_1(u) \wedge \mu_2^*(v_1, v_2)$

$(\gamma_2 \cdot \gamma_2^*)((u, v_1), (u, v_2)) = \gamma_1(u) \wedge \gamma_2^*(v_1, v_2)$ and

$(\mu_2 \cdot \mu_2^*)((u_1, v_1)(u_2, v_2)) = \mu_2(u_1, u_2) \wedge \mu_2^*(v_1, v_2)$

$(\gamma_2 \cdot \gamma_2^*)((u_1, v_1)(u_2, v_2)) = \gamma_2(u_1, u_2) \wedge \gamma_2^*(v_1, v_2)$

Definition 3.3

The strong product of two Intuitionistic fuzzy graph $G_1: ((\mu_1, \mu_2), (\gamma_1, \gamma_2))$ with crisp graph $G_1^*: (V_1, E_1)$ and G_2☹$(\mu_1^*, \mu_2^*)(\gamma_1^*, \gamma_2^*)$ with crisp graph $G_2^*: (V_2, E_2)$ where we assume that $V_1 \cap V_2 = \emptyset$ is defined to be the Intuitionistic fuzzy graph $G_1 \otimes G_2: [((\mu_1 \otimes \mu_1^*), (\gamma_1 \otimes \gamma_1^*)], [(\mu_2 \times$ *μ2, ∗,γ2⊗ γ2∗* with crisp graph *G∗: V1× V2, E* where

$E =$

$\{(u, v_1)(u, v_2): u \in V_1, (v_1, v_2) \in E_2\} \cup$

$\{(u_1, w)(u_2, w): w \in V_2, (u_1, u_2) \in E_1\} \cup$

$\{(u_1, v_1)(u_2, v_2): (u_1, u_2) \in E_1, (v_1, v_2) \in E_2\}$

$(\mu_1 \otimes \mu_1^*)(u, v) = \mu_1(u) \wedge \mu_1^*(v) \ \forall \ (u, v) \in V_1 \times V_2$

$(\gamma_1 \otimes \gamma_1^*)(u, v) = \gamma_1(u) \wedge \gamma_1^*(v) \ \forall \ (u, v) \in V_1 \times V_2$

$(\mu_2 \otimes \mu_2^*)((u, v_1), (u, v_2)) = \mu_1(u) \wedge \mu_2^*(v_1, v_2)$

$(\mu_1^* \otimes \mu_2^*)((u_1, w), (u_2, w)) = \mu_1^*(w) \wedge \mu_2(u_1, u_2)$

$(\gamma_2 \otimes \gamma_2^*)((u, v_1), (u, v_2)) = \gamma_1(u) \wedge \gamma_2^*(v_1, v_2)$

$(\gamma_1^* \otimes \gamma_2^*)((u, w), (u, w)) = \gamma_1^*(w) \wedge \gamma_2(v_1, v_2)$ and

$(\mu_2 \otimes \mu_2^*)((u_1, v_1)(u_2, v_2)) = \mu_2(u_1, u_2) \wedge \mu_2^*(v_1, v_2)$

$(\gamma_2 \otimes \gamma_2^*)((u_1, v_1)(u_2, v_2)) = \gamma_2(u_1, u_2) \wedge \gamma_2^*(v_1, v_2)$

Note: The direct product, the semi - strong product and the strong product of two complete Intuitionistic fuzzy graphs are again Intuitionistic fuzzy complete graphs.

Theorem 3.1

If $G: ((\mu_1, \mu_2)(\gamma_1, \gamma_2))$nd $G_2: (\mu_1^*)(\gamma_1^*, \gamma_2^*)$are complete Intuitionistic fuzzy graphs then $G_1 \Pi G_2$ is complete.

Proof:

If $(u_1, v_1)(u_2, v_2) \in E$. Then since G_1 and G_2 are complete.

$(\mu_1 \Pi \mu_2^*)((u_1, v_1)(u_2, v_2)) = \mu_2(u_1, u_2) \wedge \mu_2^*(v\_1, v_2)$

$= \mu_1(u_1) \wedge \mu_1(u_2) \wedge \mu_1^*(v_1) \wedge \mu_1^*(v_2)$

$= (\mu_1 \Pi \mu_1^*)((u_1, v_1)) \wedge (\mu_1 \Pi \mu_1^*)(u_2, v_2)$

$(\gamma_1 \Pi \gamma_2^*)((u_1, v_1)(u_2, v_2)) = \gamma_2(u_1, u_2) \wedge \gamma_2^*(v_1, v_2)$

$= \gamma_1(u_1) \wedge \gamma_1(u_2) \wedge \gamma_1^*(v_1) \wedge \gamma_1^*(v_2)$

$= (\gamma_1 \Pi \gamma_1^*)((u_1, v_1)) \wedge (\gamma_1 \Pi \gamma_1^*)(u_2, v_2)$

Hence $G_1 \Pi G_2$ is complete.

Theorem 3.2

If $G_1: ((\mu_1, \gamma_1)(\mu_1^*, \gamma_1^*))$and $G_2: ((\mu_2, \gamma_2), (\mu_2^*, \gamma_2^*))$ are complete Intuitionistic fuzzy graphs then $G_1 \cdot G_2$ is complete.

Proof:

If $(u, v_1)(u, v_2) \in E$. Then

$(\mu_2 \Pi \mu_2^*)((u, v_1)(u, v_2)) = \mu_1(u) \wedge \mu_2^*(v_1, v_2)$

$= \mu_1(u_1) \wedge \mu_1^*(v_1) \wedge \mu_1^*(v_2) \because G_2$is complete

$= (\mu_1 \cdot \mu_1^*)((u, v_1)) \wedge (\mu_1 \cdot \mu_1^*)(u_2, v_2)$

$(\gamma_2 \cdot \gamma_2^*)((u, v_1)(u, v_2)) = \gamma_1(u) \wedge \gamma_2^*(v_1, v_2)$

$= \gamma_1(u_1) \wedge \gamma_1^*(v_2) \wedge \gamma_1^*(v_2) \because G_2$is complete

$= (\gamma_1 \cdot \gamma_1^*)((u, v_1)) \wedge (\gamma_1 \cdot \gamma_1^*)(u_2, v_2)$

If $(u_1, v_1)(u_2, v_2) \in E$ then since G_1 and G_2 are complete.

$(\mu_2 \Pi \mu_2^*)((u_1, v_1)(u_2, v_2)) = \mu_2(u_1, u_2) \wedge \mu_2^*(v_1, v_2)$

$= \mu_1(u_1) \wedge \mu_1(u_1) \wedge \mu_1^*(v_1) \wedge \mu_1^*(v_2)$

$= ((\mu_1 \cdot \mu_1^*)(u, v_1)) \wedge (\mu_1 \cdot \mu_1^*)(u_2, v_2)$

$(\gamma_2 \cdot \gamma_2^*)((u_1, v_1)(u_2, v_2)) = \gamma_2(u_1, u_2) \wedge \gamma_2^*(v_1, v_2)$

$= \gamma_1(u_1) \wedge \gamma_1(u_2) \wedge \gamma_1^*(v_1) \wedge \gamma_1^*(v_2)$

$= ((\gamma_1 \cdot \gamma_1^*)(u_1, v_1)) \wedge (\gamma_1 \cdot \gamma_1^*)(u_2, v_2)$

Hence $G_1 \cdot G_2$ is complete.

Theorem 3.3

If $G_1: ((\mu_1, \mu_1^*)(\gamma_1, \gamma_1^*))$ and $G_2: ((\mu_2, \mu_2^*), (\gamma_1, \gamma_2^*))$ are complete Intuitionistic fuzzy graphs then $G_1 \otimes G_2$ is complete.

Proof:

If $(u, v_1)(u, v_2) \in E$. Then

$(\mu_2 \Pi \mu_2^*)((u, v_1)(u, v_2)) = \mu_1(u) \wedge \mu_2^*(v_1, v_2)$

$= \mu_1(u_1) \wedge \mu_1^*(v_1) \wedge \mu_1^*(v_2) \because G_2$is complete

$= (\mu_1 \otimes \mu_1^*)(u, v_1) \wedge ((\mu_1 \otimes \mu_1^*))(u, v_2)$

$(\gamma_2 \otimes \gamma_2^*)((u, v_1)(u, v_2)) = \gamma_1(u) \wedge \gamma_2^*(v_1, v_2)$

$= \gamma_1(u_1) \wedge \gamma_1^*(v_1) \wedge \gamma_1^*(v_2) \because G_2$is complete

$= (\gamma_1 \otimes \gamma_1^*)((u, v_1)) \wedge (\gamma_1 \otimes \gamma_1^*)(u, v_2)$

If $(u_1, w)(u_2, w) \in E$ then

$(\mu_2 \Pi \mu_2^*)((u_1, w)(u_2, w))\& = \mu_1 \wedge \mu_2(u_1, u_2)$

$= \mu_1^*(w) \wedge \mu_1(u_1) \wedge \mu_1(u_2)$ ($\because$ $G$is complete)

$= ((\mu_1 \otimes \mu_1^*)(u, w)) \wedge (\mu_1 \otimes \mu_1^*)(u_2, w)$

$(\gamma_2 \otimes \gamma_2^*)((u_1, w)(u_2, w)) = \gamma_1(w) \wedge \gamma_2^*(u_1, u_2)$

$= \gamma_1^*(w) \wedge \gamma_1(u_1) \wedge \gamma_1(u_2)$ ($\because$ $G$is complete)

$= ((\gamma_1 \otimes \gamma_1^*)(u_1, w)) \wedge (\gamma_1 \otimes \gamma_1^*)(u_2, w)$

If $(u_1, v_1)(u_2, v_2) \in E$ then since G_1 and G_2 are complete.

$(\mu_2 \Pi \mu_2^*)((u_1, v_1)(u_2, v_2)) = \mu_2(u_1, u_2) \wedge \mu_2^*(v_1, v_2)$

$= \mu_1(u) \wedge \mu_1(u_2) \wedge \mu_1^*(v_1) \wedge \mu_1^*(v_2)$

$= ((\mu_1 \otimes \mu_1^*)(u, v_1)) \wedge (\mu_1 \otimes \mu_1^*)(u_2, v_2)$

$(\gamma_2 \otimes \gamma_2^*)((u_1, v_1)(u_2, v_2)) = \gamma_1(u_1, u_2) \wedge \gamma_2^*(v_1, v_2)$

$= \gamma_1(u) \wedge \gamma_1(u_2) \wedge \gamma_1(v_1) \wedge \gamma_1^*(v_2)$

$= ((\gamma_1 \otimes \gamma_1^*)(u_1, v_1)) \wedge (\gamma_1 \otimes \gamma_1^*)(u_2, v_2)$

Hence G_1\otimes G_2 is complete.

Theorem 3.4

If $G_1: ((\mu_1, \mu_1^*)(\gamma_1, \gamma_1^*))$ and $G_2: ((\mu_2, \mu_2^*)(\gamma_2, \gamma_2^*))$ are complete Intuitionistic fuzzy graphs, then $\overline{G_1 \otimes G_2} \approx G_1 \otimes G_2 \approx \overline{G_1} \otimes \overline{G_2}$.

Proof:

Let $G: ((\mu_1, \overline{\mu_2})(\gamma_1, \overline{\gamma_2})) = \overline{G_1} \otimes \overline{G_2}\}, \bar{\mu} = \overline{\mu_2 \otimes \mu_2^*} \bar{\gamma} = \overline{\gamma_2 \otimes \gamma_2^*}$

$\overline{G^*} = (V, \bar{E}),$

$\overline{G_1}: ((\mu_1, \overline{\mu_2})(\gamma_1, \overline{\gamma_2}))$

$\overline{G_1^*} = (V_1, \overline{E_1})$

$G_2 = ((\mu_1^*, \overline{\mu_2^*}), (\gamma_1^*, \overline{\gamma_2^*}))$

$\overline{G_2^*} = (V_2, \overline{E_2})$ and $\overline{G_1} \otimes \overline{G_2}: (\mu_1 \otimes \mu_1^*, \gamma_1 \otimes \gamma_1^*), (\overline{\mu_2} \otimes \overline{\mu_2^*}, \overline{\gamma_2} \otimes \overline{\gamma_2^*})$

We only need to show that $\overline{\mu_2 \otimes \mu_2^*} = \overline{\mu_2} \otimes \overline{\mu_2^*}$

For any arc e joining nodes of V. We have the following cases:

Case: 1

$e = (u, v_1)(u, v_2)$ where $(v, v_2) \in E_2$. Then as G is complete by the above theorem.

$\overline{\mu_1^*}(e) = 0$

$(\overline{\mu_2} \otimes \overline{\mu_2^*})(e) = 0 \quad [(v_1, v_2) \in \overline{E_2}]$

Case: 2

$e = (u, v_1)(u_2, v_2)$ where $(v_1, v_2) \in E_2$ and $v_1 \neq v_2$. Since $e \in E$ and $\mu^*(e) = 0$ and

$\overline{\mu^*}(e) = \mu_1(u, v_1) \wedge \mu_1(u, v_2)$

$= \mu_1(u_1) \wedge \mu_1^*(v_1) \wedge \mu_1^*(v_1)$

$\overline{\gamma^*}(e) = \gamma_1(u, v_1) \wedge \gamma_1(u, v_2)$

$= \gamma_1(u_1) \wedge \gamma_1^*(v_1) \wedge \gamma_1^*(v_2)$ and as $(v_1, v_2) \in \overline{E_2}$

$(\overline{\mu_2} \otimes \overline{\mu_1^*})(e) = \mu_1(u) \wedge \mu_2^*(v_1, v_2)$

$(\overline{\gamma_2} \otimes \overline{\gamma_2^*})(e) = \gamma_1(u) \wedge \gamma_2^*(v_1, v_2)$ and as $\overline{G_2}$ is complete

$(\overline{\mu_2} \otimes \overline{\mu_2^*})(e) = \mu_1(u_1) \wedge \mu_1^*(v_1) \wedge \mu_1^*(v_2)$

$(\overline{\gamma_2} \otimes \overline{\gamma_2^*})(e) = \gamma_1(u_1) \wedge \gamma_1^*(v_1) \wedge \gamma_1^*(v_2)$

Case: 3

$e = (u_1, w)(u_2, w)$ where $(u_1, u_2) \in E$, since $e \in E \backslash; \overline{\mu^*}(e) = 0, \overline{\gamma^*}(e) = 0$ and as $(u_1, u_2) \notin \overline{E_1} (\overline{\mu_2} \otimes \overline{\mu_2^*}) = 0, (\overline{\gamma_2} \otimes \overline{\gamma_2^*})(e) = 0$

Case: 4

$e = (u_1, w)(u_2, w)$ where $(u_1, u_2) \notin E$, since $e \notin E, \mu^*(e) = 0$ and $\gamma^*(e) = 0$

$\overline{\mu^*}(e) = \mu_1(u_1, w) \wedge \mu_1(u_2, w) = \mu_1(u) \wedge \mu_2^*(u_2) \wedge \mu_2^*(w)$

$\overline{\gamma^*}(e) = \gamma_1(u_1, w) \wedge \gamma_1(u_2, w) = \gamma_1(u) \wedge \gamma_2^*(u_2) \wedge \gamma_2^*(w)$

and is $(u_1, u_2) \in \overline{E_1}, (\overline{\mu_2}) \otimes \overline{\mu_2^*} = \mu_1^*(w) \wedge \overline{\mu_2}(u_1, u_2) = \mu_1(u) \wedge \mu_1(u_2) \wedge \mu_1^*(w)$

$(\overline{\gamma_2} \otimes \overline{\gamma_2^*}) = \gamma_1^*(w) \wedge \overline{\gamma_2}(u_1, u_2) = \gamma_1(u) \wedge \gamma_1(u) \wedge \gamma_1^*(w)$ since $\overline{G_1}$ is complete.

Case: 5

$e = (u_1, v_1)(u_2, v_2)$ where $(u_1, u_2) \notin E_1$ and $v_1 \neq v_2$ since $e \in E$ $\overline{\mu^*}(e) = 0$ and $\overline{\gamma^*}(e) = 0$ and as $(u_1, u_2) \in \overline{E_1} (\overline{\mu_2} \otimes \overline{\mu_2^*})(e) = 0 = (\overline{\gamma_2} \otimes \overline{\gamma_2^*})$

Case: 6

$e = (u_1, v_1)(u_2, v_2)$ where $(u_1, u_2) \in E_1$ and $v_1 \notin v_2$, since $e \notin E, \overline{\mu^*}(e) = 0$, $\overline{\gamma^*}(e) = 0$ and so

$\overline{\mu^*}(e) = \mu_1(u_1, w) \wedge \mu_1(u_2, w)$

$= \mu_1(u) \wedge \mu_2^*(u_2) \wedge \mu_1^*(v_1) \wedge \mu_1^*(v_2)$

$\overline{\gamma^*}(e) = \gamma_1(u_1, w) \wedge \gamma_1(u_2, w)$

$= \gamma_1(u_1) \wedge \gamma_1(v_2) \wedge \gamma_1^*(v_1) \wedge \gamma_1^*(v_2)$

and as $(u_1, u_2) \in \overline{E_1}$ and as $\overline{G_1}$ is complete.

$(\overline{\mu_2} \otimes \overline{\mu_2^*}) = \overline{\mu_2}(u_1, u_2) \wedge \mu_1^*(v_1) \wedge \mu_1^*(v_2) = \overline{\mu_2}(e)$

$(\overline{\gamma_2} \otimes \overline{\gamma_2^*}) = \overline{\gamma_2}(u_1, u_2) \wedge \gamma_1^*(v_1) \wedge \gamma_1^*(v_2) = \overline{\gamma_2}(e)$

Case: 7

$e = (u_1, v_1)(u_2, v_2)$ where $(u_1, u_2) \notin E_1$ and $(v_1, v_2) \notin E_2$ since $e \notin E, \mu(e) = 0$ and $\gamma(e) = 0$

$\mu(e) = \mu(u_1, w) \wedge \mu(u_2, w) = \mu_1(u_1) \wedge \mu_1^*(v_1) \wedge \mu_1^*(v_2)$

$\gamma(e) = \gamma(u_1, w) \wedge \gamma(u_2, w) = \gamma_1(u_1) \wedge \gamma_1^*(v_1) \wedge \gamma_1^*(v_2)$

As, $(u_1, u_2) \in \overline{E_1}$ and if $v_1 = v_2$ then we have Case 4.

If $(u_1, u_2) \in \overline{E_1}$ and $v_1 \neq v_2$ then we have Case 6.

In all cases

$\overline{\mu_2 \otimes \mu_2^*} = \overline{\mu_2} \otimes \overline{\mu_2^*}$ and $\overline{\gamma_2 \otimes \gamma_2^*} = \overline{\gamma_2} \otimes \overline{\gamma_2^*}$

$\therefore \overline{G_1 \otimes G_2} \approx \overline{G_1} \otimes \overline{G_2}$

Theorem 3.5

If $G_1: (\mu_1, \mu_2)(\gamma_1, \gamma_2)$ and $G_2: (\mu_1^*, \mu_2^*)(\gamma_1^*, \gamma_2^*)$ are Intuitionistic fuzzy complete graphs then $\overline{G_1 \Pi G_2} \approx \overline{G_1} \Pi \overline{G_2}$ and $\overline{G\_1 \cdot G\_2} \approx \overline{G_1} \cdot \overline{G_2}$

Proof:

We show that if the direct product the semi - strong product or the strong product of two Intuitionistic fuzzy graphs is complete then at least one of hte two Intuitionistic fuzzy graphs must be complete.

Theorem 3.6

If $G\_1: ((\mu_1, \mu_2)(\gamma_1, \gamma_2))$ and $G_2: (\mu_1^*, \mu_2^*)(\gamma_1^*, \gamma_2^*)$ are Intuitionistic fuzzy graphs, such that $G_1 \Pi G_2$ is complete, then at least G_1 or G_2 must be complete.

Proof:

Suppose that G_1 and G_2 are not complete. Then there exists at least one $(u_1, v_1) \in E_1$ and $(u_2, v_2) \in E_2$ such that

$\mu_2(u_1, v_1) < \mu_1(u_1) \wedge \mu_1(v_1)$

$\mu_2^*(u_2, v_2) < \mu_1^*(u_2) \wedge \mu_1^*(v_2)$ and

$\mu_2(u_1, v_1) < \mu_1(u_1) \wedge \mu_1(v_1)$

$\mu_2^*(u_2, v_2) < \mu_1^*(u_2) \wedge \mu_1^*(v_2)$

Now

$(\mu_2 \Pi \mu_2^*)((u_1, v_1)(u_2, v_2)) = \mu_2(u_1, u_2) \wedge \mu_2^*(v_1, v_2)$

$< \mu_1(u_1) \wedge \mu_1(u_2) \wedge \mu_1^*(v_1) \wedge \mu_1^*(v_2) G_1, G_2$ are complete

$(\gamma_2 \Pi \gamma_2^*)((u_1, v_1)(u_2, v_2)) = \gamma_2(u_1, u_2) \wedge \gamma_2^*(v_1, v_2)$

$< \gamma_1(u_1) \wedge \gamma_1(u_2) \wedge \gamma_1^*(v_1) \wedge \gamma_1^*(v_2) G_1, G_2$ are complete

But

$(\mu_1 \Pi \mu_1^*)(u_1, v_1) = \mu_1(u_1) \wedge \mu_1^*(v_1)$

$(\mu_1 \Pi \mu_1^*)(u_2, v_2) = \mu_1(u_2) \wedge \mu_1^*(v_2)$

$(\gamma_1 \Pi \gamma_1^*)(u_1, v_1) = \gamma_1(u_1) \wedge \gamma_1^*(v_1)$

$(\gamma_1 \Pi \gamma_1^*)(u_2, v_2) = \gamma_1(u_2) \wedge \gamma_1^*(v_2)$

Thus

$(\mu_1 \Pi \mu_1^*)(u_1, v_1) \wedge ((\mu_1 \Pi \mu_1^*)(u_2, v_2))$

$= \mu_1(u_1) \wedge \mu_1(u_2) \wedge \mu_1^*(v_1) \wedge \mu_1^*(v_2)$

$> (\mu_2 \Pi \mu_2^*)((u_1, v_1)(u_2, v_2))$

$(\gamma_1 \Pi \gamma_1^*)(u_1, v_1) \wedge ((\gamma_1 \Pi \gamma_1^*)(u_2, v_2))$

$= \gamma_1(u_1) \wedge \gamma_1(u_2) \wedge \gamma_1^*(v_1) \wedge \gamma_1^*(v_2)$

$> (\gamma_2 \Pi \gamma_2^*)((u_1, v_1)(u_2, v_2))$

Hence $G_1 \Pi G_2$ is not complete, a contradiction.

Theorem 3.7

If $G_1: ((\mu_1, \mu_2)(\gamma_1, \gamma_2))$ and $G_2: ((\mu_1^*, \mu_2^*)(\gamma_1^*, \gamma_2^*))$ are Intuitionistic fuzzy graphs such that $G_1 \cdot G_2$ or $G_1 \otimes G_2$ is complete, then atleast G_1 or G_2 must be complete.

4. Balanced Intuitionistic fuzzy graphs

In this section we are defining the density of a Intuitionistic fuzzy graph and balanced Intuitionistic fuzzy graphs. We then show that any complete Intuitionistic fuzzy graph is balanced.

But the converse need not be true.

Definition 4.1
The density of an Intuitionistic fuzzy graph $G:(V,E)$ is

$$D(G) = 2\left[\left(\frac{\sum_{u,v\in V}\mu_2^*(u,v)}{\sum_{u,v\in V}(\mu_1(u)\wedge\mu_1(v))}\right)\left(\frac{\sum_{u,v\in V}\gamma_2(u,v)}{\sum_{u,v\in V}(\gamma_1(u)\wedge\mu_1(v))}\right)\right]$$

G is balanced if $D(H) \leq D(G)$ for all Intuitionistic fuzzy non - empty subgraphs H of G.

Theorem 4.1
Any complete Intuitionistic fuzzy graph is balanced.

Proof:
Let G be a complete Intuitionistic fuzzy graph. Then

$$D(G) = 2\left(\frac{\sum_{u,v\in V}(\mu_2(u,v),\gamma_2(u,v))}{\sum_{u,v\in V}(\mu_1(u)\wedge\mu_1(v)),(\gamma_1(u)\wedge\gamma_1(v))}\right)$$

$$= 2\left(\frac{\sum_{u,v\in V}(\mu_1(u)\wedge\mu_1(v)),(\gamma_1(u)\wedge\gamma_1(v))}{\sum_{u,v\in V}(\mu_2(u,v)),(\gamma_1(u,v))}\right) = 2$$

If H is a non - empty Intuitionistic fuzzy subgraph of G then

$$D(G) = 2\left(\frac{\sum_{u,v\in V(H)}(\mu_2(u,v),\gamma_2(u,v))}{\sum_{u,v\in V(H)}(\mu_1(u)\wedge\mu_1(v)),(\gamma_1(u)\wedge\gamma_1(v))}\right)$$

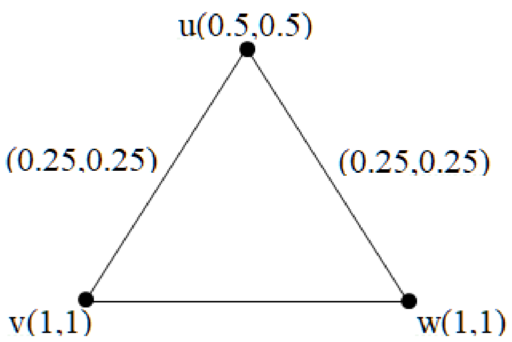
$$= 2\left(\frac{\sum_{u,v\in V(H)}(\mu_1(u)\wedge\mu_1(v)),(\gamma_1(u)\wedge\gamma_1(v))}{\sum_{u,v\in V(H)}(\mu_1(u)\wedge\mu_1(v)),(\gamma_1(u)\wedge\gamma_1(v))}\right)$$

$$= 2\left(\frac{\sum_{u,v\in V}(\mu_1(u)\wedge\mu_1(v)),(\gamma_1(u)\wedge\gamma_1(v))}{\sum_{u,v\in V}(\mu_1(u)\wedge\mu_1(v)),(\gamma_1(u)\wedge\gamma_1(v))}\right)$$

$$= 2 = D(G)$$

Thus G is balanced.
The converse of the preceding result need not be true.

Example 4.1
The following Intuitionistic fuzzy graph $G:(\mu,\gamma)$ is a balanced Intuitionistic graph that is not complete.



Two types of Intuitionistic fuzzy graph each with density equals 1.

Theorem 4.2
Every self - complementary Intuitionistic fuzzy graph has density equals to 1.

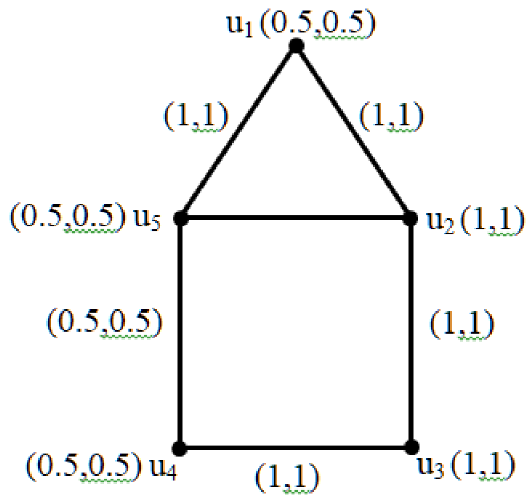
Proof:
Let G be a self-complementary Intuitionistic fuzzy graph. Then

$$D(G) = 2\left(\frac{\sum_{u,v\in V}(\mu_2(u,v),\gamma_2(u,v))}{\sum_{u,v\in V}(\mu_1(u)\wedge\mu_1(v)),(\gamma_1(u)\wedge\gamma_1(v))}\right)$$

$$= 2\left(\frac{\sum_{u,v\in V}(\mu_1(u)\wedge\mu_1(v)),(\gamma_1(u)\wedge\gamma_1(v))}{\sum_{u,v\in V}(\mu_2(u,v)),(\gamma_2(u,v))}\right) = 1$$

Hence the proof.
The converse of the preceding result need not be true.

Example 4.2
The following Intuitionistic fuzzy graph $G:(V,E)$ has density equal 1, but it is not self-complementary.



Theorem 4.3
Let $G:(\mu,\gamma)$ be an Intuitionistic fuzzy graph such that

$$\mu_2(u,v) = \left(\frac{1}{2}\right)(\mu_1(u)\wedge\mu_1(v)), \forall\, u,v \in V$$

$$\gamma_2(u,v) = \left(\frac{1}{2}\right)(\gamma_1(u)\wedge\gamma_1(v)), \forall\, u,v \in V$$

Then $D(G) = 1$

Proof:
Let $G:(\mu,\gamma)$ be an Intuitionistic fuzzy graph such that

$$\mu_2(u,v) = \left(\frac{1}{2}\right)(\mu_1(u)\wedge\mu_1(v)), \forall\, u,v \in V$$

$$\gamma_2(u,v) = \left(\frac{1}{2}\right)(\gamma_1(u)\wedge\gamma_1(v)), \forall\, u,v \in V$$

Then G is self complementary and thus by the preceding theorem $D(G) = 1$.

Next, we prove the following that we use to give necessary and sufficient conditions for the direct product, semi - strong product and strong product of two Intuitionistic fuzzy balanced graphs to be balanced.

Lemma 4.1
Let G_1 and G_2 be an Intuitionistic fuzzy graph then $D(G_1) \leq D(G_1 \Pi G_2)$ for $i = 1,2$ if and only if $D(G_1) = D(G_2) = D(G_1 \Pi G_2)$

Proof:
If $D(G_i) \leq D(G_1 \Pi G_2)$ for $i = 1,2$ then

$$D(G) = 2\left(\frac{\sum_{u,v\in V_1}\left(\mu_2(u,v),\gamma_2(u,v)\right)}{\sum_{u,v\in V(H)}(\mu_1(u_1)\wedge\mu_1(u_2)),(\gamma_1(u_1)\wedge\gamma_1(v_2))}\right)$$

$$\geq 2\left(\frac{\sum_{u_1,u_2\in V_1,v_1,v_2\in V_2}\begin{matrix}\mu_2(u_1,u_2),\gamma_2(u_1,u_2)\wedge\\ \left((\mu_1^*(v_1)\wedge\mu_1^*(v_2)),(\gamma_1^*(v_1),\gamma_1^*(v_2))\right)\end{matrix}}{\sum_{u_1,u_2\in V_1,v_1,v_2\in V_2}\begin{matrix}\mu_1(u_1)\wedge\mu_1(u_2),\\ \left(\gamma_1(u_1)\wedge\gamma_1(u_2)\wedge(\mu_1^*(u_1)\wedge\mu_1^*(u_2)),(\gamma_1^*(u_1)\wedge\gamma_1^*(u_2))\right)\end{matrix}}\right)$$

$$= 2\left(\frac{\sum_{u_1,u_2\in V_1,v_1,v_2\in V_2}\begin{matrix}\mu_2(u_1,u_2)\wedge\gamma_2^*(v_1,v_2),\\ (\gamma_2(u_1,u_2)\wedge\gamma_2^*(v_1,v_2))\end{matrix}}{\sum_{u_1,u_2\in V_1,v_1,v_2\in V_2}\begin{matrix}\mu_1(u_1)\wedge\mu_1(u_2),\\ (\gamma_1(u_1)\wedge\gamma_1(u_2))\wedge(\mu_1^*(u_1)\wedge\mu_1^*(u_2)),(\gamma_1^*(v_1)\wedge\gamma_1^*(v_2))\end{matrix}}\right)$$

$$= 2\left(\frac{\sum_{u_1,u_2\in V_1,v_1,v_2\in V_2}\left(\mu_2\Pi\ (u_1,u_2)(v_1,v_2)\right),\left(\gamma_2\Pi\ \gamma_2^*(u_1,u_2)(v_1,v_2)\right)}{\sum_{u_1,u_2\in V_1,v_1,v_2\in V_2}\left(\mu_1\Pi\mu_2((u_1,u_2)(v_1,v_2))\right)\left(\gamma_1\Pi\gamma_2((u_1,u_2)(v_1,v_2))\right)}\right)$$

$$= D(G_1\Pi G_2)$$

Hence $D(G_1) \geq D(G_1\Pi G_2)$ and thus $D(G_1) = D(G_1\Pi G_2)$ similarly $D(G_2) = D(G_1\Pi G_2)$
$\therefore\ D(G_1) = D(G_2) = D(G_1\Pi G_2)$

Theorem 4.4
Let G_1 and G_2 be an Intuitionistic fuzzy balanced graphs. Then $G_1\Pi G_2$ is balanced if and only if $D(G_1) = D(G_2) = D(G_1\Pi G_2)$

Proof:
If $G_1\Pi G_2$ is balanced, then $D(G_i) \leq D(G_1\Pi G_2)$for $i = 1,2$ $D(G_1) = D(G_2) = D(G_1\Pi G_2)$

Conversely,
If $D(G_1) = D(G_2) = D(G_1\Pi G_2)$ and H is an Intuitionistic fuzzy subgraph of $G_1\Pi G_2$. Then there exists Intuitionistic fuzzy subgraphs H_1 of $G_1$and $H_2$of G_2. As G_1 and G_2 are balanced and $D(G_1) = D(G_2) = \frac{n_1}{r_1}$ then $D(H_1) = \frac{a_1}{b_1} \leq \frac{n_1}{r_1}$ and $D(H_2) = \frac{a_2}{b_2} \leq \frac{n_1}{r_1}\}$

Thus, $a_1 r_1 + a_2 r_1 \leq b_1 n_1 + b_2 n_1$ and hence,$D(H) \leq \frac{(a_1+a_2)}{(b_1+b_2)} \leq \frac{n_1}{r_1} = D(G_1\Pi G_2)$

Therefore $G_1\Pi G_2$ is balanced.

Theorem 4.5
Let G_1 and G_2 be an Intuitionistic fuzzy balanced graphs then,

(i) $G_1 \cdot G_2$ is balanced iff$D(G_1) = D(G_2) = D(G_1 \cdot G_2)$
(ii) $G_1 \otimes G_2$is balanced iff $D(G_1) = D(G_2) = D(G_1\backslash otimes\ G_2)$

By showing that isomorphism between Intuitionistic fuzzy graphs preserve balanced.

Theorem 4.6
Let G_1 and G_2 be isomorphic Intuitionistic fuzzy graphs. If G_2 is balanced then G_1 is balanced.

Proof:
Let $h: V_1 \to V_2$ be a bijection such that $\mu_1(u) = \mu_1^*(h(u)), \gamma_1(u) = \gamma_1^*(h(u))$ and
$\mu_2(u,v) = \mu_2^*(h(u),h(v))\gamma_2(u,v) = \gamma_2^*(h(u),h(v))\forall\ u,v \in\ V_1$

$$\sum_{u\in V_1}\mu_1(u) = \sum_{u\in V_2}\mu_1^*(u),\ \sum_{u\in V_1}\gamma_1(u) = \sum_{u\in V_2}\gamma_1^*(u),$$
$$\sum_{u,v\in V_1}\mu_2(u,v) = \sum_{u,v\in V_2}\mu_2^*(u,v),\ \sum_{u,v\in V_1}\gamma_2(u,v) = \sum_{u,v\in V_2}\gamma_2^*(u,v)$$

If $H_1 = (\mu_1', \gamma_1')$ is an Intuitionistic fuzzy subgraph of G_1 with underlying set w, then $H_2 = (\mu_1^{*\prime})$ is an Intuitionistic fuzzy subgraph of G_2 with underlying set $h(w)$ where $\mu_1^{*'}(h(u)) = \mu_1'(u), \gamma_1^{*'}(h(u)) = \gamma_1'(u)$ and $\mu_2^{*'}(h(u),h(v)) = \mu_2'(u,v)\gamma_2^{*'}(h(u),h(v)) = \gamma_2'(u,v)\forall\ u,v \in\ w$

Since G_2 is balanced, $D(H_2) \leq\ D(G_2)$ and so

$$2\left(\frac{\sum_{u,v\in w}(\mu_2^{*'}(h(u),h(v)),\gamma_2^*(h(u),h(v))}{\sum_{u,v\in w}\left(\mu_1^{*'}(u)\wedge\mu_1^*(v)\right),(\gamma_1^*(u)\wedge\gamma_1^*(v))}\right)$$
$$\leq 2\left(\frac{\sum_{u,v\in V_2}(\mu_2^*(u,v),\gamma_2^*(u,v))}{\sum_{u,v\in V_2}(\mu_1^*(u)\wedge\mu_1^*(v),\gamma_1^*(u)\wedge\gamma_1^*(v)}\right)$$

and so

$$2\left(\frac{\sum_{u,v\in w}(\mu_2^{*'}(u,v),\gamma_2^*(u,v))}{\sum_{u,v\in w}\ (\mu_1^{*'}(u)\wedge\mu_1^*(v)}\right)$$
$$\leq 2\left(\frac{\sum_{u,v\in V_2}(\mu_2(u,v),\gamma_2(u,v)}{\sum_{u,v\in V_2}((\mu_1^*(u)\wedge\mu_1^*(v)),\gamma_1^*(u)\wedge\gamma_1^*(v)}\right)$$
$$\leq\ D(G_1)$$

Therefore G_1 is balanced.

5. Conclusion

In this paper we discussed three operations on Intuitionistic fuzzy graphs, direct product, semi strong product and strong product on Intuitionistic fuzzy graphs and we discussed about product of two Intuitionistic fuzzy balanced graphs to be balanced.

References

[1] K.R. Bhutani,Onauthomorphism of fuzzy graphs, Pattern Recognition Letter, 9 (1989), 159 - 162.
[2] J.N. Mordeson and C.S. Peng, Operations on fuzzy graphs, Information Sciences, 79 (1994), 381 - 384.
[3] A. NagoorGani and J. Malarvizhi, Isomorphism on fuzzy graphs, Int. J. Comp. and Math. Sci., Vol.2, 4 (2008), 190 - 196.
[4] A. NagoorGani and J. Malarvizhi, Isomorphism properties on strong fuzzy graphs, Int. J. Algorithms, Comp. and Math., Vol. 2, 1 (2009), 39 - 47.
[5] A. NagoorGani and K. Radha, On regular fuzzy graphs, J. Physical Sciences, 12 (2008), 33 - 40.
[6] A. Rosenfeld, Fuzzy graphs, in L.A. Zadeh, K.S. Fu, K. Tanaka and M. Shirmura (Eds.), Fuzzy and Their Applications to Cognitive and Decision Processes, Academic Press, New York, 1975, 77 - 95.
[7] M.S. Sunitha and A.V. Kumar, Complements of fuzzy graphs, Indian J. Pure Appl. Math., Vol. 33, 9 (2002), 1451 - 1464.
[8] L.A. Zadeh, Fuzzy sets, Inform, Control, 8 (1965), 338 - 353.