Rough Statistical Convergence on Biquadratic Sequences

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Abstract: ***In this paper, using the concept of natural density, we introduce the notion of rough statistical convergence of biquadratic sequences. We define the set of rough statistical limit points of biquadratic sequence and obtain rough statistical convergence criteria associated with this set. Later, we prove this set is closed and convex and also examine the relations between the set of rough statistical cluster points and the set of rough statistical limit points of a biquadratic sequence.***

Keywords: Rough statistical convergence, Natural density, biquadratic sequences, chi sequence.

Mathematics Subject Classification 2010: 40F05, 40J05, 40G05

1. Introduction

The idea of statistical convergence was introduced by Steinhaus and also independently by Fast [4] for real or complex sequences. Statistical convergence is a generalization of the usual notion of convergence, which parallels the theory of ordinary convergence.

A biquadratic sequence (real or complex) can be defined as a function x: N×N ×N×N → R(C), where N, R and C denote the set of natural numbers, real numbers and complex numbers respectively. The different types of notions of triple sequence and biquadratic sequences were introduced and investigated by Sahiner [3], Esi [1], Datta [2], Debnath [7] and many others.

Let K be a subset of the set $N \times N \times N \times N$, and let us denote the set $\{(m,n,k,l) \in K: m \le p, n \le q, k \le r, l \le s\}$ by K_{pqrs}. Then the natural density of K is given by $\delta(K) = lim_{pqrs \to \infty} \frac{|K_{pqrs}|}{pqra}$, where $|K_{pqrs}|$ denotes the number of elements in K_{pqrs}. Clearly, a finite subset has natural density zero and we have $\delta(K^c) = 1 - \delta(K)$, where $K^c = {}^{N}/_{K}$ is the complement of K. If $K_1 \subseteq K_2$, then $\delta(K_1) \le \delta(K_2)$.

Consider a biquadratic sequence $x = (x_{mnkl})$ such that $x_{mnkl} \in R$, $m,n, k, l \in$N. A biquadratic sequence $x = (x_{mnkl})$ is said to be statistically convergent to 0 ∈R, written as $st{-}lim\ x = 0$, provided that the set

$$\{(m,n,k,l) \in N^4: |x_{mnkl} - 0| \ge \epsilon\}$$

has natural density zero for any $\epsilon > 0$. In this case, 0 is called the statistical limit of the biquadratic sequence x.

If a biquadratic sequence is statistically convergent, then for every $\epsilon > 0$, infinitely many terms of the sequence may remain outside the ϵ –neighbourhood of the statistical limit, provided that the natural density of the set consisting of the indices of the set terms is zero. This is an important property that distinguishes statistical convergence from ordinary convergence. Because the natural density of a finite set is zero, we can say that every ordinary convergent sequence is statistically convergent.

If a biquadratic sequence $x = (x_{mnkl})$ satisfies some property P for all m, n, k, l except a set of natural density zero, then we say that the biquadratic sequence x satisfies P for "almost all (m, n, k, l)" and we abbreviate this by "*a.a. (m,n,k,l).*"

Let $(x_{m_t n_u k_v l_w})$ be a sub sequence of $x = (x_{mnk})$.If the natural density of theset

$$K = \{(m_t, n_u, k_v, l_w) \in N^4: (t,u,v,w) \in N^4\}$$

Is different from zero, then $(x_{m_t n_u k_v l_w})$ is called a non thin sub sequence of a biquadratic sequence x.

c ∈R is called a statistical cluster point of a biquadratic sequence $x = (x_{mnkl})$ provided that the natural density of the set

$$\{(m,n,k,l) \in N^4: |x_{mnkl} - c| < \epsilon\}$$

is different from zero for every ϵ>0.We denote the set of all statistical cluster points of the sequence x by Γ_x.

A biquadratic sequence $x=(x_{mnkl})$ is said to be statistically analytic if there exists a positive number M such that

$$\delta\left(\left\{(m,n,k,l) \in N^4: |x_{mnkl}|^{1/m+n+k+l} \ge M\right\}\right) = 0$$

The theory of statistical convergence has been discussed in trigonometric series, summability theory, measure theory, turnpike theory, approximation theory, fuzzy set theory and so on.

The idea of rough convergence was introduced by Phu [9], who also in- traduced the concepts of rough limit points and roughness degree. The idea of rough convergence occurs very naturally in numerical analysis and has interesting applications. Aytar [1] extended the idea of rough convergence into rough statistical convergence using the notion of natural density just as usual convergence was extended to statistical convergence. Paletal.[8] extended the notion of rough convergence using the concept of ideals which automatically extends the earlier notions of rough convergence and rough statistical convergence.

In this paper, we introduce the notion of rough statistical convergence of biquadratic sequences. Defining the set of rough statistical limit points of a biquadratic sequence, we obtain rough statistical convergence criteria associated with this set.

Throughout the paper r be a non negative real number.

2. Definitions and Preliminaries

Definition 2.1: A biquadratic sequence $x = (x_{mnkl})$ is said to be rough convergent (r *-convergent*) to a (Pringsheim's sense), denoted as $x_{mnkl} \to a$, provided that

$$\forall \epsilon > 0, \exists\, i_\epsilon \in N : m,n,k,l \geq i_\epsilon \implies |x_{mnkl} - a| < r + \epsilon$$

Or equivalently, if

$$\lim\sup |x_{mnkl} - a| \leq r.$$

Here r is called the roughness of degree. If we take r=0, then we obtain the ordinary convergence of a biquadratic sequence.

Definition 2.2: It is obvious that the r –limit set of a biquadratic sequence is not unique. The r –limit set of the biquadratic sequence $x = (x_{mnkl})$ is defined as $LIM^r x_{mnkl} = \{a \in R : x_{mnkl} \to^r a\}$.

Definition 2.3: A biquadratic sequence $x=(x_{mnkl})$is said to be r-convergent if $LIM^r x \neq 0$. In this case, r is called the convergence degree of the biquadratic sequence $x= (x_{mnkl})$. For r =0, we get the ordinary convergence.

Definition 2.4: A biquadratic sequence (x_{mnkl}) is said to be r-statistically convergent to a, denoted by $x_{mnkl} \to^{rst} a$, provided that the set

$$\{(m,n,k,l) \in N^4 : |x_{mnkl} - a| \geq r + \epsilon\}$$

has natural density zero for every ϵ>0, or equivalently, if the condition

$$\mathrm{st} - \lim\sup |x_{mnkl} - a| \leq r.$$

is satisfied.

In addition, we can write $x_{mnkl} \to^{rst} a$ if and only if the inequality

$$|x_{mnkl} - a| < r + \epsilon$$

Holds for everyϵ>0 and almost all (m,n,k,l).Here r is called the roughness of degree. If we take r =0, then we obtain the statistical convergence of biquadratic sequences.

In a similar fashion to the idea of classical rough convergence, the idea of rough statistical convergence of a biquadratic sequence can be interpreted as follows:

Assume that a biquadratic sequence $y= (y_{mnkl})$ is statistically convergent and cannot be measured or calculated exactly; one has to do with an approximated (or statistically approximated) biquadratic sequence $x = (x_{mnkl})$ satisfying $|x_{mnkl} - y_{mnkl}| \leq r$ for all m, n, k, l(or for almost all (m,n,k,l),i.e.,

$$\delta(\{(m,n,k,l) \in N^4 : |x_{mnkl} - y_{mnkl}| \geq r\}) = 0.$$

Then the biquadratic sequence x is not statistically convergent any more, but as the inclusion

$$\{(m,n,k,l) \in N^4 : |y_{mnkl} - a| \geq \epsilon\} \supseteq \{(m,n,k,l) \in N^4 : |x_{mnkl} - a| \geq r + \epsilon\}$$

Holds and we have

$$\delta(\{(m,n,k,l) \in N^4 : |x_{mnkl} - a| \geq r\}) = 0.$$

i.e., we get

$$\delta(\{(m,n,k,l) \in N^4 : |x_{mnkl} - a| \geq r + \epsilon\}) = 0.$$

i.e., the biquadratic sequence spaces x is r-statistically convergent in the sense of definition (2.3).

In general, the rough statistical limit of a biquadratic sequence may not unique for the roughness degree r >0.So we have to consider the so called r- statistical limit set of a biquadratic sequence $x = (x_{mnk})$, which is defined by

$$st - LIM^r x = \{L \in R : x_{mnkl} \to^{rst} a\}$$

The biquadratic sequence x is said to be r-statistically convergent provided that$st - LIM^r x \neq \emptyset$. It is clear that if $st - LIM^r x \neq \emptyset$ for a biquadratic sequence $x = (x_{mnkl})$ of real numbers, then we have

$$st - LIM^r x = [st - \lim sup\, x - r, st - \lim inf\, x + r]$$

We know that $LIM^r x = \emptyset$. for an unbounded biquadratic sequence $x = (x_{mnkl})$.But such a biquadratic sequence might be rough statistically convergent. For instance, define

$$x_{mnkl} = \begin{cases} (-1)^{mnkl}, & if\ (m,n,k,l) \neq (t,u,v,w)^2 \\ (mnkl), & otherwise \end{cases}$$

in R. Because the set$\{1,64,739,\cdots\}$has natural density zero, we have

$$st - LIM^r x = \begin{cases} \varphi, & if\ r < 1 \\ [1-r, r-1], & otherwise \end{cases}$$

And $LIM^r x = \varphi$ for all $r \geq 0$.

A scan be seen by the example above, the fact that $st - LIM^r x \neq \emptyset$. Does not imply $st - LIM^r x \neq \emptyset$..Because a finite set of natural numbers has natural density zero, $st - LIM^r x \neq \emptyset$. Implies $st - LIM^r x \neq \emptyset$.. Therefore, we get$LIM^r x \subseteq st - LIM^r x$.This obvious fact means $\{r \geq 0 : LIM^r x\} \subseteq \{r \geq 0 : st - LIM^r x\}$ in this language of sets and yields immediately

$$inf\ \{r \geq 0 : LIM^r x \neq \varphi\} \geq inf\{r \geq 0 : st - LIM^r x \neq \varphi\}$$

Moreover, it also yields directly $diam\ (LIM^r x) \leq diam\ (st - LIM^r x)$.

3. Main Results

Theorem 3.1: For abiquadratic sequence spaces $x = (x_{mnkl})$, we have $diam(st - LIM^r x) \leq 2r$. In general $diam\ (st - LIM^r x)$ has an upper bound.

Proof: Assume that diam (st−LIM^{r}x)>2r.Then$\exists\ w, y \in$

$st\text{-}LIM^r x$ such that $|w-y|>2r$. Take $\epsilon \in \left(0, \frac{|w-y|}{2} - r\right)$.
Because $w, y \in st\text{-}LIM^r x$ we have $\delta(K_1)=0$ and $\delta(K_2) = 0$ for every $\epsilon > 0$ where

$$K_1 = \{(m,n,k,l) \in N^4 : |x_{mnkl} - w| \geq r + \epsilon\}$$
and
$$K_2 = \{(m,n,k,l) \in N^4 : |x_{mnkl} - y| \geq r + \epsilon\}$$

Using the properties of natural density, we get $\delta(K_1^c \cap K_2^c) = 1$. Thus we can write,

$$|w-y| \leq |x_{mnkl} - w| + |x_{mnkl} - y|$$

$$< 2(r+\epsilon) = 2\left(\frac{|w-y|}{2}\right) = |w-y|$$

For all $(m,n,k,l) \in K_1^c \cap K_2^c$, which is a contradiction.

Now let us prove the second part of the theorem. Consider a triple sequence $x = (x_{mnkl})$ such that $st\text{-}\lim x_{mnkl} = a$. Let $\epsilon > 0$. Then we can write

$$\delta(\{(m,n,k,l) \in N^4 : |x_{mnkl} - a| \geq \epsilon\}) = 0.$$

We have

$$|x_{mnkl} - y| \leq |x_{mnkl} - a| + |a - y| \leq |x_{mnkl} - a| + r$$

For each $y \in B_r(a) = \{y \in R^3 : |y - a| \leq r\}$.

Then we get $|a - y| < r + \epsilon$ for each $(m,n,k,l) \in \{(m,n,k,l) \in N^4 : |xmnk - l| < \epsilon\}$.

Because the biquadratic sequence spaces x is statistically convergent to a, we have

$$\delta(\{(m,n,k,l) \in N^4 : |x_{mnkl} - a| \geq \epsilon\}) = 1.$$

Therefore we get $y \in st\text{-}LIM^r x$. Hence, we can write $st\text{-}LIM^r x = \overline{B_r}(a)$

Because $diam \overline{B_r}(a) = 2r$, this shows that in general, the upper bound $2r$ of the diameter of the set $st\text{-}LIM^r x$ is not an lower bound.

Theorem 3.2: Let $r>0$. Then a biquadratic sequence $x = (x_{mnk})$ is r-statistically convergent to a if and only if there exists a biquadratic sequence $y = (y_{mnkl})$ such that $st\text{-}\lim y = a$ and $|x_{mnkl} - y_{mnkl}| \leq r$ for each $(m,n,k,l) \in N^4$.

Proof: Necessity: Assume that $x_{mnkl} \to^{rst} a$. Then we have

(3.1) $st\text{-}\lim\sup |x_{mnkl} - a| \leq r$

Now we define,

$$y_{mnkl} = \begin{cases} a, & if\ |x_{mnkl} - a| \leq r \\ x_{mnkl} + r\left(\frac{a - x_{mnkl}}{|x_{mnkl} - a|}\right), & otherwise. \end{cases}$$

Then, we write

$$|y_{mnkl} - a| = \begin{cases} |a - r|, & if\ |x_{mnkl} - a| \leq r \\ |x_{mnkl} - a| + r\left(\frac{|a - r| - |x_{mnkl} - a|}{|x_{mnkl} - a|}\right), & otherwise. \end{cases}$$

i.e.,

$$|y_{mnkl} - a| = \begin{cases} 0, & if\ |x_{mnkl} - a| \leq r \\ |x_{mnkl} - a| - r\left(\frac{|x_{mnkl} - a|}{|x_{mnkl} - a|}\right), & otherwise. \end{cases}$$

i.e.,

$$|y_{mnkl} - a| = \begin{cases} 0, & if\ |x_{mnkl} - a| \leq r \\ |x_{mnkl} - a| - r, & otherwise \end{cases}$$

We have

$$|y_{mnkl} - a| \geq |x_{mnkl} - a| - r$$
$$\Rightarrow |x_{mnkl} - a - y_{mnkl} + a| \leq r.$$

(3.2) $|x_{mnkl} - y_{mnkl}| \leq r$

For all $m,n,k,l \in N$. By equation (3.1) and by definition of y_{mnkl}, we get
$st - \lim\sup |y_{mnkl} - a| = 0.$

$$\Rightarrow st - \lim y_{mnkl} \to^r a.$$

Sufficiency- Because $st - \lim y_{mnkl} = a$, we have

$$\delta(\{(m,n,k,l) \in N^4 : |y_{mnkl} - a| \geq \epsilon\}) = 0.$$

For each $\epsilon > 0$. It is easy to see that the inclusion

$$\{(m,n,k,l) \in N^4 : |y_{mnkl} - a| \geq \epsilon\} \supseteq \{(m,n,k,l) \in N^4 : |x_{mnkl} - a| \geq r + \epsilon\}$$

Holds. Because

$$\delta(\{(m,n,k,l) \in N^4 : |y_{mnkl} - a| \geq \epsilon\}) = 0.$$

We get

$$\delta(\{(m,n,k,l) \in N^4 : |x_{mnkl} - a| \geq r + \epsilon\}) = 0.$$

Theorem3.3: For an arbitrary $c \in \Gamma_x$ of biquadratic sequence $x = (x_{mnkl})$ we have $|a - c| \leq r$ for all $l \in st\text{-}LIM^r x$.

Proof: Assume on the contrary that there exist a point $c \in \Gamma_x$ and $l \in st\text{-}LIM^r x$ such that $|a - c| > r$.

Define $\epsilon = \frac{|a-c|-r}{3}$
Then

(3.3) $\{(m,n,k,l) \in N^4 : |y_{mnkl} - a| \geq \epsilon\} \supseteq \{(m,n,k,l) \in N^4 : |x_{mnkl} - a| \geq r + \epsilon\}$

Since $c \in \Gamma_x$, we have

$$\delta(\{(m,n,k,l) \in N^4 : |x_{mnkl} - c| < \epsilon\}) \neq 0.$$

Hence, by (3.3), we get

$$\delta(\{(m,n,k,l) \in N^4: |x_{mnkl} - a| \geq r+\epsilon\}) \neq 0,$$

Which contradicts the fact $a \in st - LIM^r x$.

Proposition 3.4: If a biquadratic sequence $x=(x_{mnk})$ is analytic, then there exist0s non-negative real number r such that $st\text{-}LIM^r x \neq \varphi$..

Proof: If we take the biquadratic sequence is to be statistically analytic, then the of proposition holds. Thus we have the following theorem.

Theorem 3.5: A biquadratic sequence $x = (x_{mnk})$ is statistically analytic if and only if there exists a non-negative real number r such that $st\text{-}LIM^r x \neq \varphi$...

Proof: Since the biquadratic sequence x is statistically analytic, there exists a positive real number M such that

$$\delta\left(\left\{(m,n,k,l) \in N^4: |x_{mnkl}|^{1/m+n+k+l} \geq M\right\}\right) = 0,$$

Define

$$r' = \sup\{|x_{mnkl}|^{1/m+n+k+l}: (m,n,k,l) \in K^c\},$$

Where

$$K = \{(m,n,k,l) \in N^4: |x_{mnkl}|^{1/m+n+k+l} \geq M\}$$

Then the set $st\text{–}LIM^r x$ contains the origin of R. So we have $st\text{-}LIM^r x \neq \varphi$.

If $st\text{-}LIM^r x \neq \varphi$ for some $r \geq 0$, then there exists a such that $a \in st\text{-}LIM^r x$ i.e.,

$$\delta\left(\left\{(m,n,k,l) \in N^4: |x_{mnkl} - a|^{1/m+n+k+l} \geq r+\epsilon\right\}\right) = 0,$$

for each $\epsilon > 0$. Then we say that almost all x_{mnkl} are contained in some ball with any radius greater than r. So the biquadratic sequence x is statistically analytic.

Remark 3.6: If $x' = (x_{m_t n_u k_v l_w})$ is a non-thin subsequence of biquadratic sequence $x = (x_{mnkl})$, then $LIM^r x \subseteq LIM^r x'$. But it is not valid for statistical convergence.

For example: we define

$$x_{mnkl} = \begin{cases}(-1)^{mnkl}, & if\ (m,n,k,l) \neq (t,u,v,w)^2 \\ (mnkl), & otherwise\end{cases}$$

Of real numbers. Then the biquadratic sequence $x = (1,64,739,\ldots)$ is a subsequence of x. we have $st - LIM^r x = [-r,r]$ and $st - LIM^r x = \varphi$.

Theorem 3.7: If $x' = (x_{m_t n_u k_v l_w})$ is a non-thin subsequence of biquadratic sequence $x = (x_{mnkl})$, then $st - LIM^r x \subseteq st - LIM^r x'$.

Proof.

Theorem 3.8: The r-statistical limit set of biquadratic sequence $x = (x_{mnkl})$, is closed.

Proof. If $st\text{-}LIM^r x \neq \varphi$., then it is true. Assume that $st\text{-}LIM^r x \neq \varphi$, then we can choose a biquadratic sequence spaces $(y_{mnkl}) \subseteq st\text{-}LIM^r x$ such that $y_{mnkl} \to^r a$ as $m,n,k,l \to \infty$. If we prove that $a \in st\text{-}LIM^r x$, then the proof will be complete.

Let $\epsilon > 0$ be given. Because $y_{mnkl} \to^r a, \forall \epsilon > 0, \exists i_\epsilon \in N: m,n,k,l \geq i_\epsilon$

Such that

$$|y_{mnkl} - a| < \frac{\epsilon}{2} \text{ for all } m,n,k,l \geq i_\epsilon$$

Now choose an $(m_0 n_0 k_0 l_0) \in N$ such that $m_0, n_0, k_0, l_0 \geq i_\epsilon$. Then we can write

$$|y_{m_0 n_0 k_0 l_0} - a| < \frac{\epsilon}{2}.$$

On the other hand, because $y_{mnkl} \in st - LIM^r x$, we have $y_{m_0 n_0 k_0 l_0} \in st - LIM^r x$,

(3.4) $\delta\left(\left\{(m,n,k,l) \in N^4: |x_{mnkl} - y_{m_0 n_0 k_0 l_0}| \geq r + \epsilon 2 = 0.\right.\right.$

Now let us show that the inclusion

(3.5)

$$\{(m,n,k,l) \in N^4: |y_{mnkl} - a| \geq r+\epsilon\} \supseteq \left\{(m,n,k,l) \in N^4: |x_{mnkl} - y_{m_0 n_0 k_0 l_0}| \geq r + \frac{\epsilon}{2}\right\}$$

Hols. Take

$$(t,u,v,w) \in \left\{(m,n,k,l) \in N^4: |x_{mnkl} - y_{m_0 n_0 k_0 l_0}| \geq r + \frac{\epsilon}{2}\right\}$$

Then we have

$$\left\{|x_{mnkl} - y_{m_0 n_0 k_0 l_0}| \geq r + \frac{\epsilon}{2}\right\}$$

And hence

$$|x_{tuvw} - a| \leq |x_{tuvw} - y_{m_0 n_0 k_0 l_0}| + |y_{m_0 n_0 k_0 l_0} - a| < r + \frac{\epsilon}{2} + \frac{\epsilon}{2} < r+\epsilon$$

i.e., $(t,u,v,w) \in \{(m,n,k,l) \in N^4: |x_{mnkl} - y_{m_0 n_0 k_0 l_0}| \geq r+\epsilon$ which proves the equation (3.5). Hence the natural density of the set on the LHS of equation (3.5) is equal to 1. So, we get $\delta(\{(m,n,k,l) \in N^4: |x_{mnkl} - y_{m_0 n_0 k_0 l_0}| \geq r+\epsilon = 0.$

Theorem 3.9: The r-statistical limit set of biquadratic sequence $x = (x_{mnkl})$, is closed.

Proof: Let $y_1, y_2 \in st\text{-}LIM^r x$ for the biquadratic sequence $x = (x_{mnkl})$ and let $\epsilon > 0$ be given. Define

$$K_1 = \{(m,n,k,l) \in N^4: |x_{mnkl} - w| \geq r+\epsilon\}$$

and

$$K_2 = \{(m,n,k,l) \in N^4: |x_{mnkl} - y| \geq r+\epsilon\}$$

Because $y_1, y_2 \in st\text{-}LIM^r x$, we have $\delta(K_1) = \delta(K_2) = 0$. Thus we have

$$|x_{mnkl} - [(1-\lambda)y_1 + \lambda y_2]| = |(1-\lambda)(x_{mnkl} - y_1) + \lambda(x_{mnkl} - y_2)| < r+\epsilon,$$

For each $(m,n,k,l) \in (K_1^c \cap K_2^c)$ and each $\lambda \in [0,1]$. Because $\delta(K_1^c \cap K_2^c) = 1$, we get

$$\delta(\{(m,n,k,l) \in N^4 : |x_{mnkl} - [(1-\lambda)y_1 + \lambda y_2]| \geq r + \epsilon\}) = 0.$$

i.e., $[(1-\lambda)y_1 + \lambda y_2] \in st - LIM^r x$, Which proves the convexity of the set $st - LIM^r x$.

Theorem 3.10: A biquadratic sequence $x = (x_{mnkl})$ statistically convergent to a if and only if $st - LIM^r x = \overline{B_r}(a)$.

Proof: For the necessity part of this theorem is in proof of the theorem (3.1).

Sufficiency: Because $st - LIM^r x = \overline{B_r}(a) \neq \varphi$, then by theorem (3.5) we can say that A biquadratic sequence $x = (x_{mnkl})$ is statistically analytic. Assume on the contrary that the A biquadratic sequence $x = (x_{mnkl})$ has another statistical cluster point a' different from a. Then the point

$$\bar{\boldsymbol{a}} = \boldsymbol{a} + \frac{\boldsymbol{r}}{|\boldsymbol{a} - \boldsymbol{a}'|}(\boldsymbol{a} - \boldsymbol{a}')$$

Satisfies

$$\bar{\boldsymbol{a}} - \boldsymbol{a}' = \boldsymbol{a} - \boldsymbol{a}' + \frac{\boldsymbol{r}}{|\boldsymbol{a} - \boldsymbol{a}'|}(\boldsymbol{a} - \boldsymbol{a}')$$

$$|\bar{\boldsymbol{a}} - \boldsymbol{a}'| = |\boldsymbol{a} - \boldsymbol{a}'| + \frac{\boldsymbol{r}}{|\boldsymbol{a} - \boldsymbol{a}'|}(\boldsymbol{a} - \boldsymbol{a}')$$

$$|\bar{\boldsymbol{a}} - \boldsymbol{a}'| = |\boldsymbol{a} - \boldsymbol{a}'| + \boldsymbol{r} > r.$$

Because a' is statistical cluster point of the biquadratic sequence spaces x, by theorem (2.4) this inequality implies that $\bar{a} \in st - LIM^r x$. This contradicts the fact $|\bar{a} - a| = r$ and $st - LIM^r x = \overline{B_r}(a)$. Therefore, a is the unique statistical cluster point of the biquadratic sequence x. Hence the statistical cluster point of statistically analytic biquadratic sequence is unique, then the biquadratic sequence spaces x is statistically convergent to a.

Theorem 3.11: (a) If $c \in \Gamma_x$ then $st - LIM^r x \subseteq \overline{B_r}(c)$.
(b) $st - LIM^r x = \bigcap_{c \in \Gamma_x} \overline{B_r}(c) = \{a \in R^4 : \Gamma_x \subseteq \overline{B_r}(a)\}$.

Proof. (a) Assume that $a \in st - LIM^r x$ and $c \in \Gamma_x$. Then by theorem 3.4, we have

$$|a - c| \leq r;$$

otherwise, we get

$$\delta(\{(m,n,k,l) \in N^4 : |x_{mnkl} - [(1-\lambda)y_1 + \lambda y_2]| \geq r + \epsilon\}) \neq 0.$$

For $\epsilon = \frac{|a-c|-r}{3}$. This contradicts the fact $a \in st - LIM^r x$.

(b) (3.6) $\quad st - LIM^r x \subseteq \bigcap_{c \in \Gamma_x} \overline{B_r}(c)$

Now assume that $y \in \bigcap_{c \in \Gamma_x} \overline{B_r}(c)$. Then we have

$$|y - c| \leq r$$

For all $c \in \Gamma_x$, which is equivalent to $\Gamma_x \subseteq \overline{B_r}(y)$, i.e,

(3.7) $\bigcap_{c \in \Gamma_x} \overline{B_r}(c) \subseteq \{a \in R : \Gamma_x \subseteq \overline{B_r}(a)\}$.

Now let $y \notin st - LIM^r x$. Then there exists an $\epsilon > 0$ such that

$$\delta(\{(m,n,k,l) \in N^4 : |a - y| \geq r + \epsilon\}) \neq 0,$$

The existence of a statistical cluster point c of the biquadratic sequence spaces x with $|y - c| \geq r + \epsilon$, i.e., $\Gamma_x \subseteq \overline{B_r}(y)$ and $y \notin \{a \in R : \Gamma_x \subseteq \overline{B_r}(a)\}$.

Hence $y \in st - LIM^r x$ follows from $y \in \{a \in R : \Gamma_x \subseteq$ *Bra*, i.e.,

$$\{a \in R : \Gamma_x \subseteq \overline{B_r}(a)\} \subseteq st - LIM^r x.$$

Therefore the inclusion (3.7)- (3.8) ensure that (3.6) holds .

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