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Some New Applications of Sumudu Transform for Solution of Linear Volterra Type Integral Equations

Eltaib M Abd Elmohmoud

Mathematics Department, Faculty of Sciences and Arts –Alkamil, University of Jeddah, Jeddah, Saudi Arabia

Abstract: ***The applications of Volterra type integral equations in science and different fields utilizing integral equations to model physical eventsis not a novel idea. Similar to this, a lot of research is being done to develop precise and effective integral equation solution techniques. A few of Adomian Decomposition Method (ADM), Variational Iteration Method (VIM), and Method of Successive Approximation are notable techniques the Laplace transform method, the Galerkin method, and more. This study is focused on illustrating the usage of the Sumudu transform to solve linear Volterra integral equations, which also include equations of the convolutional kind as a single equational system. The chosen issues can be found in the literature. Using diverse analytical, semi-analytical, and numerical techniques. Techniques: Results from the use of the Sumudu transform have been compared to answers gained from other well-known.***

Keywords: Sumudu Transform, Linear Volterra Integral Equation, Adomian Decomposition Method, Method of Successive Approximation, Laplace Sumudu Duality, LSD

1.Introduction

Finance, the physical sciences, and other fields all use integral equations. Water waves, scattering issues, and diffraction problems I used integral equations to model [1]. The unidentified function that is necessary to be determined is represented by the symbol $v(y)$, and it appears with the integration symbol.

The limits of integration are $f(y)$ and $m(y)$, while the "kernel" function is $l(y,r)$. Integral equations have a great introduction in [1], along with solutions techniques. An integral equation's general form is as follows:

$$v(y) = g(y) + \mu \int_{m(x)}^{f(x)} l(y,r)\, v(r) dr \qquad (1)$$

As opposed to an integral equation, an integrato-differential equation contains the unidentified function $v(y)$, which is displayed under the integral sign, and it also regular derivative of an unidentified function. the equation for the integral differential, The general form is as follows:

$$v^n(y) = g(y) + \mu \int_{m(x)}^{f(x)} l(y,r)\, v(r) dr \qquad (2)$$

Two or more equations with two or more variables that must be determined make up the system of integral or integralo-differential equations. Volterra a system of equations including at least one integral, integro-differential, and associated term contains at integration limit of at least one variable. a lot of mathematical methods for locating analytical (exact), appropriate analytical, and numerical integral equations' answers. Techniques like the Adomian Decomposition Method (ADM), Homotopy Perturbation Method (HPM), Homotopy Analysis Method (HAM), and Variational Iteration Method (VIE), among others, are only a few of the available ones. Integral equations of various sorts and classes are also solved using transformation techniques such as the Sumudu transform and Laplace transform, among others.

All across the world, though, new techniques are always being investigated. Sumudu transform was first presented by [2], and there is increasing interest in Researchers are investigating potential uses for the aforementioned transformation technique. Sumudu transform method has been used to solve many mathematical issues. However, in terms of discovering new uses, there are a lot of promise is present. Earlier integral equations of the Volterra type solved utilizing one of the many mathematical techniques that are available. In this essay Sumudu transform has some fresh applications that have been talked about to discover analytical Convolution is a part of the solution of linear Volterra type integral equations type and an equationary system. The issues chosen for demonstration Applications of the Sumudu transform are those that have not yet been solved utilizing gestated conversion process. The conclusions drawn from the analyses following the use of Results from the Sumudu transform have been compared to those from semi-analytic techniques, such as ADM (which have only been allowed in the first (4 iterations total). The findings demonstrate the Sumudu transform method's precision and ease of use, as well as witness to its compatibility with the Laplace transform.

2.Literature Review

Has studied integral-differential equations involving the bulge function. Trapezoidal rule has been used to generate [3] a numerical solution. For locating Taylor series, the Sumudu and inverse Sumudu transforms, and the precise answer.

Theorems of expansion and convolution have been applied. Through illustrations, it has it has been demonstrated that the roughly obtained answers by the trapezoidal rule are in good agreement with solutions that were precisely calculated using the transformation method. [4] carried out a comparison of the Adomian Decomposition Method Sumudu and (ADM) change. Linear partial differential equations with constant coefficients have been solved using both techniques. Method of Sumudu transform has been applied by [5] to

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solve linear integral-differential problems with constant coefficients. Basic characteristics of the Sumudu transform have been covered by [6] and an extensive list of functions using the Sumudu transform is available. Additionally, broader shift theorems have been introduced. A complex inverse Sumudu was triggered by the Laplace- Sumudu Duality (LSD).

As a Bromwich contour integral formula, transform. [7] Conducted useful research formulas used in differential equations for differentiating integral transforms whose coefficients are changeable. Laplace, and have been tested as transforms.

Sumudu and Elzaki. Additionally, it has been asserted that suggested formulas can be used with practically all equations. To address initial value issues that sometimes arise in control engineering, [8] proposed shifting theorems for the Sumudu transform. The Time shifting and the u-shifting theorem make up the suggested theorems.

The evidence is contrasted to precedent-setting ones. Romberg technique was developed for solving second-class linear Volterra integral equations. Examples given in the study demonstrate the advantages of the Romberg approach over the quadrature method. [10] employed the differential transform technique to resolve Volterra integral equations using separable kernels. The approximate solution has been determined as a sequence whose terms are simple to calculate. Exact answers for both linear and nonlinear equations. Equations that are integral have been presented. Results demonstrate the validity of the using a differential transform. A method for collocation has been given by [11].

For the second type of linear Volterra integral equation utilizing the Sinc basis functions. There are presented approximate solutions and auxiliary basis functions.

Satisfy four unique boundary requirements. A set of numerical results has been included to demonstrate the method's effectiveness and precision. [12] Volterra's solution integral Using the Sumudu method, second-kind integral equations of the convolution type transform. Using the Sumudu transform, integro-differential equations are solved [13] has discussed this. A first-order system of two equations is solved numerically.

Volterra integro-differential equations resulting from the theory of the ultimate ruin have been [14] discusses this. Existence and Uniqueness of Volterra Integral Solution [15] has researched equations. Finite Difference Method was utilized by [16] for Linear Volterra Integral Equations with Smooth Solutions. Numerical Remedies for Galerkin technique for Volterra integral equations with Hermite polynomials having been talked about by [17].

3. Mathematical Foundations of Sumudu Transform

Sumudu transform is a variation of the conventional Fourier Integral.

Sumudu transform is defined by definition for a given set A as

$$B = \left\{ f(t) | \exists M, \tau_1, \text{and} \backslash or\ \tau_2 > 0\ such\ that\ |f(t)| < Me{-}t\tau j, if\ t \in {-}1j \times [0,\infty) \right. \tag{3}$$

For the given function M should be finite however τ_1 and τ_2 may be finite or infinite. Sumudu transform is denoted by $S(.)$ and is given as

$$G(u) = S[f(t)] = \frac{1}{u}\int_0^\infty f(t)e^{-t/u}dt, t \geq 0, \tau_1 \leq u \leq \tau_2 \tag{4}$$

Here $G(u)$Sumudu transform of integral function $f(t)$.

Assumed to exist is the integral shown in Equation (4).

Let $f(t) = 1$

$$G(u) = S[1] = \frac{1}{u}\int_0^\infty 1. e^{-t/u}dt = 1 \tag{5}$$

Let $f(t) = t$

$$G(u) = S[t] = \frac{1}{u}\int_0^\infty t. e^{-t/u}dt = u \tag{6}$$

For the nth order

$$S[t^n] = n!\, u^n \tag{7}$$

3.1. Sumudu Transform of Common Functions

Sumudu transform of some common functions is given as under Sumudu transform of exponential function

$$G(u) = S[e^{at}] = \frac{1}{u}\int_0^\infty e^{at}. e^{-t/u}dt = \frac{1}{1-au} \tag{8}$$

Sumudu transform of Sin function

$$S\left[\frac{1}{a}\sin(at)\right] = \frac{u}{1+u^2a^2} \tag{9}$$

Sumudu transform of Cosine function

$$S[\cos at] = \frac{1}{1+u^2a^2} \tag{10}$$

Sumudu transform of Cosine hyperbolic function

$$S[\cosh at] = \frac{1}{1-u^2a^2} \tag{11}$$

Similarly, Sumudu transform for derivatives of a function can be given as

$$S[f'(t)] = \frac{G(u)}{u} - \frac{f(0)}{u} \tag{12}$$

$$S[f''(t)] = \frac{1}{u^2}G(u) - \frac{1}{u^2}f(0) - \frac{1}{u}f'(0) \tag{13}$$

$$S[f^n(t)] = u^{-n}\left[G(u) - \sum_{k=0}^{n-1} u^k f^{(k)}(0)\right] \tag{14}$$

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3.2. Laplace Sumudu Duality (LSD)

For function $f(t)$ which belongs to set B

$$f(t) \in B = \left\{ f(t) | \exists M, \tau_1, \text{and} \backslash or\ \tau_2 > 0\ such\ that\ |f(t)| < Met\tau j, if\ t \in -1j \times [0,\infty) \right. \quad (15)$$

Since Laplace transform is given as

$$F(r) = L\big(f(t)\big) = \int_0^\infty e^{-rt} f(t)dt \quad (16)$$

You can provide an interconversion between the Laplace and Sumudu transform by

$$G(w) = \frac{F\left(\frac{1}{w}\right)}{w} \quad (17)$$

$$F(s) = \frac{G_s\left(\frac{1}{s}\right)}{s} \quad (18)$$

4. Examples

Examples 4.1

$$v(x) = 4x + 2x^2 - \int_0^x v(t)dt \quad (19)$$

This problem has been taken from [18].

Taking Sumudu transform on both sides of Equation (19)

$$S[v(x)] = 4[S(x)] + 2[S(x^2)] - S\left[\int_0^x v(t)dt\right] \quad (20)$$

$$G(u) = 4u + 2(2!\,u^2) - uG(u) \quad (21)$$

$$G(u) + uG(u) = 4u + 4u^2 \quad (22)$$

$$G(u) = \frac{4u(1+u)}{(1+u)} \quad (23)$$

After being streamlined

$$G(u) = 4u \quad (24)$$

Taking Inverse Sumudu transform on both sides of Equation (24), we get,

$$v(x) = 4x \quad (25)$$

As a result, the necessary analytical answer is found.

Example 4.2

$$v(x) = x^2 + \frac{1}{12}x^4 - \int_0^x (t-x)v(t))dt \quad (26)$$

This problem has been discussed by [18].

Taking derivatives and applying Leibniz rule to equation (26)

$$v'(x) = 2x + \frac{1}{3}x^3 - \int_0^x v(t)dt \quad (27)$$

$$v''(x) = 2 + x^2 - v(x) \quad (28)$$

$$v''(x) + v(x) = 2 + x^2, \text{ with } v(0) = 0, v'(0) = 0 \quad (29)$$

Since,

$$S[v''(x)] = \frac{1}{u^2}G(u) - \frac{1}{u^2}f(0) - \frac{1}{u}f'(0) \quad (30)$$

Plugging initial values into the result, we get

$$S[v''(x)] = \frac{1}{u^2}G(u) \quad (31)$$

Consequently, the given equation's Sumudu transform will be

$$S\,[v''(x)] + S\,[v(x)] = S\,[2] + S\,[x^2] \quad (32)$$

$$\frac{1}{u^2}G(u) + G(u) = 2 + 2u^2 \quad (33)$$

$$G(u) + u^2G(u) = 2u^2 + 2u^4 \quad (34)$$

$$G(u) = \frac{2u^2\,(1+u^2)}{1+u^2} \quad (35)$$

After being streamlined

$$G(u) = 2u^2 \quad (36)$$

Taking Inverse Sumudu transform on both sides of Equation (36), we get,

$$v(x) = x^2 \quad (37)$$

As a result, the necessary analytical answer is found.

Example 4.3

Consider Volterra Integral equation of Convolution type

$$v(x) = x + \int_0^x (x-t)v(t))dt \quad (38)$$

$$S[v(x)] = S[x] + S\left[\int_0^x (x-t)v(t))dt\right] \quad (39)$$

$$G(u) = 2 + u^2G(u) \quad (40)$$

$$G(u) - u^2G(u) = 2 \quad (41)$$

$$G(u) = \frac{2}{1-u^2} \quad (42)$$

Taking Inverse Sumudu transform on both sides of Equation (36), we get,

$$v(x) = 2\cos hx \quad (43)$$

As a result, the necessary analytical answer is found.

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Example 4.4

The solution to this problem was found [11] by employing Sinc basis functions,

$$v(x) = x - x^2 + \frac{x^3}{6} + \frac{x^4}{12} - \int_0^x (t-x)v(t))dt\,, 0 \leq x \leq 1 \quad (44)$$

Taking Inverse Sumudu transform on both sides of Equation (44), we get,

$$S[v(x)] = S[x] - S[x^2] + \frac{1}{6}S\,[x^3] - \frac{1}{12}S\,[x^4] - S[\int_0^x (t-x)v(t))dt] \quad (45)$$

$$G(u) = u - 2u^2 - u^3 - 2u^4 - u^2G(u) \quad (46)$$

$$G(u) = u - 2u^2 \quad (47)$$

Taking Inverse Sumudu transform on both sides,

$v(x) = x - x^2$ (48)

Hence, required analytic solution is obtain.

5.Result

As shown by the numerous examples in section 4, it has been demonstrated that the Sumudu transform is simple to use to locate an analytical solution of Volterra type integral equations. We will present the answer to domain 0 in this part.

In order to compare semi-analytic techniques (ADM and MSA with zero as initial guess) solutions (restricted to the first four iterations), have also given. Conclusions from semi-analytic methods will probably converge to analytical results.

Solution if more iterations are done, yet the same thing will cost more money.

Compute-intensive work. Mathematica version 9 has been used for all computational tasks.

$v(x) = 4x$ is the analytical solution to Example 4.1, which has also been achieved by employing the Sumudu transform while the error for LDM and SDM once restricted to the first four iterations is evident (Table 1).

The analytical solution for Example 4.2 is $v(u) = 2u^2$, which was likewise reached by the use of the Sumudu transform, whereas the error for LDM and SDM once limited to the first four iterations is evident (Table 2).

The analytical solution for Example 4.3 is $v(x) = 2\cos hx$, which was also accomplished by using the Sumudu transform. LDM accuracy is marginally superior to SDM evident (Table 3).

The Sumudu transform was used to arrive at the analytical solution for Example 4.4, which is $v(x) = x - x^2$. Table 4 displays the error for LDM and SDM after the first four iterations were excluded. Table 4

Table 1: Solution of Example 4.1 using different methods for approximate analytic method

x	LDM	SDM	Exact
0	0.000000	0.000000	0.000000
0.1	0.400000	0.400000	0.400000
0.2	0.799989	0.800000	0.800000
0.3	1.199919	1.200000	1.200000
0.4	1.599659	1.600000	1.600000
0.5	1.998958	2.000000	2.000000
0.6	2.397408	2.400000	2.400000
0.7	2.794398	2.800000	2.800000
0.8	3.189077	3.200000	3.200000
0.9	3.580317	3.600000	3.600000
1	3.966667	4.000000	4.000000

Table 2: Solution of Example 4.2 using different methods for approximate analytic method

x	LDM	SDM	Exact
0	0.000000	0	0.000000
0.1	0.010000	0.01	0.010000
0.2	0.040002	0.04	0.040000
0.3	0.090027	0.09	0.090000
0.4	0.160146	0.16	0.160000
0.5	0.250531	0.25	0.250000
0.6	0.361484	0.36	0.360000
0.7	0.493435	0.49	0.490000
0.8	0.646858	0.64	0.640000
0.9	0.822055	0.81	0.810000
1.	1.018750	1.	1.000000

Table 3: Solution of Example 4.3 using different methods for approximate analytic method

x	LDM	SDM	Exact
0	2.000000	2	2.000000
0.1	2.010008	2.01001	2.010008
0.2	2.040134	2.04013	2.040134
0.3	2.090677	2.09068	2.090677
0.4	2.162145	2.16214	2.162145
0.5	2.255252	2.25525	2.255252
0.6	2.370930	2.37093	2.370930
0.7	2.510335	2.51034	2.510338
0.8	2.674862	2.67487	2.674870
0.9	2.866151	2.86617	2.866173
1.	3.086111	3.08616	3.086161

Table 4: Solution of Example 4.4 using different methods for approximate analytic method

x	LDM	SDM	Exact
0	0.000000	0.000000	0.000000
0.1	0.090001	0.090000	0.090000
0.2	0.160021	0.160000	0.160000
0.3	0.210147	0.210000	0.210000
0.4	0.240560	0.240000	0.240000
0.5	0.251516	0.250000	0.250000
0.6	0.243278	0.240000	0.240000
0.7	0.215998	0.210000	0.210000
0.8	0.169574	0.160000	0.160000
0.9	0.103489	0.090000	0.090000
1.	0.016667	0.000000	0.000000

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6.Conclusion

In this study, linear integral equations of the Volterra type were effectively solved analytically using the Sumudu transform. Sumudu transform has been proven to be a reliable, compatible alternative to other well-known analytical techniques. Additionally, it is not only accurate but frequently simpler to use when compared to well-known semi/approximate analytical methods as the Adomian Decomposition Method and the Method of Successive Approximations. This claim is supported by study findings. The study can be furthered by talking about the use of the Sumudu transform for linear Volterra type integral equations with separate solutions.

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