Energy of Gravitational Waves in Black Hole Collisions

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Abstract: ***Gravitational energy of the black hole is derived from equations of relativity and energy equations of the black hole. The equation is derived by considering the escape of a photon from the black hole. When observed at infinity, the mass of the black hole beyond its horizon is found to be twice its mass for a non-rotating black hole. The equation also infers that for a photon to escape the gravitational pull of the black hole, it expends half of its energy. The effect of the recent detection of Gravitational waves, from collision of black holes, is visualized with our results, we were able to relate with data obtained from LIGO. A simple but thorough analysis is made. This work shows that the theoretical calculation is comparable with the observed data.***

Keywords: Black hole, Mass energy, Relativity, Photon, Gravitational waves

1. Introduction

What is a Black Hole? It is a region in space, where the gravitational field is so strong that not even light (i.e., Electromagnetic Waves), which are having the highest measurable speed known, cannot escape. Black holes are not much easy to envisage as other planets or stars, we cannot physically see a black hole, since no waves will escape from its horizon and reflect.

Such entities are formed during the collapse of a massive star with mass greater than at least 5 times the solar mass. It is presumed that there exists a super massive black hole in the center of most of the galaxies.

Why do we study black holes? Black holes are one the most mysterious object we can find in our universe, encountering a primordial black hole may possibly unravel the mysteries of the universe, its creation, and evolution with more certainty. Therefore, it is an important discovery of space research.

How are these black holes detected? Black hole emits (or reflects) no light, therefore, it cannot be observed or imaged directly. Some black holes can be detected from their strong gravity, which attracts other matters. These black holes may have a disk-shaped collection of matter (dust and gases) around them spiraling into the black hole, at the same time emitting a huge amount of X-Rays or other high energy radiations.

Black holes can also be detected when it exists as a binary star system. When a black hole and a star are close together, high-energy light is ejected (This is due to the gravitational pull of the black hole swallowing the matter of the star and settles as its accretion disk).

Special Theory of Relativity: Einstein derived his famous $E = mc^2$, called the law of mass-energy equivalence, from the principles of special theory. This indicated that mass of a body is a measure of its energy content.

Schwarzschild radius: The physical radius of event horizon of a black hole. This equation is derived from Einstein's equation [2].Event Horizon is the no escape zone of the black hole, once a particle is within this zone it cannot escape the gravitational pull of the black hole.

$$R_g = 2GM/c^2$$

where,
G is the universal gravitational constant, 6.673 x 10^{-11} Nm^2kg^{-2}
M is mass of the object
c is speed of light, 3 x 10^8

Relativistic Mass: According to Einstein's Theory of relativity, when an object is at motion its mass is converted to energy, and thus the rest mass is different from its actual mass in motion.

$$\boldsymbol{m = \frac{m_0}{\sqrt{1 - \frac{v^2}{c^2}}} = m_0\gamma}$$

m_0 is the rest mass, v is the velocity at which the mass travels and $\boldsymbol{\gamma} = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$, is the Lorentz Factor.

Hawking Radiation / Black Hole Evaporation

Any matter or particles will lose its energy as well as its mass according to Einstein's theory of mass-energy equivalence. In this case, black hole loses its energy by radiation. This was predicted by Dr. Stephen W Hawking in1974, and the phenomenon was named as Hawking Radiation. It is a black-body radiation that is predicted to be released by black holes, due to quantum effects near the Event Horizon [5].

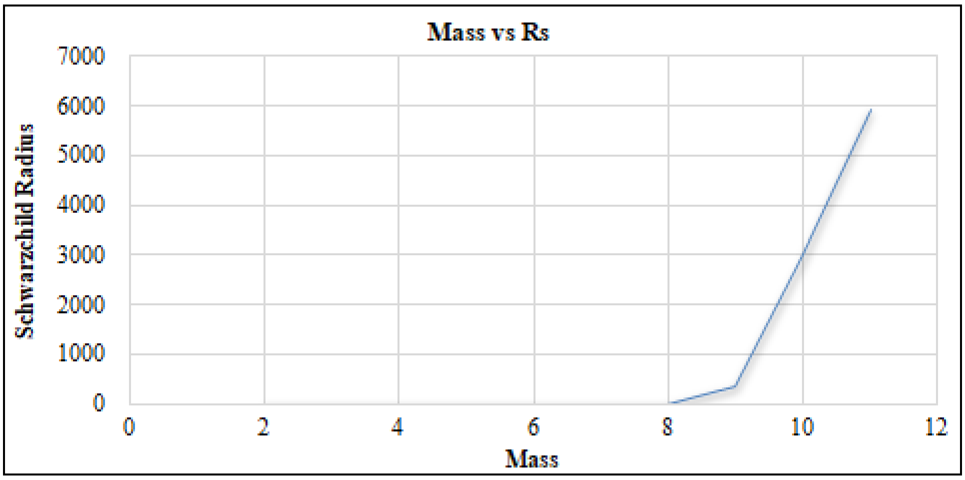
By analyzing the thermal equilibrium, with extreme red shifting effects close to event horizon, a pair of virtual particles arises beyond the event horizon due to some quantum effects. At extremely near to the event horizon a pair of photons is produced. One may pass beyond the event horizon, while other may escapes into the Universe. Due to extreme red shifting, escaping photon almost tears it, and makes a partner to carry a negative energy, where it remains trapped by the black hole. The escaping photons adds equal amount of positive energy which escapes to the universe. Thus, the energy of the black hole is reduced. Hawking radiation thus reduces the energy and rotational energy of a black hole. This is also called as Black hole evaporation [3]. As black hole continuously loses its energy, it shrinks and eventually completely vanishes i.e., evaporated.

Creation from Death

In the final stage of a star, certain mass is deposited in a region of space. If the mass is more than five to eight times the mass of sun ($M_{\odot} = 1.989 \times 10^{30}\ kg$), the fabric of these region (space-time fabric) begins to bend to infinite. This bending or infinite dip is so called as a black hole.

The gravitation radius (R_s) below which the dip between the particles is irreversible is called as Schwarzs child radius [2]. Radius of the Event Horizon is given by R_s.

Graph (1) is a graph for mass vs Schwarzschild radius, and we can infer that heavier the mass, the larger its horizon radius will be. The size of the black hole for varied sizes of objects is seen, as the size of black hole increases for objects of heavier mass.



Graph 1: Mass vs Radius Curve

If we want an infinite dip of electron,

$$m_e = 9.1 \times 10^{-31} Kg$$

Then the electron must be compressed to a radius of,

$$R_s = \frac{2 \times 6.673 \times 10^{-11} \times 9.1 \times 10^{-31}}{9 \times 10^{16}} = 1.3 \times 10^{-57} m$$

This value is indefinite in a region of space-time because it violates the value of **Planck length [5].**

Planck length is the smallest length at which the space can be defined, its value is $1.6 \times 10^{-36}\ m$.

Similarly, for high scale masses the radius R_s varies.

Energy of the Black Hole

Black hole continuously losses its energy (in form of heat), so it also losses it is mass due to the escaping of particles.
This is related by,

$$E = mc^2$$

When a photon of particular energy is emitted outside the horizon of a black hole, as it reaches infinity its energy becomes zero. It can be said that the entire energy will be used by the photon just to escape the gravitational pull of the black hole [1].

Suppose if the photon comes due to the annihilation of a particle of mass m near the horizon, it means the total mass is used to make the photon to escape from the black hole. Hence the energy required to escape a mass m outside the horizon region to an infinity distance is also mc^2.

Pair Production is a theoretical concept that a particle and its antiparticle are created without supply of external energy, and it gets annihilated, therefore the energy remains conserved. We assume such occurrence occurs just near the event horizon and one of the two (either particle or its antiparticle) particle falls into the black hole and other escapes the black hole's gravitational pull.

Let us consider a mass m is ejected from the horizon, it travels to infinite distance very slowly by an external agent. Therefore, no kinetic energy is produced in this process and the energy required is same. It travels to infinity as a free mass. Let us continue this situation for another particle produced outside the horizon packed up with sufficient energy to escape to infinity and end up as a free particle of mass.

The total energy for this process is

$$mc^2 + mc^2 = 2mc^2$$

As a result, entire black hole lose energy of $2mc^2$ for each particle of mass m released at horizon an observed at infinity and independent of its mass.

Considering M as mass of the black hole within its horizon whose energy is given by,

$$E = \frac{1}{2}Mc^2 \text{ (At infinity)}$$

If a photon is emitted at a coordinate (r) with energy E_r and the energy at infinity be E_∞.

The relativistic energy,

$$E_\infty = E_r \sqrt{(1 - \frac{v^2}{c^2})} = Mc^2$$

Using Schwarzschild radius,

$$r = \frac{2GM}{c^2} \Rightarrow \quad E_\infty = E_r \sqrt{(1 - \frac{2GM}{rc^2})}$$

$$\therefore\Rightarrow E_\infty = E_r \sqrt{(1 - \frac{2GM}{rc^2})}$$

Let the Above Equation be ((1)).

Difference between the energy of photon at two locations,

$$E_\infty - E_r = E_r - E_r(\sqrt{(1 - \frac{2GM}{rc^2})}$$

$$\Rightarrow E_\infty - E_r = E_r(1 - (\sqrt{(1 - \frac{2GM}{rc^2})}) \qquad ((2))$$

Equation ((2)) is the measure of change of Gravitational potential energy of black hole as a function of coordinates (r).

Let us introduce a function f(r), interpolating between the surface of black hole and infinity, hence energy of black hole also becomes a function of co-ordinate (r),

E_r be total energy i.e $(E_r - E_\infty)$

$$\Rightarrow \mathbf{E(r)} = \mathbf{f(r)}\{1 - \sqrt{1 - \frac{2GM}{rc^2}}\} \qquad ((3))$$

To find f(r),

1) The total energy E(r) must be positive between R_s and ∞.
2) E(r) decreases smoothly between R_s and ∞, i.e $\frac{dE}{dR}$ is always negative.
3) At large distance, E(r) approaches asymptotic values, I.e.,$\frac{dE}{dR} \simeq 0$ at very large

References
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Distance

((3)) → at long distance becomes

$$\frac{dF(r)}{dr} = \frac{1}{r} f(r)$$

Rewriting the above equation,

$$\frac{dF(r)}{f(r)} = \frac{dr}{r}$$

Solving,
log(f(r)) = log(r) + log(C)
log(f(r)) = log (r.C) (C is constant)
f(r) = r. C (Taking Exponential on both sides) **((4))**

To find C, at large distance, using Maclaurin series,

$$\sqrt{1 - x} = 1 - \frac{1}{2}x - \frac{1}{8}x^2 - \cdots.$$

$$((3)) \rightarrow \sqrt{1 - \frac{2GM}{rc^2}} = 1 - \frac{1}{2}\left(\frac{2GM}{rc^2}\right) \simeq 1 - \frac{GM}{rc^2} \quad ((5))$$

((5)) equals Mc^2 as it observed by observer at distance,

$E(r) = f(r)\frac{GM}{rc^2} \rightarrow Mc^2$ as $r \rightarrow \infty$

$$f(r) = \frac{rc^4}{G} \qquad ((6))$$

Hence overall energy of black hole

$$\mathbf{E(r)} = \frac{rc^4}{G}\left[1 - \sqrt{1 - \frac{2GM}{rc^2}}\right] \qquad ((7))$$

$$\mathbf{E(r} = R_s) = 2Mc^2$$

(Case) When r $=R_s \rightarrow \frac{2GM}{rc^2}$

Hence, we arrived at desired equation.

Application in G-Wave

Gravitational waves are the disturbance (or ripples) in space-time curvature. These waves are generated by accelerated masses and are propagates as waves. These waves travel at the speed of light. Existence of Gravitational waves was predicted by Albert Einstein in his general theory of relativity mathematically and in 2016 gravitational waves were physically observed by LIGO (Laser Interferometer Gravitational-Wave Observatory). These G-waves are presumed to carry information about their origins, as well as clues to nature of gravity itself.

Every object produces gravitational waves, but only massive catastrophic events produce significant, detectable, ripples. Events like supernova, black hole collision. Etc. are main sources of gravitational waves. When two gigantic masses collide each other, there is loss of mass which in turn we experienced in the form of energy.

Let us consider a collision of two Black holes of different masses. After collision there is a formation of single black hole. The newly formed black hole has a mass greater than individual masses but less than the combined mass of two black holes. Thus, the disappeared mass is converted and blown out to the cosmos in a form of Gravitational Waves. The energy of the gravitational waves depends on the mass disappeared during collisions.

Let M_1 and M_2 be the mass of two different black holes and M_R be the resultant mass after collision.

$M_1+M_2 = M_R$(ideal collision)

But, $M_1+M_2 > M_R$

Also, the difference in two different masses results in the form of G-Wave.

i.e.,$M_1 + M_2 - M_R = M_G$

M_G be the disappeared mass which in turn converted into energy which is sufficient enough to disturb the Space-Time fabrics i.e., Gravity. Hence, we were able to detect those disturbances in the form of G-waves.

In 2015 the first G-wave (GW150914) is observed in LIGO and VIRGO. This wave is a result of merging of two black holes of mass of 36 and 29 solar masses. The energy output is

$$3^{+0.5}_{-0.5}\ M_{sol}C^2=5.4x10^{47}J.$$

By calculation,
(36-29) = 3(solar mass) = $3M_{sun}c^2$
= 5.1×10^{47}J

We can clearly see that the calculated and observed energy values are almost identical.

The following data [6] was collected from LIGO website (https://www.ligo.caltech.edu/page/ligo-data) are the confident detections from "GWTC-1: A Gravitational-Wave Transient Catalog of Compact Binary Mergers Observed by LIGO and Virgo during the First and Second Observing Runs."

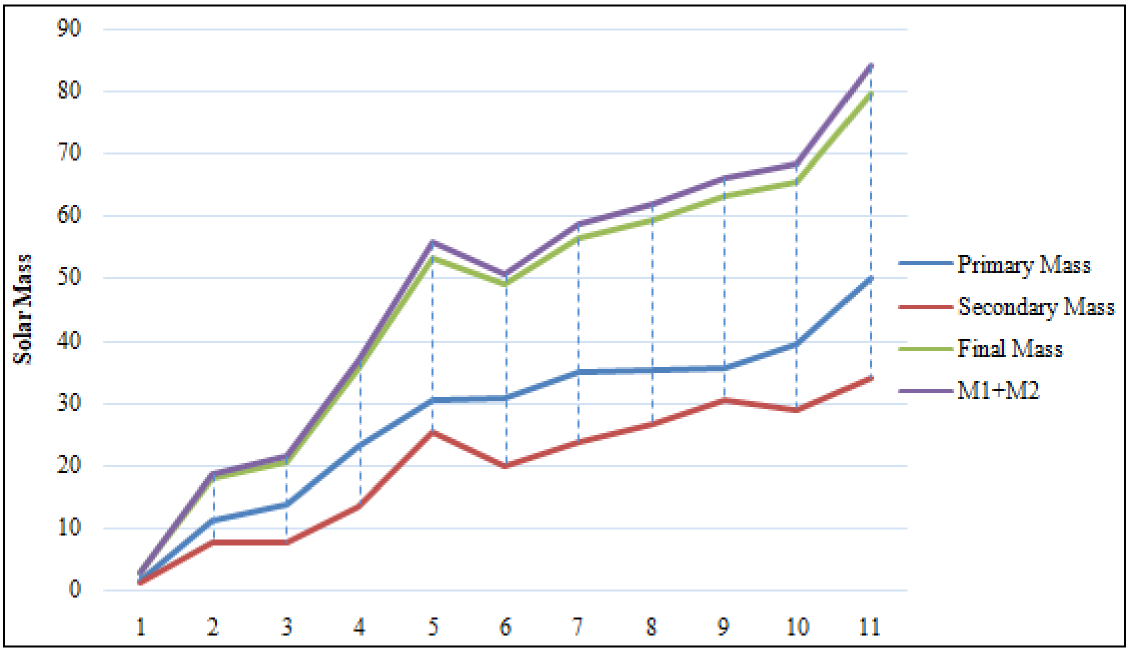
Table 1: Data Generated from generated from LIGO where all the observations of gravitational waves, until date can be found

Event	Primary Mass	Secondary Mass	Final Mass	Luminosity Distance	Radiated Energy ($M_{sun}C^2$)	Calculated Energy ($M_{sun}C^2$)
GW150914	35.6	30.6	63.1	440	3.1±0.4	**3.1**
GW151012	23.2	13.6	35.6	1080	1.6±0.5	**1.2**
GW151226	13.7	7.7	20.5	450	1.0±0.2	**0.9**
GW170104	30.8	20.0	48.9	990	2.2±0.5	**1.9**
GW170608	11.0	7.6	17.8	320	0.9±0.1	**0.8**
GW170729	50.2	34.0	79.5	2840	4.8±1.7	**4.7**
GW170809	35.0	23.8	56.3	1030	2.7±0.4	**2.5**
GW170814	30.6	25.2	53.2	600	2.7±0.4	**2.6**
GW170817	1.46	1.27	≤ 2.8	40	≥ 0.04	**≥ 0**
GW170818	35.4	26.7	59.4	1060	2.7±0.5	**2.7**
GW170823	39.5	29.0	65.4	1940	3.3±1	**3.1**

Source: https://www.gw-openscience.org/catalog/GWTC-1-confident/html/

The collected data from LIGO with additional column of calculated energy, where we calculated the energy value using the given masses. The value of energy is given in term of Solar Mass Energy (Mass of Sun times c^2).

Here we can see that the calculated value and observed values are almost comparable and identical. We can hereby conclude that almost half of the energy of the particle is lost just to escape the gravitational attraction of the black hole, and what is observe is Mc^2 instead of $2Mc^2$.



Graph 2: Variance in individual mass and combined mass

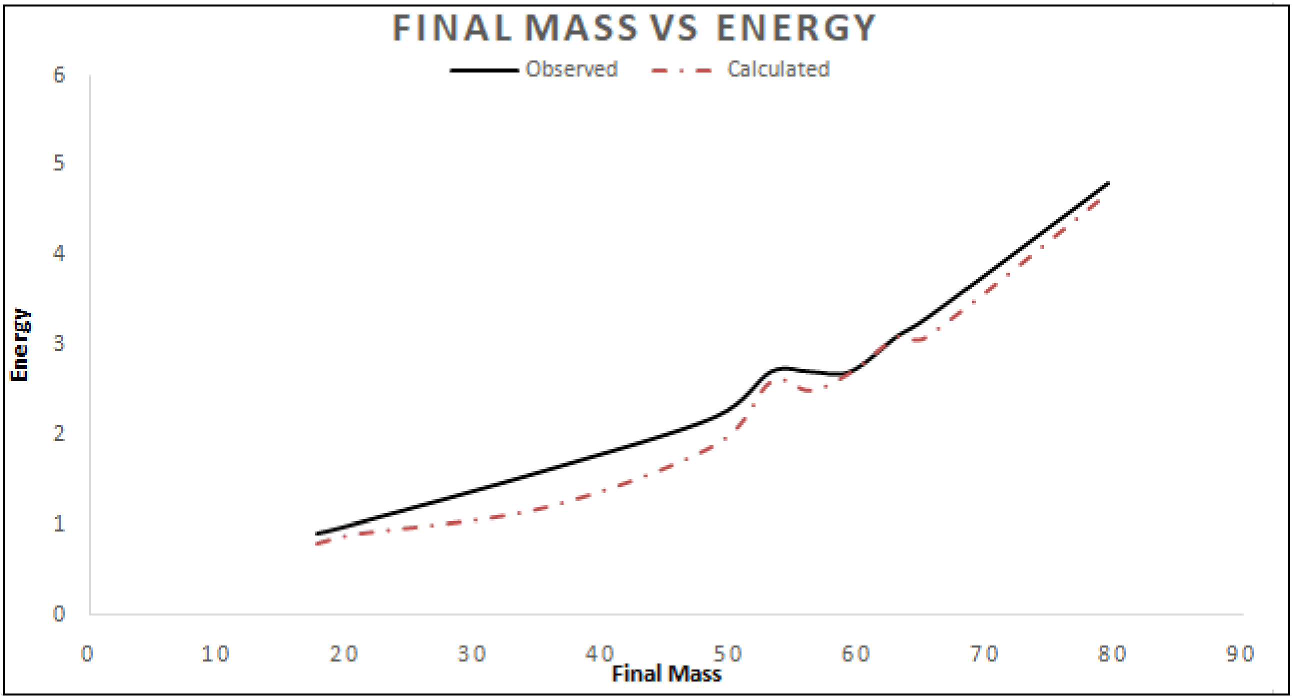
When two black holes are merged, the final mass of the black hole is greater than that of individual masses and lesser than combined mass; this can be seen in Graph 2.

A graph is plotted between the final mass of the black hole and energy values.

Graph 3is the plot of energies, where the dotted curve is the observed energy and solid line curve denotes the plot for calculated energy.

2. Result and Discussion

Deriving the energy equation and substituting different values of masses of black hole. A comparison is done between collected data and calculated energy value, and it was found to be comparable. Hence, it can be concluded, for **Photon emitted from large stellar bodies, some energy is lost in travel from the event source to infinity distance.**



Graph 3: Plotted between the final mass of the black hole and energy values

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