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On the Diophantine Equations $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=\frac{1}{4}$ and $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}+\frac{1}{t}=\frac{1}{4}$

Dr. Rajive Atri

Associate Professor and Head, Department of Mathematics, C.S.S.S. (PG) College, Machhra, Meerut, India
rajiveatri20[at]gmail.com

Abstract: In this paper, the above Diophantine equations have been discussed for integral solutions.

Keywords: Diophantine equation and integral solution.

1. Introduction

Erdos & Straus (1950) conjectured that for all integers $n \geq 2$, the rational number $\frac{4}{n}$ can be expressed as sum three unit fractions. Thus the conjecture formally states that for every integer $n \geq 4$, there exists positive integers x, y and z such that $\frac{4}{n}=\frac{1}{x}+\frac{1}{y}+\frac{1}{z}$.

Monks, M. & Velingker, A. examined the solutions of the above equation for prime n. **Hari Kishan et. al.** (2011) discussed the Diophantine equations of second and higher degree of the form $3xy = n(x+y)$ and $3xyz = n(xy+yz+zx)$ etc. **Rabago, J. F.T. & Tagle, R.P.** (1913) discussed the area and volume of a certain regular solid and the Diophantine equation $\frac{1}{2}=\frac{1}{x}+\frac{1}{y}+\frac{1}{z}$. **Sander, J.** (1913) discussed the Diophantine equation $\frac{1}{2}=\frac{1}{x}+\frac{1}{y}+\frac{1}{z}$ and obtained solutions of this Diophantine equation.

In this paper, the above problem has been extended for the Diophantine equations $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=\frac{1}{4}$ and $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}+\frac{1}{t}=\frac{1}{4}$.

Analysis:

Case 1: First, the Diophantine equation $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=\frac{1}{4}$ has been discussed. Suppose $x \leq y \leq z$. Then $\frac{1}{x}<\frac{1}{4}$. This implies that $x \geq 5$. Also $\frac{3}{x} \geq \frac{1}{4}$. This implies that $x \leq 12$. Thus $x = \{5,6,7,8,9,10,11,12\}$. Thus there may be the following cases:

For $x = 5$, $\frac{1}{y}+\frac{1}{z}=\frac{1}{20}$, ...(1)

For $x = 6$, $\frac{1}{y}+\frac{1}{z}=\frac{1}{12}$, ...(2)

For $x = 7$, $\frac{1}{y}+\frac{1}{z}=\frac{3}{28}$, ...(3)

For $x = 8$, $\frac{1}{y}+\frac{1}{z}=\frac{1}{8}$, ...(4)

For $x = 9$, $\frac{1}{y}+\frac{1}{z}=\frac{5}{36}$, ...(5)

For $x = 10$, $\frac{1}{y}+\frac{1}{z}=\frac{3}{20}$, ...(6)

For $x = 11$, $\frac{1}{y}+\frac{1}{z}=\frac{7}{44}$, ...(7)

For $x = 12$, $\frac{1}{y}+\frac{1}{z}=\frac{1}{6}$. ...(8)

Equations (1), (2), (4) and (8) imply the following relations:

$(y-20)(z-20) = 400$, ...(9)

$(y-12)(z-12) = 144$, ...(10)

$(y-8)(z-8) = 64$, ...(11)

$(y-6)(z-6) = 36$. ...(12)

From (9), the following cases may be possible:
$(y-20)=1, (z-20)=400; (y-20)=2, (z-20)=200;$
$(y-20)=4, (z-20)=100; (y-20)=5, (z-20)=80;$
$(y-20)=8, (z-20)=50; (y-20)=10, (z-20)=40;$
$(y-20)=20, (z-20)=20.$

Thus we have the following solutions:
$(x,y,z) = (5,21,420), (5,22,220), (5,24,120), (5,25,100), (5,28,70),$
$(5,30,60), (5,40,40).$

From (10), the following cases may be possible:
$(y-12)=1, (z-12)=144; (y-12)=2, (z-12)=72;$
$(y-12)=3, (z-12)=48; (y-12)=4, (z-12)=36;$
$(y-12)=6, (z-12)=24; (y-12)=8, (z-12)=16;$
$(y-12)=12, (z-12)=12.$

Thus we have the following solutions:
$(x,y,z) = (6,13,156), (6,14,84), (6,15,60), (6,16,48), (6,18,36),$
$(6,20,28), (6,24,24).$

From (11), the following cases may be possible:
$(y-8)=1, (z-8)=64; (y-8)=2, (z-8)=32;$
$(y-8)=4, (z-8)=16; (y-8)=8, (z-8)=8.$

Thus we have the following solutions:
$(x,y,z) = (8,9,72), (8,10,40), (8,12,24), (8,16,16),$

From (12), the following cases may be possible:
$(y-6)=1, (z-6)=36; (y-6)=2, (z-6)=18;$
$(y-6)=3, (z-6)=12; (y-6)=4, (z-6)=9;$
$(y-6)=6, (z-6)=6.$

Thus we have the following solutions:
$(x,y,z) = (12,7,42), (12,8,24), (12,9,18), (12,10,15), (12,12,12).$

Similarly the solutions the solutions for $x = 6, 7, 8, 9, 10, 11$ and 12 can be obtained. The solutions can further be obtained by using the equations (3), (5), (6) and (7). Since x,

y and z are symmetric in the Diophantine equation there are $3! = 6$ cases for each solution.

Case 2: Second, the Diophantine equation $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}+\frac{1}{t}=\frac{1}{4}$ has been discussed. Suppose $x \le y \le z \le t$. Then $\frac{1}{x} < \frac{1}{4}$. This implies that $x \ge 5$. Also $\frac{4}{x} \ge \frac{1}{4}$. This implies that $x \le 16$. Therefore $x = \{5,6,7,8,9,10,11,12,13,14,15,16\}$. Thus there may be the following cases:

For $x = 5,\ \frac{1}{y}+\frac{1}{z}+\frac{1}{t}=\frac{1}{20},$ ...(13)

For $x = 6,\ \frac{1}{y}+\frac{1}{z}+\frac{1}{t}=\frac{1}{12},$ ...(14)

For $x = 7,\ \frac{1}{y}+\frac{1}{z}+\frac{1}{t}=\frac{3}{28},$ ...(15)

For $x = 8,\ \frac{1}{y}+\frac{1}{z}+\frac{1}{t}=\frac{1}{8},$ ...(16)

For $x = 9,\ \frac{1}{y}+\frac{1}{z}+\frac{1}{t}=\frac{5}{36},$ ...(17)

For $x = 10,\ \frac{1}{y}+\frac{1}{z}+\frac{1}{t}=\frac{3}{20},$ ...(18)

For $x = 11,\ \frac{1}{y}+\frac{1}{z}+\frac{1}{t}=\frac{7}{44},$ ...(19)

For $x = 12,\ \frac{1}{y}+\frac{1}{z}+\frac{1}{t}=\frac{1}{6},$ ...(20)

For $x = 13,\ \frac{1}{y}+\frac{1}{z}+\frac{1}{t}=\frac{7}{78},$ ...(21)

For $x = 14,\ \frac{1}{y}+\frac{1}{z}+\frac{1}{t}=\frac{2}{21},$ ...(22)

For $x = 15,\ \frac{1}{y}+\frac{1}{z}+\frac{1}{t}=\frac{1}{10},$ ...(23)

For $x = 16,\ \frac{1}{y}+\frac{1}{z}+\frac{1}{t}=\frac{1}{16},$ ...(24)

Equation (13) gives $\frac{1}{y} < \frac{1}{20}$. This implies that $y \ge 21$. Also $\frac{3}{y} \ge \frac{1}{20}$. This implies that $y \le 60$. Therefore $y = \{21, 22, 23, \ldots, 60\}$. Thus there may be the following cases:

For $y = 21,\ \frac{1}{z}+\frac{1}{t}=\frac{1}{420},$ ...(25)

For $y = 22,\ \frac{1}{z}+\frac{1}{t}=\frac{1}{220},$ ...(26)

For $y = 23,\ \frac{1}{z}+\frac{1}{t}=\frac{3}{460},$ ...(27)

For $y = 24,\ \frac{1}{z}+\frac{1}{t}=\frac{1}{120},$ ...(28)

For $y = 28,\ \frac{1}{z}+\frac{1}{t}=\frac{1}{70},$ ...(29)

For $y = 32,\ \frac{1}{z}+\frac{1}{t}=\frac{3}{160},$ ...(30)

For $y = 40,\ \frac{1}{z}+\frac{1}{t}=\frac{1}{40},$ ...(31)

For $y = 56,\ \frac{1}{z}+\frac{1}{t}=\frac{9}{280},$ ...(32)

For $y = 60,\ \frac{1}{z}+\frac{1}{t}=\frac{1}{30},$ ...(33)

Equations (25), (26), (28), (29), (31) and (33) imply the following relations:

$(z-420)(t-420) = 176400,$ ...(34)

$(z-220)(t-220) = 48400,$ ...(35)

$(z-120)(t-120) = 14400,$ ...(36)

$(z-70)(t-70) = 4900,$ ...(37)

$(z-40)(t-40) = 1600,$ ...(38)

$(z-30)(t-30) = 900.$ ...(39)

From equation (37), the following cases may be possible:
$(z-70) = 1, (t-70) = 4900; (z-70) = 2, (t-70) = 2450;$
$(z-70) = 4, (t-70) = 1225; (z-70) = 5, (t-70) = 980;$
$(z-70) = 7, (t-70) = 700; (z-70) = 10, (t-70) = 490;$
$(z-70) = 14, (t-70) = 350; (z-70) = 20, (t-70) = 245;$
$(z-70) = 28, (t-70) = 175; (z-70) = 35, (t-70) = 140;$
$(z-70) = 49, (t-70) = 100; (z-70) = 70, (t-70) = 70.$

Thus we have the following solutions:

$$(x,y,z,t) = (5,28,71,4970), (5,28,72,2520), (5,28,74,1295), (5,28,75,1050), (5,28,77,770), (5,28,80,560), (5,28,84,420), (5,28,90,325), (5,28,98,245), (5,28,105,210), (5,28,119,170), (5,28,140,140).$$

Similarly the other solutions can be obtained for other values of x and y and for relations (25), (26), (29), (31) and (33). Since x, y, z and $t$are symmetric in the Diophantine equation there are $4! = 24$ cases for each solution.

2. Concluding Remarks

Here the two Diophantine equations have been discussed for integral solutions for certain values of variables x and y. More solutions can further be obtained for other values of variables x and y and for other relations in the variables.

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