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On Integrability Conditions of a Framed Algebraic ε-Structure Manifold

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Abstract: ***The generalized para (ε,r) - contact structure manifolds have been defined and studied by the authors in their paper . In this paper we have studied the integrability conditions of a framed algebraic ε -structure manifold. Integrability of distributions has also been studied in this paper.***

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1. Preliminaries

Let M^{n+r} be an $(n+r)$-dimensional differentiable manifold of class C^∞. Suppose there exists on M^{n+r}, a tensor field $f(\neq 0)$ of type (1,1) $r(C^\infty)$ contravariant vector fields ξ^p, $r(C^\infty)$ l-forms n_p and a scalar ε satisfying

$$f^2 = a^2 I - \sqrt{\varepsilon}\sum_{p=1}^{r} n_P \otimes \xi^p \qquad (1.1)$$

'a' being a complex. Also

$$\begin{aligned} &(i) f\xi^p + \sqrt{\varepsilon}\sum_{q=1}^{r}\theta_q^p\,\xi^q = 0 \\ &(ii) n_p of + \sqrt{\varepsilon}\sum_{q=1}^{r}\theta_p^q\, n_q = 0 \\ &(iii) n_p(\xi^q) + \sqrt{\varepsilon}\sum_{m=1}^{r}\theta_m^q\,\theta_p^m = \frac{a^2}{\sqrt{\epsilon}}\delta_p^q \end{aligned} \qquad (1.2)$$

where p,q,m take the values 1,2,....r δ_p^q the Kronecker delta and θ_q^p are scalar fields.

Taking the scalor fields θ_q^p equal to zero the equations (1.1) and (1.2) take the form

$$f^2 = a^2 r - \sqrt{\varepsilon}\sum_{p=1}^{r} n_p\,\xi^p \qquad (1.3)$$

$$\begin{aligned} &(i) f^p = 0, \\ &(ii) n_p of = 0 \\ &and \\ &(iii) n_p(q) = \frac{a^2}{\sqrt{\epsilon}}\delta_p^q \end{aligned} \qquad (1.4)$$

Let us call such a manifold M^{n+r} satisfying the equations (1.3) and (1.4) as the framed algebraic ε- manifold .

Theorem 1. *LetM^{n+r} be an (n+r) - dimensional differentiate manifold admitting the framed algebraic ϵ -structure . Then there exist s eigen values each a and r eigen values each equal to zero of f.*

Proof

Let λ be the eigen value of f and P the corresponding eigen vector. So

$$fP = \lambda P \qquad (1.5)$$

Operating the above equation (1.5) with f again and using the equations (1.1) and (1.5) we get

$$\lambda^2 P = a^2 P - \sqrt{\varepsilon}\sum_{p=1}^{r} n_p\,(P)^p \qquad (1.6)$$

Case I Suppose P=ξ^q, q= 1,2,3.......r. Then in view of the equation (1.4)(iii) , the equation (1.6) takes the form

$$\lambda^2 p = 0 = \lambda^2 = 0 = \lambda = 0$$

Hence there are r eigen values each equal to zero of f.

Case II Suppose that vectors P and ξ^P are linearly independent . Hence in view of the equation (1.6), we get

$$\lambda^2 = a^2 \Rightarrow \lambda = \pm a$$

Thus if s eigen values are each equal to 'a' (n-s) values are each -a so that their sum is n. Thus the theorem is proved.

2. Integrability Conditions

As we have seen in the previous section that M^{n+r} admits the framed algebraic ε-structure, if and only if there are s eigen values each s and each 'a' (n-s) values each '-a' and r values each zero of f. Let $U^1, U_2, \ldots . U^s$ be the eigen vectors for the eigen value 'a', $V^1, V_2, \ldots . V^{n-s}$ vectors for the eigen values '-a' of f. We prove the following theorem.

Theorem 2. *In order that M^{n+r} be a framed algebraic ε-manifold, it is necessary and sufficient that it possesses a tangent subbundle. πs of dimension s, a subbundle $\pi_{(n-s)}$ of dimension (n-s and π_r of dimension r such that they are mutually disjoint and span together a manifold of dimension (n+r). Projections on subbundles π_s, $\pi_{(n-s)}$ and π_r are given by*

$$\begin{aligned} &(i) 2L = \frac{f^{2x}}{a^2} + \frac{f^x}{a} \\ &(ii) 2M = \frac{f^{2x}}{a^2} + \frac{f^x}{a} \\ &and \\ &(iii) N = I - \frac{f^{2x}}{a^2} \text{ ,r some finite integer} \end{aligned} \qquad (2.1)$$

Proof.
Suppose the manifold M^{n+r} admits the framed algebraic ε structure. Hence there exists s eigen vectors $U^1, U^2, \ldots . U^s$ corresponding to the eigen value a(n-s) vectors $V^1, V^2, \ldots, V^{n-s}$ for the eigen values -a and r eigen vectors 1,2,....,r for the eigen value zero of f. As the vectors are linealy independent so

$$(i) a_x U^x = 0 \Longrightarrow a_x = 0, x = 1,2, \ldots . s$$
$$(ii) b_y V^y = 0 \Longrightarrow b_y = 0, y = 1,2, \ldots \ldots, (n-s) \quad (2.2)$$
$$and$$
$$(iii) c_z W^z = 0 \Longrightarrow c_z = 0, z = 1,2, \ldots \ldots, r.$$

Suppose

$$a_x U^x + b_y V^Y + c_z W^z = 0 \quad (2.3)$$

Operating the above equation by f and using the fact that U^x, V^Y and W^z are eigen vector for the eigen values a, -a and o, we have

$$a_x U^x - b_y V^y = 0$$

Premultiplying the above equation by f and using the same fact that U^x, V^y, W^z are eigen vectors corresponding to eigen values as -a and O respectively we obtain

$$a_x U^x + b_y V^y = 0. \quad (2.4)$$

Thus we have from above equations

$$a_x = 0, x = 1,2, \ldots . s \quad and$$
$$b_y = 0, y = 1,2, \ldots . (x-s)$$

Hence from the equation (2.3), it follows that $c_z = 0$, z=1,2,...r. So the set of vectors U^x, V^y, W^z is linearly independent. Now in view of the equations (2.1), it follows that

$$(i) LU^x = U^x, MU^x = 0, NU^x = 0$$
$$(ii) LV^y = 0, MV^y = V^y, NV^y = 0 \quad (2.5)$$
$$and$$
$$(iii) LW^z = 0, MW^z = 0, NW^z = 0$$

Thus there exist tangent subbundles π_s of dimenstion s, $\pi_n - s$ of dimenstion (n-s) and π_r of the dimension r such that they are mutually disjoint and span together the manifold M^{n+r}.

Supoose convertly that for M^{n+r}, there exist tangent subbundles π_s, π_{n-s} and π_r as said earlier. Let U^x be the set of s eigen vectors in π_s, V^y, $(n-s)$ eigen vectors in π_{n-s} and $\varepsilon^{1/2} R^2$, r eigen vectors in the distributions π_r. Such that they are largly independent and span together a manifold of dimension $(n+r)$.

If u_x, v_y, $\frac{\varepsilon^{1/4}}{a}(r_z)$ be the set dual to U^x, V^y, $\frac{\varepsilon^{1/4}}{a}(R^z)$. Then

$$u_x U^x + v_y V^y + \frac{\sqrt{\varepsilon}}{a^2} r_z R^Z = I \quad (2.6)$$

I denote the unit tensor field. Let us now put

$$f = a u_x U^x - v_y V^Y \quad (2.7)$$

Operating above equation (2.7) by f both sides and using the fact that U^x and V^y are eigen vectors for the eigen values 'a' and '-a' of f we get

$$f^2 = a^2 u_x U^x + v_y V^y \quad (2.8)$$

In view of the equations (2.6) and (2.8), it follows that

$$f^2 = a^2 I - \sqrt{\varepsilon} r_z R^z$$

Hence the manifold M^{n+r} admits the framed algebraic ε structure.

3. Integrability of distributions

In this section we shall establish some theorems on the integrability of distributions π_s, π_{n-s} and π_x.

Theorem 3. *In order that the distribution π_x be integrable it is necessary and sufficient that for arbitrary vector fields X and Y.*

$$\overset{2r}{\overline{[X,Y]}} = \pm a^r \overset{r}{\overline{[X,Y]}} \quad (3.1)$$
$$where \ \overset{r}{\overline{[X,Y]}} = f^r([X,Y]) \quad etc.$$

Proof

The distribution π_r is given by

$$L(X) = 0, M(X) = 0, \quad and \quad N(x) = X,$$

Hence in order that the distribution π_r be integrable , it is necessary and sufficient that L(X) =0 are M(X) =0 be completely integrable. Thus

$$(i)(dL)(X,Y) = 0$$
$$and \quad (3.2)$$
$$(ii)(dM)(X,Y) = 0$$

Thus we have

$$L[X,Y] = M[X,Y] = 0 \quad (3.3)$$

In view of the equations (2.1) and (3.1) we get the desired result

Theorem 4. *For the integrability of the distributions π_s and π_{n-s} the necessary and sufficient conditions are*

$$(i) \overset{2r}{\overline{[X,Y]}} = a^r \overset{r}{\overline{[X,Y]}} = a^{2r}[X,Y]$$
$$and \quad (3.4)$$
$$(ii) \overset{2r}{\overline{[X,Y]}} = -a^r \overset{r}{\overline{[X,Y]}} = a^{2r}[X,Y]$$

Proof.
The distribution π_s is given by

$$L(X) = X, M(X) = 0, N(X) = 0 \quad (3.5)$$

Hence for the integrability of π_s, the necessary and sufficient conditions are

$$(dM)(X,Y) = 0 \text{and} (dN)(X,Y) = 0 \quad (3.6)$$

In a way similar to the previous theorem, above equation takes the form

$$\overset{2r}{\overline{[X,Y]}} = a^r \overset{r}{\overline{[X,Y]}} = a^{(2r)}[X,Y]$$

which proves (3.4)(i). The condition (3.4)(ii) for π_{n-s} can also be obtained in a similar manner.

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