The Birch and Swinnerton-Dyer Conjecture

Adriko Bosco

Abstract: *This is one of the most challenging Mathematical problems. The conjecture was chosen as one of the seven millennium prize problems listed by the Clay Mathematics Institute, which has offered a $1, 000, 000 prize for the first correct proof. It is named after Mathematicians Bryan Birch and Peter Swinnerton-Dyer, who developed the conjecture during the first half of the 1960s with the help of machine computation. The conjecture is of great economic importance used in cryptology to develop the passwords in machines used for cash transactions and in the study of planetary motions in physics. The conjecture is in number theory about elliptical curves which are equations where one side you have a quadratic equation in y and the other a cubic equation in x. For example,* $y^2 = x^3 - 2$*. The challenge is to find numbers (x, y) which solve this equation. Here one could take x = 2 and y = 3.*

Keywords: *Birch and Swinnerton-Dyer conjecture, Elliptic curve, Boscomplex method, Quadratic equation, Concentric circles*

1. Method / Boscomplex Approach

Using the above given equation $y^2 = x^3 - 2$ in the introduction.

From $y^2 = x^3 - 2$ (1)

$\Rightarrow x - y^2 - 2 = 0$ (2)

Reduce the power of x to 2 by factorizing x^3 to obtain a quadratic form of the equation.

$x(x^2) - y^2 - 2 = 0$ (3)

$\Rightarrow x(x^2) - y^2x^0 - 2 = 0$, since $x^0 = 1$ (4)

c.f: $ax^2 + bx + c = 0$ (5)

$\Rightarrow a = x, b = -y^2, c = -2$ (6)

From the formula for solving the quadratic equation,

$$x = \frac{-b \pm \sqrt{(b^2 - 4ac)}}{2a} \quad (7)$$

$$\Rightarrow x = \frac{-(-y^2) \pm \sqrt{\{(-y^2)^2 - 4.x(-2)\}}}{2x} \quad (8)$$

$$x = \frac{y^2 \pm \sqrt{(y^4 + 8x)}}{2x} \quad (9)$$

$$2x^2 = y^2 \pm \sqrt{(y^4 + 8x)} \quad (10)$$

Square both sides

$(2x^2)^2 = \{(y^2 \pm \sqrt{(y^4+8x)}\}^2$

$4x^4 = y^4 \pm 2y^2 \sqrt{(y^4+8x)} \pm y^4+8x$ (11)

Either $4x^4= y^4+ 2y^2 \sqrt{(y^4+8x)} + y^4+ 8x$ (12)

Or $4x^4 = y^4 - 2y^2 \sqrt{(y^4+ 8x)} - y^4+ 8x$ (13)

(12) + (13)

$4x^4 = y^4+ 2y^2 \sqrt{(y^4+ 8x)} + y^4+ 8x$

$+ \{4x^4 = y^4- 2y^2 \sqrt{(y^4+ 8x)} - y^4+ 8x\}$

$=> 8x^4 = 2y^4 + 0 - 0 +16x$

$8x^4 = 2y^4+ 16x$

$=> y^4 = 4x^4- 8x$ (14)

(12) - (13)

$4x^4 = y^4 + 2y^2\sqrt{(y^4+8x)} + y^4+ 8x$

$- \{4x^4 = y^4 - 2y^2\sqrt{(y^4+8x)} - y^4+ 8x\}$

$0 = 0 + 4y^2\sqrt{(y^4+ 8x)} +2y^4 + 0$

$=>2y^4 = -4y^2\sqrt{(y^4+ 8x)}$

$\therefore y^4 = -2y^2\sqrt{(y^4+ 8x)}$ (15)

(15) - (14)

$y^4 = -4y^2\sqrt{(y^4+ 8x)}$

$-(y^4 = 4x^4-8x)$

$0 = -4y^4\sqrt{(y^4+ 8x)}- 4x^4+ 8x$

$=>4x^4- 8x = -4y^2\sqrt{(y^4+ 8x)}$ (16)

Divide through by 4

$x^4 - 2x = -y^2\sqrt{(y^4 + 8x)}$

$=> x(x^3-2) = -y^2\sqrt{(y^4+ 8x)}$

But $y^2 = x^3 - 2$

$=.> x(y^2) = -y^2\sqrt{(y^4+ 8x)}$

$\therefore x = - \sqrt{(y^4+ 8x)}$ (17)

Squaring both sides of (17), i have

$x^2= \{- \sqrt{(y^4+ 8x)}\}^2$

$x^2 = y^4+ 8x$

$=>y^4= x^2 - 8x$ (18)

But (18) = (14) or solve them simultaneously.

=> $x^2 - 8x = 4x^4 - 8x$

Thus $x^2 = 4x^4$
$1 = 4x^2$
$x^2 = 1/4$
=> $x = \pm\sqrt{1/4}$ (19)

Either x = 0.5 or 1/2

Or x = -0.5 or -1/2
Substitute for x in (1)
From $y^2 = x^3 - 2$

When x = 0.5
=> $y^2 = (0.5)^3 - 2$
$= (1/2)^3 - 2$
$= 1/8-2$
$y^2 = -15/8$
$\therefore y = \pm\sqrt{-15/8}$

Thus $y = \pm i\sqrt{30}/4$ (20)
$\therefore (x, y) = (0.5, i\sqrt{30}/4)$ (21)
Or $(x, y) = (0.5, - i\sqrt{30}/4)$ (22)

When x = -0.5
=> $y^2 = (-0.5)^3-2$
$= (-1/2)^3-2$
$= -1/8-2$
$y^2 = -17/8$
$\therefore y = \pm\sqrt{-17/8}$

Thus $y = \pm i\sqrt{34}/4$ (23)
$\therefore (x, y) = (-0.5, i\sqrt{34}/4)$ (24)
Or $(x, y) = (-0.5, - i\sqrt{34}/4)$ (25)
$\therefore$ The elliptic equation $y^2 = x^3 - 2$ has $(x, y) = (0.5, i\sqrt{30}/4)$, $(0.5, - i\sqrt{30}/4)$, $(-0.5, i\sqrt{34}/4)$ and $(-0.5, - i\sqrt{34}/4)$

Prove:
From $y^2 = x^3 - 2$
=> $(i\sqrt{30}/4)^2 = (1/2)^3-2$ (26)
$i^2(30)/16 = 1/8-2$
$-1(30)/16 = (1-16)/8$
$-1(15)/8 = -15/8$

Paper ID: MR21826001842 DOI: 10.21275/MR21826001842

-15/8 = -15/8 (27)

1.1 Sketches of the solution of $y^2 = x^3 - 2$

Table 1(a)

X	-0.5	-0.5	0.5	0.5
Y	i√34/4	-i√34/4	i√30/4	-i√30/4

OR

Table 1(a): above can also be recorded as below.

X	-0.5	-0.5	0.5	0.5
Y	+√-17/8	- √-17/8	+√-15/8	- √-15/8

Sketch of the points that satisfy $y^2 = x^3 - 2$ as its solutions using table 1(a).

Scales:

Horizontal; 1 cm : 0.125 units

Vertical; 1 cm : i√1 units

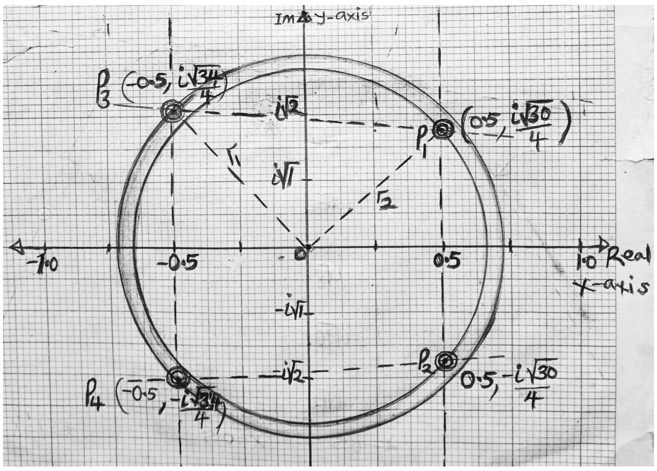

Fig. 1(a)

1.2 Description of the graph above.

The points P_1, P_2, P_3 and P_4 arrange themselves inform of matrix i.e.

$$\begin{pmatrix} -0.5, i\sqrt{34}/4 & 0.5, i\sqrt{30}/4 \\ -0.5, -i\sqrt{34}/4 & 0.5, -i\sqrt{30}/4 \end{pmatrix} \quad (28)$$

These are the solutions to the elliptic curve $y^2 = x^3 - 2$. When the diagonals of the four points are drawn, they intercept at C (0, 0) the center of the two circles as in fig.1. (a) above.

General graph of $y^2 = x^3 - 2$

Table 1(b).

Table values for y² = x³ - 2.

X	- 2	- 1	0	1	2
Y	±√-10	±√-3	±√-2	±√-1	±√6

OR

Table 1(b) above can also be recorded as below.

X	- 2	- 1	0	1	2
Y	±i√10	±i√3	±i√2	±i√1	±√6

NB: ±√6 is a real root. It's therefore, at infinity hence not plotted. This means the y values are real for x ≥ 2. The others are non real roots (irrational) for x < 2, they are therefore, used to plot the graph since the y-values of the solutions to the elliptic curve are irrational.

Sketch of elliptic curve y² = x³ - 2.

Scales:
Horizontal; 1cm: 0.25 units
Vertical; 1cm: i√1 units

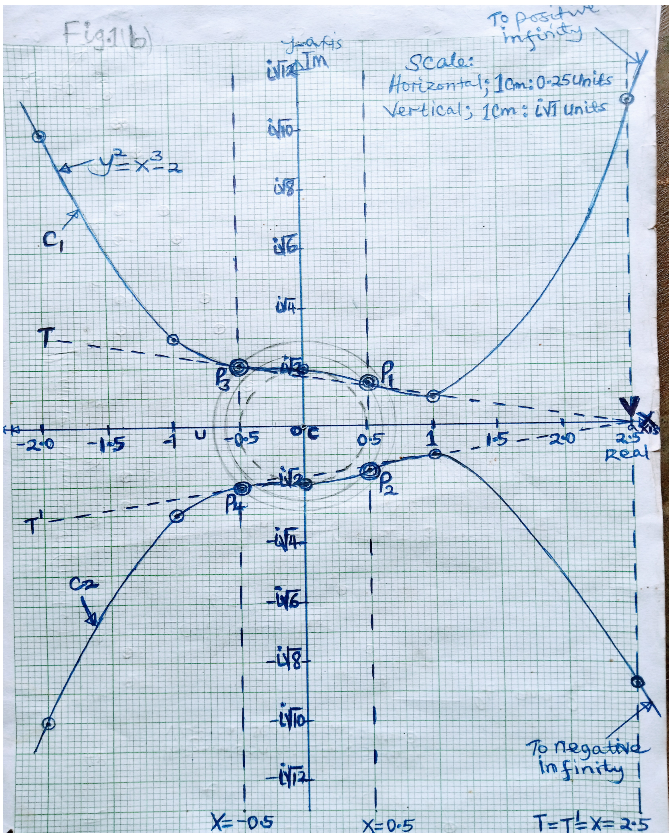

Fig. 1 (b)

1.3 Description of the graph:

Each curve is an image of the other. Let C_2 be the image of C_1. C_1 has minimum turning points at P_3 (-0.5, i√34/4) and (1, +i√1) and has its maximum turning point at (0, i√2). C_2 has minimum turning point at (0, i√2) and its maximum turning points are at P_4 (-0.5, -i√34/4) and (1, -i√1). Two tangents T and T' are drawn through the turning points of each curve which intersect at V= x = 2.5 units, thus forming ellipse VP_3UP_4V. Therefore, at x = 2.5 units, T = T' = x = 2.5 units. If P_1, P_2, P_3 and P_4 are joined diagonally, they intersect at point C (0, 0) as shown in fig. 1(a) above. Two concentric circles whose radii are r_1 and r_2 are drawn with their centre at C(0, 0), the smaller one passes through two points P_1 and P_2 and the larger one passes through P_3 and P_4 of the points P_1, P_2, P_3, P_4 that solve the elliptic curve $y^2 = x^3 - 2$. r_1 and r_2 are at right angle to each other. r_1 measures 2.5 cm, r_2 measures 3.0 cm. Therefore, the thickness t of the ring formed by the two concentric circles is given by $t = r_2 - r_1$. Hence t= 3.0 - 2.5 = 0.5 cm. Another circle whose radius is 0.5 is drawn. It touches the maximum turning point of C_1 at i√2 and the minimum turning point of C_2 at -i√2 which passes through ±0.5.The hypotenuse of triangle P_1CP_3 is calculated from Pythagoras theorem.

Hence
$$\overline{P_1P_3}^2 = \overline{P_1C}^2 + \overline{CP_3}^2$$
$$\Rightarrow \overline{P_1P_3}^2 = 2.5^2 + 3.0^2$$
$$= 6.25 + 9.0$$
$$= \sqrt{15.25} = 3.905$$
:. Length $P_1P_3 \approx 4.0$cm

NB: The points (0, √-2) and (0, - √-2) are the turning points which can also be determined by differentiating the function $y^2 = x^3 - 2$ implicitly as shown below.

From $y^2 = x^3 - 2$
$2y\,dy/dx = 3x^2$
$\Rightarrow dy/dy = 3x^2/2y$ (29)

But $dy/dx = 0$ (30)
$\Rightarrow 3x^2/2y = 0$

Hence $3x^2 = 0$
$\therefore x = 0$ (31)

Substituting for x = 0 in $y^2 = x^3 - 2$
$\Rightarrow y^2 = 0^3 - 2$
$y^2 = -2$
$\therefore y = \pm\sqrt{-2}$ as in table 2 above. (32)

Therefore, (0, +√-2) or (0, +i√2) and (0, - √2) or (0, -i√2) are the turning points where (0, - √-2) is the point image of point (0, √-2) which is the same as in table 2 above.

2. The elliptic curve E may be described as E(C): $y^2 = x^3 + Ax + B$ (33), where A and B are fixed rational numbers

Comparing (33) with (1)

=> $x^3 + Ax + B = x^3 - 2$ (34)
=> $Ax = -2 - B$ (35)

Substituting for Ax in (33), i have;

$y^2 = x^3 - 2 - B + B.$ (36)
=> $y^2 = x^3 - 2$ as in (1).This is then solved by using the equation (8);

$$x = \frac{-b \pm \sqrt{(b^2-4ac)}}{2a}$$

This yields the values of x and y as summarized in (28) above.

I.e. $x = \pm 0.5$, $y = \pm\sqrt{30}/4$ and $y = \pm\sqrt{34}/4$. Hence the points P_1, P_2, P_3 and P_4 as in table 1(a) above.
Substitute for x in (35)
When $x = 0.5$
From $Ax = -2 - B$
=>$A(0.5) = -2 - B$
$1/2A = -2 - B$
$A = -4 - 2B$
$A + 2B = -4$ (37)
When $x = -0.5$
=> $A(-1/2) = -2 - B$
$-A = -4 - 2B$
$A - 2B = 4$ (38)

Solving (37) and (38) simultaneously, i have

$$\begin{array}{rl} & A + 2B = -4 \\ + & (A - 2B = 4) \\ \hline & 2A = 0 \end{array}$$

$\therefore A = 0$

Substituting for A in (37), i have;
$0 + 2B = -4$
$\therefore B = -2$

Hence $y^2 = x^3 + Ax + B \Leftrightarrow y^2 = x^3 - 2$ as in (1).

2.1 Fig.2. (a). Sketch of the elliptic curve E(C): $y^2 = x^3 + Ax + \ldots$

Table values of solutions to $y^2 = x^3 - 2$.

Table 2(a).

X	-0.5	-0.5	0.5	0.5
Y	i√34/4	-i√34/4	i√30/4	-i√30/4

OR

X	-0.5	-0.5	0.5	0.5
Y	√-17/8	-√-17/8	√-15/8	-√-15/8

Table 2(b). Table values for $y^2=x^3-2$

X	-2	-1	0	1	2
Y	±√-10	±√-3	±√-2	±√-1	±√6

OR

X	-2	-1	0	1	2
Y	±i√10	±i√3	±i√2	±i√1	±√6

Sketch

Scales:

Horizontal; 1cm: 0.25 units

Vertical; 1cm: i√1 units

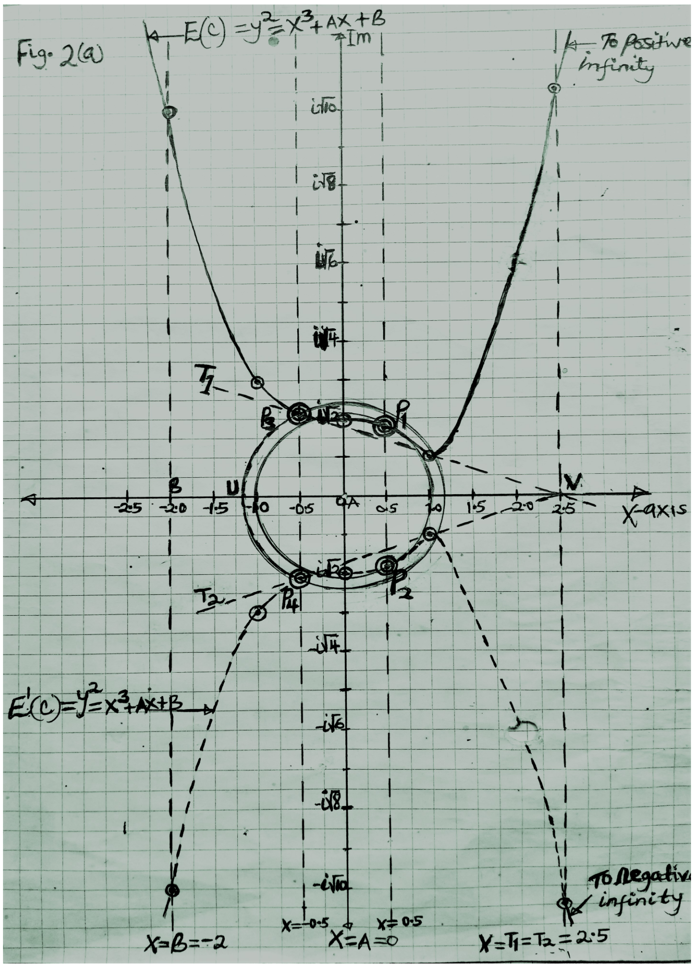

Fig. 2 (a)

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NB: $T = T' = X = 2.5$ cm (38)

From $y^2 = x^3 + Ax + B$

$\Rightarrow y^2 = (2.5)^3 + Ax + B$

$= 15\frac{5}{8} + 0(2.5) - 2$

$= \pm\sqrt{218/4}$

$\therefore y = \pm 3.69121$

Therefore, the line passing through T = T' = X = 2.5 intersects with the two curves E(C) and E'(C) at y = +3.69121 and y = -3.69121 respectively.

2.2 Description of the graph.

The same explanation in 1 above applies to the graph in 2 above, except point A = 0 when x = 0, a line passing through the y-axis i.e. the imaginary (Im) for this case. Line passing through B = -2 when x = -2 is drawn that intersects with E(C) and E'(C) the image of E(C) as shown in the graph above. An ellipse is formed by drawing two tangents T and T' which intersect at V where x = 2.5 cm, y = 0. (39)

Scales:

Horizontal; 1 cm: 0.25 units

Vertical; 1 cm: i√1

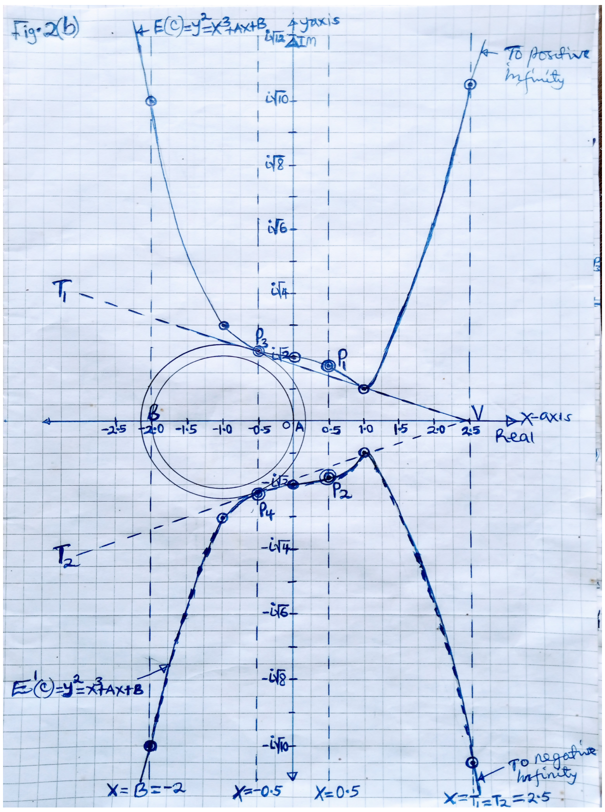

Fig. 2(b)

2.3 Description of the graph

Two concentric circles are super imposed at the center of the graph. Both circles have their centers at point A. The smaller circle passes through two points P_1 and P_2 which are the solutions to the elliptic curve $y^2 = x^3 + Ax + B$. It cuts x-axis at x = 1 and x = -1. The larger circle passes through the other two points P_3 and P_4 which are also solutions to the elliptic curve $y^2 = x^3 + Ax + B$. Hence the two concentric circles form a ring. Therefore, the base of the ellipse is thick in ring form with its apex at x = 2.5. Hence a right circular cone is formed which can be rotated through an angle of 360° along AV (i.e. about x-axis). The circle passes through the two fixed rational numbers A = 0 and B = -2 and y-axis where x = 0 and -2 respectively. The line passing through B intersects with the ellipse at $y = \pm i\sqrt{10}$. The radius, r_1 of the smaller circle is AP_1 which is measured to be 2.5 cm. The radius r_2 of the larger circle AP_3 is measured to be 3.0 cm. Therefore, the thickness of the ring $t = r_2 - r_1 = 3.0$ cm - 2.5 cm = 0.5 cm. (40)

2.4 Lengths

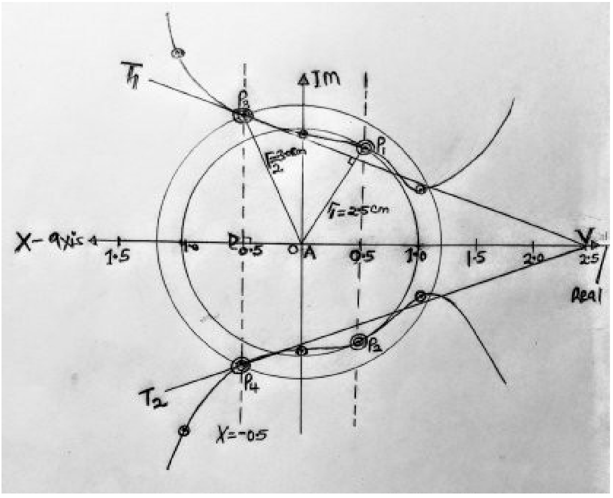

Fig. 2 c (i)

In the above diagram extracted from diagram 2(b), triangle P_3DV is a right angled triangle.

$\overline{P_3V} = 2.5$ cm, $\overline{DV} = \overline{DA} + \overline{AV} = 0.5 + 2.5 = 3.0$ cm.

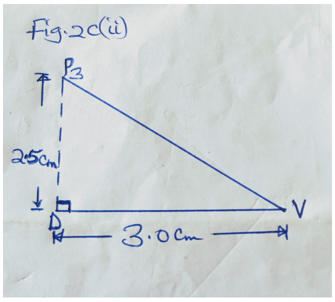

Fig. 2 c(ii)

From Pythagoras theorem,

$\overline{P_3V}^2 = \overline{P_3D}^2 + \overline{DV}^2$ (41)

$= 2.5^2 + 3.0^2$

$=15.25$

$\therefore \overline{P_3V} = \sqrt{15.5} = 3.905$

≈ 4.0 cm (42)

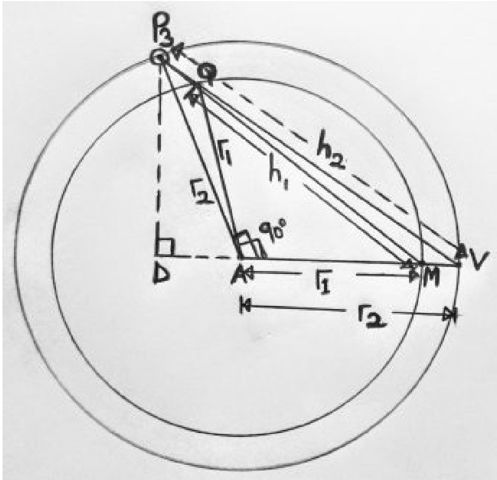

Fig. 2 (iii)

In fig.2c (iii), r_1 = 2.5 cm, r_2 = 3.0 cm. From ΔMAQ and ΔVAP3, $\overline{AM} = \overline{AQ} = r_1$, $\overline{AV} = \overline{AP_3} = r_2$ respectively. (43)
From Pythagoras theorem, in ΔP3AM;

$\overline{AM}^2 + \overline{AP_3}^2 = \overline{MP_3}^2$ (44)

$\Rightarrow r_1^2 + r_2^2 = \overline{MP_3}^2$

$2.5^2 + 3.0^2 = (\overline{MP_3})^2$

$15.25 = (MP_3)^2$

$\overline{MP_3} = \sqrt{15.25} = 3.905$

∴ Length $MP_3 \approx 4.0$ cm as in (42) above (45)

From ΔVAQ, $\overline{VA}^2 + \overline{AQ}^2 = \overline{QV}^2$ (46)

$\Rightarrow r_2^2 + r_1^2 = \overline{QV}^2$

$3.0^2 + 2.5^2 = \overline{QV}^2$

$15.25 = \overline{QV}^2$

$\overline{QV} = 3.905$ cm

∴ Length QV≈4.0 cm as in (42) and (45) above (47)

NB: In the fig. 2c (iii) above, the ratio of $r_1 : r_2$ can be expressed in terms of h_1 and h_2. Hence their respective values can be calculated. i.e. $r_1 : r_2 = h : h_2$. (48)

From diagram 2c (i) above, we can extract the fig. below.

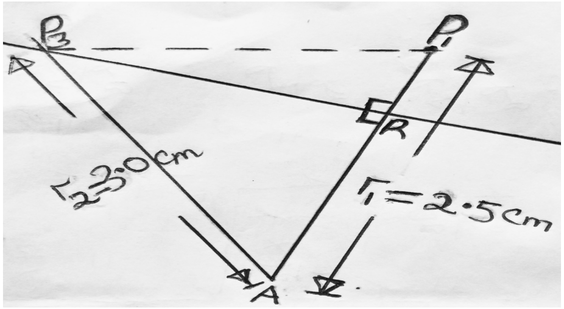

Fig. 2c (iv)

The tangent T_1 to the ellipse $y^2 = x^3 + Ax + B$ intersects perpedicularly with the line AP_1 at the

Point, R. Therefore, $\overline{RP_3} = \overline{AP_1} = r_1 = 2.5$ cm (49)

From ΔRAP_3; $\overline{AR}^2 = \overline{AP_3}^2 - \overline{RP_3}^2$ (50)

$\Rightarrow \overline{AR}^2 = 3.0^2 - 2.5^2 = 9.0 - 6.25$

$\overline{AR}^2 = 2.75$

$\therefore$ Length AR = 1.6583 cm

$\overline{AR} \approx 1.7$ cm

Hence $\overline{RP_1} = \overline{AP_1} - \overline{AR}$

$= 2.5 - 1.6583$

$\therefore$ Length $RP_1 = 0.842$ cm (51)

≈ 1 to the nearest whole number.

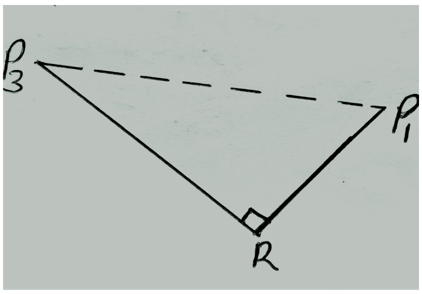

Fig. 2 c (v)

From Δ P_1P_3R, we can calculate $\overline{RP_3}$

$\overline{P_1P_3}^2 = \overline{RP_3}^2 + \overline{RP_1}^2$ (52)

But $\overline{RP_3} = r_1 = 2.5$cm, $\overline{RP_1} = 0.842$ cm (53)

$\Rightarrow \overline{P_1P_3}^2 = 2.5^2 + 0.842^2$

$= 6.25 + 0.7089$

$\overline{P_1P_3} = \sqrt{6.96}$

$= 2.638$ cm

$\therefore$ Length $P_1P_3 \approx 2.64$cm (54)

From fig.2c (iv), $\overline{P_1R} : \overline{P_1A} = 1.6583 : 2.5$

$=> \overline{P_1R} / \overline{P_1A} = 1.6583/2.5 = 1/1.507$ (55)

$\overline{P_1R}/\overline{P_1A} = 1/1.5$

$=> \overline{P_1R}/\overline{P_1A} = 1/ (3/2)$

$= 1 x 2/3$

$= \overline{P_1R} / \overline{P_1A} = 2/3$

$=> \overline{P_1R} : \overline{P_1A} = 2:3$

Similarly, $\overline{RP_3} : \overline{P_1P_3} = 2.5 : 2.64$

$=> \overline{RP_3}/\overline{P_1P_3} = 2.5/2.64$

$= 1/0.947$

$\approx 1/1$

$=> \overline{RP_3} : \overline{P_1P_3} = 1:1$ (56)

Also $\overline{AP_1} : \overline{RA} = 0.842 : 1.6583$ (57)

$=> \overline{AP_1}/\overline{RA} = 0.842/1.6583 = 1/1.969$ (58)

$\overline{AP_1} / \overline{RA} \approx 1/2$

$\therefore \overline{AP_1} : \overline{RA} = 1:2$

$\therefore \overline{AP_1} = \overline{AR} + \overline{RP_1}$

$=> \overline{AP_1} = 1.6583 + 0.842 = 2.5003$

$\therefore$ Length $AP_1 \approx 2.5$ cm as in (49) above (59)

Figure below shows the summary of ratios in fig.2 c (iv) above

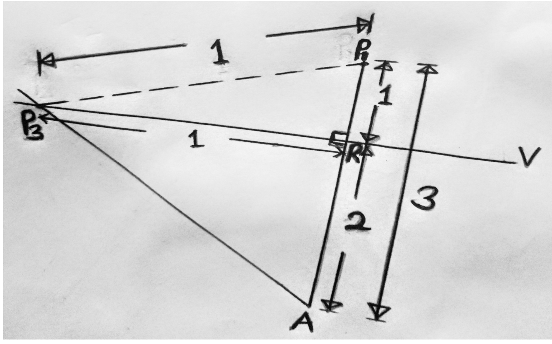

Fig. 2 c (vi)

This means one part of AR needs two parts of RP_1, AP_1 needs 2 parts of AR and 3 parts of RP_1. Length P_1P_3 = Length P_3R. (60)

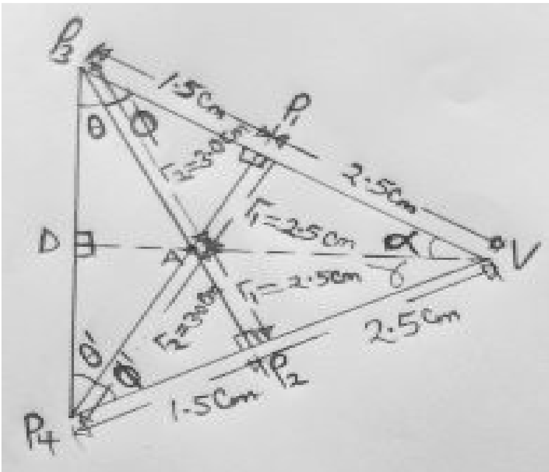

Fig.2. c (vii)

From the above figure, length P_3D can be calculated from ΔAP_3D by using Pythagoras theorem.

$\therefore \overline{DP_3}^2 = \overline{AP_3}^2 - \overline{DA}^2$

$= 32 - 0.52 = 8.75$

$\overline{DP_3} = \sqrt{8.75}$

$\therefore \overline{DP_3} \approx 2.96$

From the above geometry, $\overline{P_1A} + \overline{AP_3} = \overline{P_3P_4}$ (61)

$r_1 + r_2 = P_3D + DP_4$

But $\overline{P_3D} = \overline{DP_4}$

$2.5 + 3.0 = \overline{P_3D} + \overline{P_3D}$

$5.5 = 2\overline{P_3D}$

= 2(2.96)

5.5 ≈5.92

∴ 6 = 6 (to whole number)

Therefore, from $P_1A + AP_3 = P_3P_4$

=> $P_1 + P_3 = P_4$ (62)

From the above geometry, angles θ, α, γ, ϕ by using appropriate formulae.

From Δ D AP_3, tan θ = DA/ DP_3 = 0.5/2.96

θ = $\tan^{-1}$= (0.5/2.96) ≈ 9.59°

Note, θ = θ' = 9.59°

In fig. 2c (ii), P_3V = 4.0 cm, see equation (42)

$P_1V = P_3V - P_3V_1$ = 4.0-2.5

P_1V = 1.5 cm

sin α = opposite/hypotenuse = DP_3 / VP_3 = 2.96/4

α = $\sin^{-1}$(2.96/4)

α = 47.74°

Or from ΔVP_3D,

$DV^2 = VP_3^2 - DP_3^2$

$4^2 - 2.96^2$

∴ Length DV = 2.96 cm

From tan α = opposite/hypotenuse = DP_3 /DV

= 2.96/2.69

α = $\tan^{-1}$(2.96/2.69)

∴ α ≈ 47.74°

NB: α = γ = 47.73°

θ + θ' + ϕ +ϕ' + α + γ = 180°

But θ = θ', ϕ = ϕ', α = γ

2θ + 2ϕ + 2α =180°

2ϕ + 2(9.59) + 2(47.73) = 180°

2ϕ + 19.18° + 95.46° = 180°

2ϕ + 114.64 = 180°

ϕ = 32.68°

∴ < P_4P_3V = θ + ϕ = 9.59 + 32.68

= 42.27°

< P_3VP_4 = α + γ = 2α

= 2(47.73)

∴ < P_3VP_4 = 95.46°

NOTE: Sine or cosine rules or double angle formula can be used to calculate the angles. From α = γ, --> α + γ = α + α = 2α, hence sine rule can be used to calculate the angles i.e.

$$\frac{v}{\sin(\alpha+\gamma)} = \frac{p_3}{\sin(\theta+\phi)} = \frac{p_4}{\sin(\theta'+\phi')}$$

$$\frac{5.92}{\sin 2\alpha} = \frac{4}{\sin(\theta+\phi)} = \frac{4}{\sin(\theta'+\phi)}$$

The above expression can be used to find the required angles.

2.5 Areas of the Concentric Circles

The areas A_s and A_l of the concentric circles that form the part of the ellipse can be calculated from the formulae $A_s = \pi r_1^2$ and $A_l = \pi r_2^2$ respectively. Where A_s is the area of smaller circle, A_l is the area of the larger circle, $\pi = 22/7$. From the above data, $r_1 = 2.5$ cm, $r_2 = 3.0$ cm (63)

Therefore, $A_s = \pi r^2 = 22 / 7(2.5)$

$A_s = 1.9643$ cm² (64)

$A_l = \pi r_2^2 = 21/ 7(3.0)^2$

$= 28.286$ cm² (65)

The ratio of A_s: A_l = 19.643 cm²: 28.286 cm² (66)

=> A_s / A_l = 19.643 cm² / 28.286 cm²

$A_s / A_l = 1/1.44$

=> $A_s : A_l = 1 : 1.44$

$\therefore A_s : A_l \approx 1 : 1.4$ (67)

=> $A_s : A_l = 1 : 1$ to the nearest whole numbers. (68)

2.6 The Circumferences of the Two Concentric Circles

The circumference of each of the circles that form the ring (i.e. the concentric circles) are calculated from $C_s = 2\pi r_1$ and $C_l = 2\pi r_2$ where C_s and C_l are the circumferences of the smaller circle and the larger circle respectively, r_1 and r_2 are the radii of the smaller and larger circles respectively. (69)

$C_s = 2\pi r_1 = 2 \times 22/7 \times 2.5 = 15.714$ cm

$C_l = 2\pi r_2 = 2 \times 22/7 \times 3.0 = 18.857$ cm

∴The ratio of C_s: C_l = 15.714 cm: 18.857 cm

=> C_s / C_l = 15.714 cm / 18.857 cm

$C_s / C_l = 1/1.2$ (70)

=> $C_s / C_l \approx 1/1$

$\therefore C_s : C_l = 1:1$ (71)

NB: similarly the ratio $C_s : C_l$ can be calculated from the ratio $C_s : C_l = r_1 : r_2$.

I.e. from $C_s : C_l = 2\pi r_1 : 2\pi r_2$ (72)

=> $C_s / C_l = 2\pi r_1 / 2\pi r_2$

=> $C_s / C_l = r_1 / r_2$

Thus C_s / C_l = 2.5 cm/3.0 cm

$C_s / C_l = 1/1.2 \approx 1/1$

$\therefore C_s : C_l = 1 : 1$ as above (73)

2.7 The Shape of the Ellipse Formed Between the Two Curves

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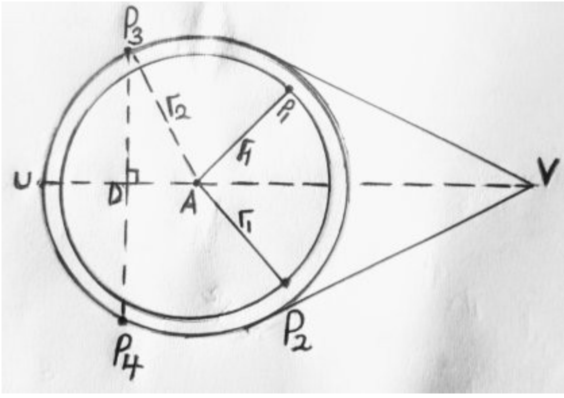

Fig. 2(d)

From the fig. above, $\overline{Ap_1} = \overline{AP_2} = \overline{P_1P_3}$ (74)

$\therefore \overline{P_1A} + \overline{AP_3} = \overline{P_3P_4}$ (75)

$\Rightarrow r_1 + r_2 = \overline{P_3P_4}$

$\therefore P_1 + P_3 = P_4$ (76)

$\therefore$ 2.5 cm + 3.0 cm ≈ 5.92 cm (77)

$\therefore$ 6.0 cm = 6.0 cm as in fig. 2 c (vii) above.

Similarly $P_2A + AP_3 = P_3P_4$

$r_1+r_2 = P_3D + DP_4$

But $P_3D = DP_4$

$r_1 + r_2 = 2(P_3D)$

2.5 cm + 3.5 cm = 2(2.96)

5.5 cm ≈ 5. 92 cm

$\therefore$ 6.0 cm = 6.0 cm

This implies that $P_2 + P_3 = P_4$

The ellipse formed between the two curves is a cone in shape whose thickness is 0.5 cm and height of 2.5 cm whose inner radius r_1 is 2.5 cm and the outer radius r_2 is 3.0 cm. Therefore, ratio theorem can be used to calculate any quantity required i.e. radius, height, area, circumference, volume etc. Therefore, an aspect of rates of change of quantities comes in i.e. differential equations. (78)

2.8 The volume of the cone formed.

The volume of the solid (the cone) formed between the two curves E (C) and E'(C) can be rotated about the x-axis, an expression for the area under $y^2 = x^3 + Ax + B$ as a general equation from x = 0 to x = 2.5 is obtained. Since A = 0, B = -2, therefore, $y^2 = x^3 - 2$.

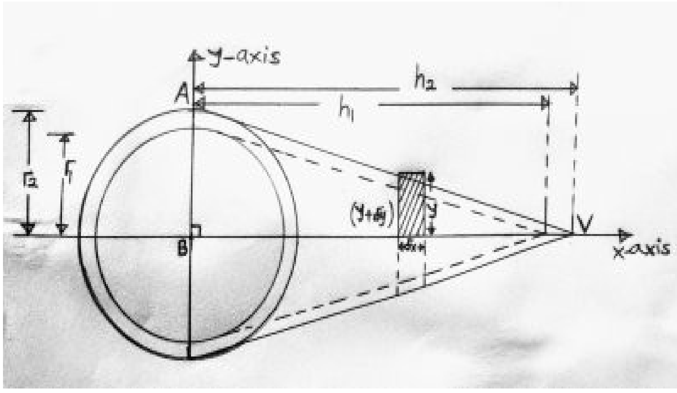

Fig.3

The element of area under $y^2 = x^3 - 2$ is y δx shown by the shaded area. Rotating this area about the x-axis generates an element of volume $\pi y^2 \delta x$. (79)

From volume = A x h; where A = area = πy^2, h = height = δx.

=> $V = \pi y^2 \delta x$ (80)

The cone (solid) has two circular surfaces of radii, $r_1 = y$ and $r_2 = y + \delta y$ (81)

Therefore, its volume lies between that of inside cylinder $\pi y^2 \delta x$ and an outside cylinder

$\pi (y + \delta y)^2 \delta x$ (82)

The sum of the volume of all the inside and outside cylinders is an approximation to the volume required.

As δx approaches zero, we can approximate the volume of the cone (solid) of revolution.

This is written as the limit, as δx-->0 of $\sum_{X=0}^{x=2.5}$

$\pi y^2 \delta x$ which can be evaluated as $\int \pi y^2 \delta x$ (83)

Since $y^2 = x^3 - 2$, => $\int_0^{2.5} \pi (x^3-2)\, \delta x$ (84)

∴The required volume $V = \int_0^{2.5} \pi (x^3-2)\, \delta x$ (85)

$V = \pi [x^4/4 - 2x]_0^{2.5}$ (86)

$= \pi\ [2.5^4/4 - 2(2.5)] - (0)$

$= \pi\ (305/64) = 14.972$

$V \approx 14.97$ unit^3

Method 2

The volume of the cone is the inside volume where the radius $r_1 = 2.5$ cm.

From volume of cone = $1/3\ (\pi r^2 h)$. (87)
=> $V = 1/3\ \pi\ (2.5^2) \times 2 = 25\pi/6\ cm^3$
Or V = 13.08997
$\approx 13.09\ cm^3$

2.9 Volume of the space between the inner and the outer surfaces of the cone.

Volume of the inner surface, $V_1 = 1/3\pi(r_1)^2 h_1$, where $r_1 = 2.5$ cm, $h_1 = 2$ cm. (88)
=> $V_1 = 1/3\ \pi(2.5)^2 \times 2 = 25\pi/6\ cm^3$
$\approx 13.09\ cm^3$
Volume of the outer surface, $V_2 = 1/3\pi(r_2)^2 h_2$, where $r_2 = 3.0$ cm, $h_2 = 2.5$ cm. (89)
=> $V_2 = 1/3\ \pi(3.0^2) \times 2.5 = 15\pi/2\ cm^3$
$\approx 23.56\ cm^3$ is the volume of the cone.
OR
Volume of the space between the two surfaces = $V_2 - V_1 = 15\pi/2 - 25\pi/6 = 10\pi/3$ cm $\approx 10.47\ cm^3$. (90)
∴ Volume of cone = (Volume of inner surface) + (Volume of space between inner and outer surfaces). (91)
= 13.09 + 10.47 = 23.56 cm^3 as above.

3. Some Worked Examples

Example1. Given the elliptic curve $y^2 = x^3 + Ax + B$. Find the values (x, y) which satisfy the equation when A = B = 0 and sketch the graph.

Solution:
Given $y^2 = x^3 + Ax + B$
From A = B = 0 (92)
Substituting for A and B is have
$y^2 = x^3 + (0)x + 0$
=> $y^2 = x^3$ (93)
$y^2 = x(x^2)$
=> $x(x^2) - y^2 = 0$
∴ $x(x^2) - y^2 x^\circ = 0$ (94)
c.f $ax^2 + bx + c = 0$
=> $a = x,\ b = -y^2,\ c = 0$

$$\text{From } x = \frac{-b \pm \sqrt{(b^2 - 4ac)}}{2a}$$

$$x = \frac{-(-y^2) \pm \sqrt{\{(-y^2)^2 - 4.x.0\}}}{2x}$$

$$x = \frac{y^2 \pm \sqrt{y^4}}{2x}$$

$\Rightarrow 2x^2 = y^2 \pm y^4$ (95)

Either $2x^2 = y^2 + y^4$ (96)

$2x^2 = y^2 - y^4$ (97)

(83) - (84)

$\Rightarrow 0 = 0 + 2y^4$

$\Rightarrow y^4 = 0$

$\therefore y = 0$ (98)

(83) + (84)

$\Rightarrow 4x^2 = 2y^2 + 0$

Substituting for y = 0

Hence x = 0 (99)

x = 0, y = 0

Therefore, the values (x, y) = (0, 0) are the solutions to the elliptic curve $y^2 = x^3 + Ax + B$ or $y^2 = x^3$ i.e. when A = B = 0.

Table 3

X	-2	-1	0	1	2	3	4
Y	$\pm\sqrt{-8}$	$\pm\sqrt{-1}$	0	$\pm\sqrt{1}$	$\pm\sqrt{8}$	$\pm\sqrt{27}$	± 8

OR

X	-2	-1	0	1	2	3	4
Y	$\pm i\sqrt{8}$	$\pm i\sqrt{1}$	0	± 1	± 2.83	5.20	8

NB: the real values of y are obtained when $x \geq 0$. Therefore, the imaginary values of y are left out while plotting although they can be plotted against the x-values while leaving out the real values of y.

A graph of $y^2 = x^3 + Ax + B$, where A = B = 0

$\therefore$ Sketch of $y^2 = x^3$

Scales:

Horizontal; 1 cm: 0.5 cm

Vertical; 1 cm: 0.5 cm

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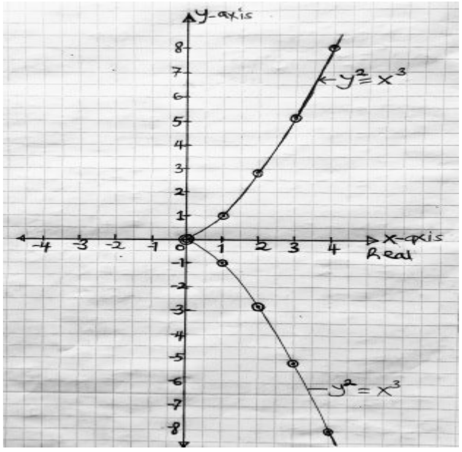

Fig.4

Description of the graph

Since A = B = 0, are points at the center, C (0, 0), the two curves E(C): $y^2 = x^3$ and E(C'): $y^2 = x^3$ the image of E(C) have their turning points at C (0, 0) where A = B = 0, hence they touch one another and form capital letter K. Therefore, the ellipse formed is a point (dot) at the center C (0, 0).

Example 2

If E(C) defines elliptic curve $y^2 = x^3 + 2$.

(a) Find the values of x, y, A and B

(b) Sketch the curve.

(c) Calculate the volume of the cone (solid) formed.

Solution:

(a) From $y^2 = x^3 + 2$ (100)

$x^3 - y^2 + 2 = 0$ (101)

$\Rightarrow x(x^2) - y^2x^0 + 2 = 0$

c.f; $ax^2 + bx + c = 0$

$\Rightarrow a = x, b = -y^2, c = 2$

$$\text{From } x = \frac{-b \pm \sqrt{(b^2 - 4ac)}}{2a}$$

$$x = \frac{-(-y^2) \pm \sqrt{(-y^2)^2\_4.x.2)}}{2x}$$

$$x = \frac{y^2 \pm \sqrt{(y^4 - 8)}}{2x}$$

$2x^2 = y^2 \pm \sqrt{(y^4 - 8)}$ (102)

Square both sides

$(2x^2)^2 = [y^2 \pm \sqrt{(y^4 - 8)}]^2$ (103)

$4x^4 = y^4 \pm 2y^2 \sqrt{(y^4 - 8)} \pm y^4 - 8$ (104)

Either $4x^4 = y^4 + 2y^2 \sqrt{(y^4 - 8)} + y^4 - 8$ (105)

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Or $4x^4 = y^4 - 2y^2 \sqrt{(y^4 - 8)} - y^4 - 8$ (**106**)

(105) + (**106**)

$8x^4 = 2y^4 - 16$

$4x^4 = y^4 - 8$

$\therefore y^4 = 4x^4 + 8$ (**107**)

(**105**) - (**106**)

$0 = 0 + 4y^2 \sqrt{y^4 - 8)} + 2y^4 + 0$

$\Rightarrow 4y^2\sqrt{(y^4 - 8)} + 2y^4 = 0$

$2y^4 = -4y^2\sqrt{(y^4 - 8)}$

$\therefore y^4 = -2y^2\sqrt{(y^4 - 8)}$ (**108**)

(**108**) = (**107**)

$\Rightarrow -2y^2\sqrt{(y^4 - 8)} = 4x^4 + 8$

Square both sides

$4y^4(y^4 - 8) = (4x^4 + 8)^2.$ (**109**)

Substitute (**107**) in (**109**)

$4\{(4x^4 + 8)\ [(4x^4 + 8 - 8)]\} = (4x^4 + 8)(4x^4 + 8)$

$4(4x^4) = (4x^4 + 8)$

$16x^4 = 4x^4 + 8$

$12x^4 = 8$

$\Rightarrow x^4 = 2/3$ (**110**)

$x = \pm \sqrt[4]{2/3}$

$x = \pm 0.9036$

$\therefore x \approx \pm 0.9$

Substitute for x values in (**107**)

I.e. from $y^4 = 4x^4 + 8$

$\Rightarrow y^4 = 4(\pm\sqrt[4]{2/3})^4 + 8$

$y^4 = 4(\pm 2/3) + 8$ (**111**)

Either $y^4 = 4 + 2/3 + 8$ (**112**)

$= (12 + 2 + 24)/3$

$y^4 = 38/3$

$\therefore y = \pm \sqrt[4]{38/3} = \pm 1.8865$

$y \approx \pm 1.9$ (**113**)

Hence $(x, y) = (\pm 0.9, \pm 1.9)$

Or $y^4 = 4 - 2/3 + 8$ (**114**)

$= (12 - 2 + 24)/3$

$y^4 = 34/3$

$\therefore y = \pm \sqrt[4]{34/3} \approx 1.8$ (1d.p) (**115**)

$\therefore (x, y) = (\pm 0.9, \pm 1.8)$

(b)

Table 4(a)

X	-0.9	-0.9	+0.9	+0.9
Y	±1.8	±1.9	±1.8	±1.9

Table 4(b)

X	-2	-1	0	1	2	3
Y	$\pm\sqrt{-6}$	±1.00	±1.41	±1.73	±3.16	±5.39

NB: the y-values are real for values of x $\geq$ -1. Therefore, y = $\sqrt{-6}$ is an imaginary value for x = -2 hence y values are imaginary for x $\leq$ - 2 (that is x < -1). (116)

Graph of E(C): $y^2 = x^3 + 2$
Scales:
Horizontal; 1cm: 0.1 units
Vertical; 1cm: 0.1 units

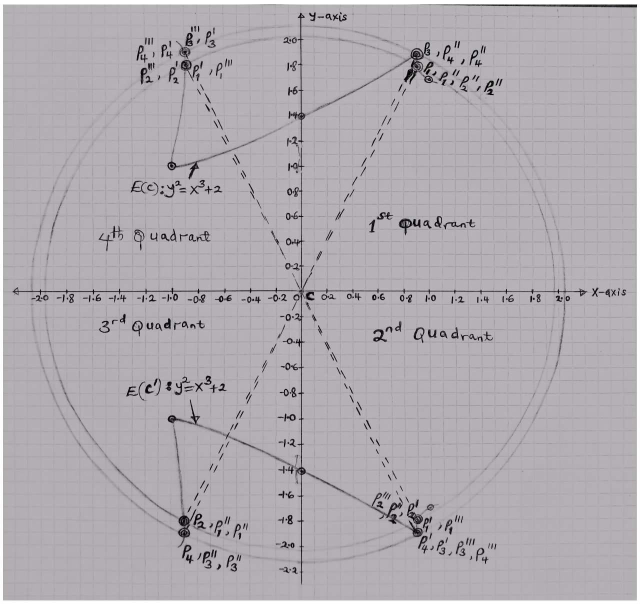

Fig.5

Description of the graph

The two curves are drawn as shown fig.5 above. When the points P_1, P_2 and P_3, P_4 are joined diagonally, they intersect at point C (0, 0). Two concentric circles are drawn. The smaller circle passes through P_1 and P_2 whilst the larger circle passes through P_3 and P_4. The larger circle passes through the minimum turning point of curve E(C') and the maximum turning point of curve E(C). The space between the two concentric circles has a thickness, t = 0.5 cm. Two tangents T_1 and T_2 each passing through the turning points of each circle. They intersect at a point x = -3, as the vertex of the right angle cone formed. Both x and y axes act as mirrors on to which many point images are formed due to multiple reflections on each of them. The points p_1 and p_3 in the first quadrant are reflected by both x and y to form images p'_1 and p'_3 in the 2nd and 4th quadrants, which are then reflected by y and x to form p''_1 and p''_3 in the 3rd quadrant, which are then reflected y and x axes to form p'''_1 and p'''_3 in the 2nd and the 4th quadrants and the process repeatedly occurs. The point p_2 and p_4 also undergo the same multiple reflections to produce many images. Hence several images are formed. (117)

(c) Volume of the cone formed is given by $\int \pi y^2 \delta x$

$$V = \pi\left[\frac{x^4}{4} + 2x\right]_0^3 = \pi[3^4/4 + 2(3) - (0)] = 105\pi/4 \text{ cm}^3 \text{ or } 82.47 \text{ cm}^3. \quad (118)$$

Volume of the cone = (volume of inner surface) + (volume of space between inner and the outer surfaces).

Volume of inner surface $v_1 = \int_0^{2.5} \pi y^2 \delta x$

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$$= \pi[(x^4/4 + 2(x)]_{0}^{2.5}$$

$= \pi[2.5^4/4 + 2(2.5) - (0)]$

$= 945\pi/64$

$V_1 \approx 46.39\ cm^3$ (119)

Volume of outer surface, $V_2 = \int_0^3 \pi y^2 \delta x = [\pi(x^4/4 + 2x]_0^3$ (120)

$[\pi(3^4/4 + 2(3)] - (0) = 105\pi/4$

$V_2 \approx 82.47\ cm^3$ (121)

∴ Volume of the space, $V_s = V_2 - V_1 = 105\pi/4 - 945\pi/64\ cm^3 = 735\pi\ cm^3$. (122)

Or $Vs = 82.47 - 46.39 = 36.08\ cm^3$

∴ Volume of the cone, V = Volume of the inner surface + volume of the space

$= 46.39 + 36.08 = 82.47\ cm^3$ as above.

NB: the volume of the cone formed is simply equal to the volume of the outer surface (larger volume).

Boscomplex Method for Finding Y-Intercept

The elliptic curve $y^2 = x^3 + Ax + B$ where A and B are fixed constants can be solved by using Boscomplex method

$$y = \pm \sqrt{\left[\frac{-b \pm \sqrt{(b^2 - 4ac)}}{2a}\right]}.$$ This can be simplified as $$y = \pm \left[\frac{-b \pm \sqrt{(b^2 - 4ac)}}{2a}\right]^{½}$$ (123)

Where ½ = Boscomplex constant, β

$$\therefore y = \pm \left[\frac{-b \pm \sqrt{(b^2 - 4ac)}^{\beta}}{2a}\right]$$ (124)

This can still be simplified further as below;

From $$y = \pm \left[\frac{-b \pm \sqrt{(b - 4ac)}}{2a}\right]^{1/2}$$

$$y = \pm \frac{-b½ \pm \sqrt{(b^2 - 4ac)^{½}}}{(2a)^{1/2}}$$

Or $$y = \pm \frac{\sqrt{-b} \pm (b^2 - 4ac)^{1/4}}{\sqrt{2a}} \equiv y = \pm \frac{\sqrt{-A} \pm \sqrt[4]{(A^2 - 4xB)}}{\sqrt{2x}}$$, since b = A, a = x, c = B (125)

Where x is the coefficient of x^2 after factorizing x^3, A is the coefficient of x to power 1, and B is a constant.

Derivation of Boscomplex Formula for Finding the Y-Intercept of Elliptic Equations

From $y^2 = x^3 + Ax + B$

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$\Rightarrow y^2 = x(x^2) + Ax + B$

c.f on the R.H.S with $ax^2 + bx + c = 0$, the general form of quadratic equation.

$$\Rightarrow y = \pm\sqrt{[x(x^2) + Ax + B]} \qquad (126)$$

Where a = x, b = A, c = B

$$\Rightarrow y = \pm\sqrt{\left[\frac{-b \pm \sqrt{(b^2 - 4ac)}}{2a}\right]} \qquad (127)$$

This is Boscomplex formula which is the square root of the formula for solving ordinary quadratic equation. Substituting for a, b and c i have

$$\Rightarrow y = \pm\sqrt{\left[\frac{-A \pm \sqrt{(A^2 - 4xB)}}{2x}\right]} \qquad (128)$$

$$\text{OR } y = \pm\left[\frac{-A \pm \sqrt{(A^2 - 4xB)}}{2x}\right]^{1/2} \qquad (129)$$

$$\therefore y = \pm\left[\frac{(-A \pm \sqrt{(A^2 - 4xB)}}{2x}\right]^{\beta}, \quad \text{where } \beta = ½. \qquad (130)$$

Worked Examples on Boscomplex Method

Example 3

Using example 1 above, solve the elliptic curve given by $y^2 = x^3 + Ax + B$, where A = 0, B = -2.

Solution:

From Boscomplex formula $y = \pm\sqrt{\left[\frac{-b \pm \sqrt{(b^2 - 4ac)}}{2a}\right]}$

Substituting for a = x, b = A, and c = B

$$\Rightarrow y = \pm\sqrt{\left[\frac{-A \pm \sqrt{(A^2 - 4xB)}}{2x}\right]}$$

$$\text{Either } y = +\sqrt{\left[\frac{-A + \sqrt{(A^2 - 4xB)}}{2x}\right]}$$

$$\text{Or } y = -\sqrt{\left[\frac{-A - \sqrt{(A^2 - 4xB)}}{2x}\right]}$$

$$\Rightarrow +\sqrt{\left[\frac{-A + \sqrt{(A^2 - 4xB)}}{2x}\right]} = -\sqrt{\left[\frac{-A - \sqrt{(A^2 - 4xB)}}{2x}\right]}$$

Squaring both sides, the first square roots cross.

$$\Rightarrow \frac{-A + \sqrt{(A^2 - 4xB)}}{2x} = \frac{-A - \sqrt{(A^2 - 4xB)}}{2x}$$

Hence $-A + \sqrt{(A - 4xB)} = -A - \sqrt{(A^2 - 4xB)}$

$2\sqrt{(A^2 - 4xB)} = 0$

Square both sides

$\Rightarrow A^2 - 4xB = 0$

Substitute for A = 0, B = -2 as given in the question.

Hence $0^2 - 4 x (-2) = 0$

$8x = 0$

$\therefore x = 0$

Substituting for x = 0, A = 0, and B = -2 in $y^2 = x^3 + Ax + B$, i have

$y^2 = 0 + 0(0) - 2$

$\Rightarrow y^2 = -2$

$\therefore y = \pm\sqrt{-2}$

$\therefore (x, y) = (0, \pm\sqrt{-2})$ as in table 2(b) above.

Hence $(x, y) = (0, +\sqrt{-2}), (0, \sqrt{-2})$ as before.

Or $(x, y) = (0, +i\sqrt{2}), (0, -i\sqrt{2})$.

OR

Boscomplex formula is also given by

$$y = \left[\frac{-A \pm \sqrt{(A^2 - 4xB)}}{2x}\right]^{\beta}$$, where Boscomplex constant, β=½.

$$y = \pm\left[\frac{-A \pm \sqrt{(A^2 - 4xB)}}{2x}\right]^{1/2}$$

$$\text{Either } y = +\left[\frac{-A + \sqrt{(A^2 - 4xB)}}{2x}\right]^{1/2}$$

$$\text{Or } y = -\left[\frac{-(A - \sqrt{(A^2 - 4xB)}}{2x}\right]^{1/2}$$

$$\Rightarrow +\left[\frac{-A + \sqrt{(A^2 - 4xB)}}{2x}\right]^{1/2} = -\left[\frac{-A - \sqrt{(A^2 - 4xB)}}{2x}\right]^{1/2}$$

Square both sides of the equation.

$$\Rightarrow + \frac{(-A + \sqrt{(A^2 - 4xB)}}{2x} = \frac{-A - \sqrt{(A^2 - 4xB)}}{2x}$$

$-A + \sqrt{(A^2 - 4xB)} = -A - \sqrt{(A^2 - 4xB)}$

$2\sqrt{(A^2 - 4xB)} = 0$

Square both sides

$\Rightarrow A^2 - 4xB = 0$

Substitute for A = 0, B = -2 as given in the question.

$\Rightarrow 0^2 -4x(-2) = 0$

$8x = 0$

$\therefore X = 0$ as before.

Substituting for x = 0, A = 0, and B = -2 in $y^2 = x^3 + Ax + B$

$\Rightarrow y^2 = 0^3 + 0(0) + (-2)$

$y^2 = -2$

$\therefore y = \pm\sqrt{-2}$ as before.

$\therefore (x, y) = (0, +\sqrt{-2}), (0, -\sqrt{-2})$

Or $(x, y) = (0, +i\sqrt{2}), (0, -i\sqrt{2})$ as in table 2(b) above.

NB: the following method can also be used.

From $y^2 = x^3 + Ax + B$

$\Rightarrow y^2 = x(x^2) + Ax + B$

c.f R.H.S with $ax^2 + bx + c$

$\Rightarrow a = x, b = A, c = B$

$$\text{From } y^2 = \frac{-b \pm \sqrt{(b^2 - 4ac)}}{2a}$$

Substituting for a, b, and c, i have,

$$y^2 = \frac{-A \pm \sqrt{(A^2 - 4xB)}}{2x}$$

$$\text{Either } y^2 = \frac{-A + \sqrt{(A^2 - 4xB)}}{2x}$$

$$\text{Or } y^2 = \frac{-A - \sqrt{(A^2 - 4xB)}}{2x}$$

$$\Rightarrow \frac{-A + \sqrt{(A^2 - 4xB)}}{2x} = \frac{-A - \sqrt{(A^2 - 4xB)}}{2x}$$

Thus $-A + \sqrt{(A^2 - 4xB)} = -A - \sqrt{(A^2 - 4xB)}$

$-A + \sqrt{(A^2 - 4xB)} = -A - \sqrt{(A^2 - 4xB)}$

$\sqrt{(A^2 - 4xB)} = -\sqrt{(A^2 - 4xB)}$

$\Rightarrow \sqrt{(A^2 - 4xB)} + \sqrt{(A^2 - 4xB)} = 0$

$2\sqrt{(A^2 - 4xB)} = 0$

$\sqrt{(A^2 - 4xB)} = 0$

Square both sides

$\Rightarrow A^2 - 4xB = 0$

But $A = 0, B = -2$

$\Rightarrow 0^2 - 4x(-2) = 0$

$8x = 0$

$\therefore x = 0$ as above.

Substituting for $x = 0$, $A = 0$, and $B = -2$ in $y^2 = x^3 + Ax + B$, i have

$y^2 = (0)^3 + 0(0) + (-2)$

$y^2 = -2$

$\therefore y = \pm\sqrt{-2}$ as in table 2(b) above

$\therefore (x, y) = (0, +\sqrt{-2}), (0, -\sqrt{-2})$

Or $(x, y) = (0, +i\sqrt{2}), (0, -i\sqrt{2})$ as before.

Example 4

Use Boscomplex formula to solve the elliptic curve $y^2 = x^3 + 2$ in example 2 above.

Solution:

From $y^2 = x^3 + 2$

$\Rightarrow y^2 = x(x^2) + 2$

c.f $y^2 = x(x^2) + Ax + B$

The R.H.S is in form of $ax^2 + bx + c = 0$

$\Rightarrow a = x, b = A, c = B$

From Boscomplex formula $y = \pm \left[\frac{-A \pm \sqrt{(A^2 - 4xB)}}{2x}\right]^{\beta}$, where $\beta=½$

$$\Rightarrow y = \pm \left[\frac{-A \pm \sqrt{(A^2 - 4xB)}}{2x}\right]^{1/2}$$

Square both sides

$$y^2 = \left[\frac{-A \pm \sqrt{(A^2 - 4xB)}}{2x}\right]$$

Substitute for A = 0, B = 2

$$y^2 = \pm \left[\frac{-0 \pm \sqrt{(0^2 - 4x(2)}}{2x}\right]$$

$$y^2 = \pm \left[\frac{\sqrt{-8x}}{2x}\right]$$

Either $y^2 = + \left[\frac{\sqrt{-8x}}{2x}\right]$

Or $y^2 = - \left[\frac{\sqrt{-8x}}{2x}\right]$

$$\Rightarrow + \left[\frac{\sqrt{-8x}}{2x}\right] = - \left[\frac{\sqrt{-8x}}{2x}\right]$$

Thus $+ \sqrt{-8x} = - \sqrt{-8x}$

Hence $+ \sqrt{-8x} + \sqrt{-8x} = 0$

$2\sqrt{-8x} = 0$

$\sqrt{-8x} = 0$

Square both sides

$\Rightarrow -8x = 0$

$\therefore x = 0$

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From $y^2 = x^3 + Ax + B$
Substitute for x = 0, A = 0, B = 2
=> $y^2 = 0^3 + 0(0) + 2$
$y^2 = 2$
∴ $y = \pm \sqrt{2}$ as before in example 2.
∴ (x, y) = (0, +√2), (0, -√2).
Therefore, it's observed in all these calculations that y is given by $y = \pm \sqrt{B}$.

Example 5

Solve the elliptic curve $y^2 = x^3$

Solution:

Given $y^2 = x^3$
From $y^2 = x^3 + Ax + B$
=> $y^2 = x(x^2) + Ax + B$
c.f on the R.H.S with $ax^2 + bx + c$
=> a = x, b= A = 0, C = B = 0
From $y = \pm \sqrt{B}$
=> $y = \pm \sqrt{0}$
∴ y=0 as before.
From $y^2 = x^3 + Ax + B$
=> $0^2 = x^3 + 0(x) + 0$
$0 = x^3$
∴ $x = \sqrt[3]{0} = 0$
∴ (x, y) = (0, 0)

Method 2: (Boscomplex formula)

Boscomplex formula can be used to calculate the values of x and the corresponding values of y as below.

$$y = \pm \left[\frac{-A \pm \sqrt{(A^2 - 4xB)}}{2x}\right]^{\beta}$$

$$y = \pm \left[\frac{-A \pm \sqrt{(A^2 - 4xB)}}{2x}\right]^{1/2}$$

A = 0, B = 0

$$y = \pm \left[\frac{-0 \pm \sqrt{(0^2 - 4x(0)}}{2x}\right]^{1/2}$$

$y = \pm (0)^{1/2}$
∴ y = 0 as above.
Substituting for y = 0 in $y^2 = x^3$, i have $y^2 = 0^3$
∴ y = 0

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Hence (x, y) = (0, 0).

4. Conclusion

It's noted that from (8), the general formula for solving elliptic curves with quadratic equation in y and the other in cubic equation in x is given by (7) i.e. $x = \frac{-b \pm \sqrt{(b^2 - 4ac)}}{2a}$

and by using Boscomplex Formula

$$x = \sqrt{\left[\frac{-b \pm \sqrt{(b^2 - 4ac)}}{2a}\right]} \quad \text{or} \quad x = \left[\frac{-b \pm \sqrt{(b^2 - 4ac)}}{2a}\right]^{\beta}$$

where β = 1/2. See equation (124)

Therefore, the formula for solving the ordinary quadratic equation is used to solve elliptic curves by using Boscomplex approach above which involves the use of both quadratic and simultaneous equations. y-values which are solutions to the elliptic curves are either rational or irrational or both. Negative numbers are obtained if **$(b^2 - 4ac) < 0$** as summarized in the table1 above. Points P_1, P_2, P_3 and P_4 are the solutions (x, y) to the curve $C = y^2 = x^3 - 2$.Therefore, the conjecture is true since the elliptic curve cuts the x-axis at x = 1 and x = -1 by the smaller circle and the elliptic curves $y^2 = x^3 - 2$ and $y^2 = x^3 + Ax + B$ have solutions (x, y) = (-0.5, i√34/4), (0.5, i√30/4) i.e. (-0.5, √-17/8), (0.5, √-15/8), that's when x = ±0.5. The images of the two curves have solutions (x, y) = (-0.5, -i√34/4), (0.5, -i√30/4) i.e. (-0.5, - √-17/8), (0.5, - √-15/8). This means the solutions to the elliptic curves $y^2 = x^3 - 2$ and $y^2 = x^3 + Ax + B$ are at the points of intersections between the two lines passing through x = ±0.5 and the curves. (131)

5. Applicability of Elliptic Curves

The application of elliptic curves is of greater economic importance than the Mathematics itself because of their application in cryptology. They are used to develop passwords for machines used for cash transactions. In Physics, it's used in the study of planetary motions. Hence it's of great importance to modern computer programs and physics at large. Fig. 5. can be used as a model to develop eight (8) stroke, twelve (12) stroke or more jet engines. In the field of Biology, figure 5 can also be used as a model for developing artificial heart. (132)

6. Future Scope

Further improvement is possible in case new ideas are developed in order to improve the literature of this research work.

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Author Biography

Adriko Bosco is a Teregean from Terego District Uganda, an independent researcher in Mathematics, more especially the Millennium Mathematics and the Hilbert David's 23 Mathematics problems. The main career objective of the author is to work for the development of Mathematics in Uganda and the World at large. He went to Otumbari secondary School, St. Joseph's college Ombaci and got certificate in ICT from Makerere University Uganda. The author taught in many Secondary schools in Uganda and South Sudan for seventeen years. He was once the head-teacher of Wulu secondary School 2008 - 2010. He has many manuscripts in Mathematics since May 2020 including; 1. Goldbach's conjecture, 2. The Beal conjecture, 3. The twin prime conjecture, 4. Fermat's Last Theory, 5. Diophantine equations, 6. Solvability of a Diophantine equation, 7. Arbitrary quadratic forms, 8. Reciprocity laws and Algebraic number fields, 9. Deal with Pi (π) and Euler's constant, e, 10. He derived some formulae for solving Arithmetic mean etc which are due to be published. The author is preparing a book entitled "THE BOOK OF WISDOM AND GINIUSENESS".

Paper ID: MR21826001842 DOI: 10.21275/MR21826001842