Certain Transformation and Summation Formulae for Poly - Basic Hypergeometric Series

Dr. Brijesh Pratap Singh

Assistant Professor Department of Mathematics Raja Harpal Singh Mahavidhyalaya, Singramau, Jaunpur (U. P.), India

Abstract: ***We offer an overview of some of the main findings from the hypergeometric sequence theories and integrals associated with root systems. In particular, for such multiple series and integrals, we list a number of summations, transformations and explicit evaluations. Interesting transformation formulas for poly - basic hypergeometric using some known summation formulae and the identity defined herein. In particular, for such multiple series and integrals, we list a number of summations, transformations and explicit evaluations. Interesting transformation formulas for poly - basic hypergeometric sequence have been constructed using some known summation formulae and the identity set out herein.***

Keywords: summation formula, transformation formula, basic hypergeometric series, poly - basic hypergeometric series

1. Introduction

Due to their applications in various fields, such as additive number theory, combinatorial analysis, statistical and quantum mechanics, vector spaces, etc, simple hypergeometric series have assumed considerable importance over the last four decades or so. They also developed a very useful method for analysts to unify and sub - sum various isolated findings under a common umbrella in the theory of numbers. The enormous mass of literature on basic hypergeometric series has become so important and important (or q - hypergeometric series as we sometimes call it) that their analysis has acquired its own separate, reputable status rather than being viewed merely as a generalization of the ordinary hypergeometric series.

The discovery of Ramanujan's 'Lost' Note book by G. E. Andrews in 1976 aroused a new interest in these functions. He gave a beautiful account of the discovery of the 'Lost' Notebook and its contents in the American Mathematical Monthly in 1979.

3.1 W. N. Bailey in 1944, gave the following result:

If

$$\beta_n = \sum_{r=0}^{n} \alpha_r u_{n-r} v_{n+r}$$

and

$$\gamma_n = \sum_{r=n}^{\infty} \delta_r u_{r-n} v_{r+n}$$

Where α_r, δ_r, μ_r and v_r are any function of r only, such that the series exists, then

$$\sum_{n=0}^{\infty} \alpha_n \gamma_n = \sum_{n=0}^{\infty} \beta_n \delta_n . \qquad (1)$$

The above transformation leads to multiple outcomes that play important roles in hypergeometric series number theory and transformation theory. We demonstrate here that this transformation can be used to define some poly - basic hypergeometric series transformations.

If we take $u_r=v_r=1$ and $\delta_r=z^r$ in (3.1.1), we get:

If

$$\beta_n = \sum_{r=0}^{n} \alpha_r \qquad (2)$$

then $$\sum_{n=0}^{\infty} \alpha_n z^n = (1-z)\sum_{n=0}^{\infty} \beta_n z^n , \qquad (3)$$

because $\gamma_n = \dfrac{z^n}{1-z}, |z|<1.$

We shall use the following known sums of truncated series to derive our transformations.

$${}_2\Phi_1\left[\begin{matrix} a, y; q; q \\ ayq \end{matrix}\right]_N = \frac{[aq, yq; q]_N}{[q, ayq; q]_N} . \qquad (4)$$

[Agarwal, R.P. 5; App. II (8)]

$${}_4\Phi_3\left[\begin{matrix} a, q\sqrt{a}, -q\sqrt{a}, e; q; 1/e \\ \sqrt{a}, -\sqrt{a}, aq/e \end{matrix}\right]_N = \frac{[aq, eq; q]_N}{[q, aq/e; q]_N \, e^N} \qquad (5)$$

[Agarwal, R.P. 5; App. II (23)]

$${}_6\Phi_5\left[\begin{matrix} a, q\sqrt{a}, -q\sqrt{a}, b, c, d \quad ; q; q \\ \sqrt{a}, -\sqrt{a}, aq/b, aq/c, aq/d \end{matrix}\right]_N$$

where α_r, δ_r, u_r and v_r are any function of r only, such that the series γ_n exists,

Then

$$= \frac{[aq, bq, cq, dq; q]_N}{[q, aq/b, aq/c, aq/d; q]_N} , \text{ (a=bcd)}. \qquad (6)$$

[Agarwal, R.P. 5; App. II (25)]

$$ {}_3\Phi_2\left[\begin{matrix} a,b,q;q;q \\ e,f \end{matrix}\right]_N = \frac{(q-e)(e-abq)}{(aq-e)(e-bq)}\left[1-\frac{[a,b;q]_{N+1}}{[e/q,abq/e;q]_{N+1}}\right], \quad (7) $$

where $ef=abq^2$.
[Srivastava, A.K. 3; (4.2)]

$$ \sum_{k=0}^{n} \frac{(1-ap^kq^k)[a;p]_k[c;q]_k\, c^{-k}}{(1-a)[q;q]_n[ap/c;p]_n} = \frac{[ap;p]_n[cq;q]_n\, c^{-n}}{[q;q]_n[ap/c;p]_n}. \quad (8) $$

[Gasper & Rahman 1; App. II (34)]

$$ \sum_{k=0}^{n} \frac{(1-ap^kq^k)(1-bp^kq^{-k})[a,b;p]_k[c,a/bc;q]_k\, q^k}{(1-a)(1-b)[q,aq/b;q]_k[ap/c,bcp;p]_k} = \frac{[ap,bp;p]_n[cq,aq/bc;q]_n}{[q,aq/b;q]_n[ap/c,bcp;p]_n}. \quad (9) $$

[Gasper & Rahman 1; App. II (35)]

$$ \sum_{k=0}^{n} \frac{(1-adp^kq^k)(1-\frac{b}{d}p^kq^{-k})[a,b;p]_k[c,ad^2/bc;q]_k\, q^k}{(1-ad)(1-b/d)[dq,adq/b;q]_k[adp/c,bcp/d;p]_k} = \frac{(1-a)(1-b)(1-c)(1-ad^2/bc)}{d(1-ad)(1-b/d)(1-c/d)(1-ad/bc)} \times $$

$$ \times\frac{[ap,bp;p]_n[cq,ad^2q/bc;q]_n}{[dq,adq/b;q]_n[adp/c,bcp/d;p]_n} + \frac{a^2d(1-d)(1-c/ad)(1-d/bc)(1-b/cd)}{(1-ad)(1-b/d)(1-c/d)(1-ad/bc)}. \quad (10) $$

[Gasper & Rahman 1; App. II (II.36), with m=0]

$$ \sum_{k=0}^{n} \frac{(1-adp^kq^kP^kQ^k)(c-dP^kQ^k/p^kq^k)(1-bp^kP^k/dq^kQ^k)\, q^{2k}}{(1-ad)(c-d)(1-b/d)} \times $$

$$ \times\frac{(1-adp^kQ^k/bcq^kP^k)q^{2k}[a;p^2]_k[c;q^2]_k[b;P^2]_k[ad^2/bc;Q^2]}{(1-ad/bc)\left[d\frac{qPQ}{p};\frac{qPQ}{p}\right]_k\left[\frac{ad}{c}\frac{pPQ}{q};\frac{pPQ}{q}\right]_k\left[\frac{ad}{b}\frac{pqQ}{P};\frac{pqQ}{P}\right]_k} \times $$

$$ \times\frac{1}{\left[\frac{bc}{d}\frac{pqP}{Q};\frac{pqP}{Q}\right]_k} = \frac{(1-a)(1-b)(1-c)(1-ad^2/bc)}{(1-ad)(c-d)(1-b/d)(1-ad/bc)} \times $$

$$ \times\frac{[ap^2;p^2]_n[cq^2;q^2]_n[bP^2;P^2]_n[ad^2Q^2/bc;Q^2]_n}{\left[d\frac{qPQ}{p};\frac{qPQ}{p}\right]_n\left[\frac{ad}{c}\frac{pPQ}{q};\frac{pPQ}{q}\right]_n\left[\frac{ad}{b}\frac{pqQ}{P};\frac{pqQ}{P}\right]_n} \times $$

$$ \frac{1}{\left[\frac{bc}{d}\frac{pqP}{Q};\frac{pqP}{Q}\right]_n} + \frac{a^2(1-d)(1-c/ad)(1-b/ad)(1-}{(1-ad)(c-d)(1-b/d)(1-ad/} $$

[Verma, A. 3; (18) p. 89, with m=0]

$$ \sum_{k=0}^{n} \frac{[\beta;p]_k[c;q]_k[y;P]_k[\beta yc/d^2;pP/q]_k\, q^k}{[dq;q]_k[\beta cp/d;p]_k\left[\frac{\beta y}{d},\frac{pP}{q};\frac{pP}{q}\right]_k[cyP/d;P]_k} \times $$

$$ \times\left(1-\frac{\beta cy}{d}p^kP^k\right)\left(1-\frac{y}{d}P^kq^{-k}\right)\left(1-\frac{\beta}{d}p^kq^{-k}\right) $$

$$ = \frac{(1-d)(1-cy/d)(1-\beta y/d)}{(c-d)} - \frac{(1-\beta)(1-c)(1-y)(1-\beta cy/d^2)}{(c-d)} \times $$

$$ \times\frac{[\beta p;p]_n[cq;q]_n[yP;P]_n\left[\frac{\beta cy}{d^2}\frac{pP}{q};\frac{pP}{q}\right]_n}{\left[\frac{\beta cp}{d};p\right]_n[dp;q]_n\left[\frac{cyP}{d};P\right]_n\left[\frac{\beta y}{d}\frac{pP}{q};\frac{pP}{q}\right]_n} \quad (12) $$

[Verma 3; 12 (A) with m=0]

4.1 Transformation and Summation Formulae:

Bailey [1] established a simple but very useful identity:

If $$ \beta_n = \sum_{r=0}^{n} u_{n-r}\, v_{n+r}\, \alpha_r \quad (1) $$

and $$ \gamma_n = \sum_{r=n}^{\infty} u_{r-n}\, v_{r+n}\, \delta_r \quad (2) $$

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where α_r, δ_r, u_r and v_r are any functions of r only such that series γ_n exists, then subject to the convergence of the series.

$$\sum_{n=0}^{\infty} \alpha_n \gamma_n = \sum_{n=0}^{\infty} \beta_n \delta_n \qquad (3)$$

Making use of (3), Slater [1] provided a long list of identities of the Rogers - Ramanujan form. Later on, a number of mathematicians, notably Verma [2], Verma and Jain [1], Singh, U. B. [3], Denis, R. Y. [12], Singh, S. P. [4] and others, used the identity of Bailey (3) and created a number of transformation formulas and identities of various modules of the Rogers - Ramanujan type. Making use of this paper. In this paper, an attempt was made to develop certain very interesting transformation formulae for q - hypergeometry series using certain established summation formulae due to Verma and Jain [1] and identity (3).

In the last section of this paper, making use of the following identity due to Verma [1], viz.,

$$\sum_{n=0}^{\infty} \frac{(-z)^n q^{n(n-1)/2}}{[q,q]_n [\gamma q^n;q]_n} \sum_{k=0}^{\infty} \frac{[\alpha,\beta;q]_{n+k} b_{n+k} z^k}{[q;q]_k [\gamma q^{2n+1};q]_k} \sum_{j=0}^{n} \frac{[q^{-n},\gamma q^n;q]_j A_j (wq)^j}{[q,q]_j [\alpha,\beta;q]_j} = \sum_{n=0}^{\infty} A_n B_n \frac{(zw)^n}{[q;q]_n}, \qquad (4)$$

and summation formulae due to Verma and Jain [9], an attempt has been made to establish certain new summation and transformation formulae for basic hypergeometric series.

Result and Discussion

In this section we shall establish our main results.

(i) Taking

$$\alpha_r = \frac{(1-ap^r q^r)[a;p]_r [c;q]_r c^{-r}}{(1-a)[q;q]_r [ap/c;p]_r}$$

in (3.1.2) and making use of (3.1.8) we get :

$$\beta_n = \frac{[ap;p]_n [cq;q]_n c^{-n}}{[q;q]_n [ap/c;p]_n}.$$

Now, putting these values in (3.1.3) we have the following transformation after some simplifications

$${}_3\varphi_2\left[\begin{matrix} c:a;apq;q,p,pq;z \\ -:ap/c;a \end{matrix}\right] = (1-cz)\,{}_2\varphi_1\left[\begin{matrix} cq;ap;q,p;z \\ -;ap/c \end{matrix}\right], \qquad (1)$$

where ${}_3\varphi_2$ is on three bases and ${}_2\varphi_1$ is an only two bases.

(ii) Next, if we set

$$\alpha_r = \frac{(1-ap^r q^r)(1-ap^r q^{-r})[a,b;p]_r [c,a/bc;q]_r q^{-r}}{(1-a)(1-b)[q,aq/b;q]_r [ap/c,bcp;p]_r}$$

in (3.1.2) and make use of (3.1.9) we get :

$$\beta_n = \frac{[ap,bp;p]_n [cq,aq/bc;q]_n}{[q,aq/b;q]_n [ap/c,bcp;p]_n}.$$

Now, putting these values in (3.1.3) we have the following transformation :

$${}_6\varphi_5\left[\begin{matrix} c,a/bc:a,b;apq;bp/q;q,p,pq,p/q;zq \\ aq/b;ap/c,bcp;a;b \end{matrix}\right] = (1-z)\,{}_4\varphi_3\left[\begin{matrix} cq,aq/bc;ap,bp;q,p;z \\ aq/b;ap/c,bcp; \end{matrix}\right], \qquad (2)$$

where ${}_6\varphi_5$ is on four bases and ${}_4\varphi_3$ is on two bases only.

(iii) Taking

$$a_r = \frac{(1-ap^r q^r)(1-bp^r q^{-r})[a,b;p]_r [c,ad^2/bc;q]_r q^r}{(1-a)(1-b)[dq,adq/b;q]_r [adp/c,bcp/d;p]_r}$$

in (3.1.2) and making use of (3.1.10) we get :

$$\beta_n = \frac{(1-a)(1-b)(1-c)(1-ad^2/bc)}{d(1-ad)(1-b/d)(1-c/d)(1-ad/bc)} \times \frac{[ap,bp;p]_n [cq,ad^2q/bc;q]_n}{[dq,adq/b;q]_n [adp/c,bcp/d;p]_n} + \frac{a^2 d(1-d)(1-c/ad)(1-d/bc)(1-b/ad)}{(1-ad)(1-b/d)(1-c/d)(1-ad/bc)}$$

Now, putting these values in (3.1.3) we get :

$${}_7\varphi_5\left[\begin{matrix} c,ad^2/bc,q;a,b;adpq;bp/dq;q,p,pq,p/q;zq \\ dq,adq/b;adp/c,bcp/d;ad;b/d; \end{matrix}\right] = \frac{(1-a)(1-b)(1-c)(1-ad^2/bc)(1-z)}{d(1-ad)(1-b/d)(1-c/d)(1-ad/bc)} \times {}_5\Phi_4\left[\begin{matrix} cq,ad^2q/bc;q;ap,bp \quad ;q,p;z \\ dq,adq/b;adp/c,bcp/d; \end{matrix}\right] + \frac{a^2(1-d)(1-c/ad)(1-d/bc)(1-b/ad)}{(1-ad)(1-b/d)(1-c/d)(1-ad/bc)} \qquad (3)$$

(iv) If we take

$$\alpha_r = \frac{(1-adp^r q^r P^r Q^r)\left(c-d\dfrac{P^r Q^r}{p^r q^r}\right)\left(1-\dfrac{b}{d}\dfrac{p^r P^r}{q^r Q^r}\right)}{(1-ad)(c-d)(1-b/d)} \times$$

$$\times\frac{\left(1-\frac{ad}{bc}\frac{p^rQ^r}{q^rP^r}\right)[a;p^2]_r[c;q^2]_r[b;P^2]_r}{(1-ad/bc)\left(d\frac{qPQ}{p};\frac{qPQ}{p}\right)_r\left(\frac{ad}{c}\frac{pPQ}{q};\frac{pPQ}{q}\right)_r}\times$$

$$\times\frac{[ad^2/bc;Q^2]_r}{\left(\frac{ad}{b}\frac{pqQ}{P};\frac{pqQ}{P}\right)_r\left(\frac{bc}{d}\frac{pqP}{Q};\frac{pqP}{Q}\right)_r}$$

$$\beta_n=\frac{(1-a)(1-b)(1-c)(1-ad^2/bc)}{(1-ad)(1-b/d)(c-d)(1-ad/bc)}\times$$

$$\times\frac{[ap^2;p^2]_n[cq^2;q^2]_n[bP^2;P^2]_n}{\left[d\frac{qPQ}{p};\frac{qPQ}{p}\right]_n\left[\frac{ad}{c}\frac{pPQ}{q};\frac{pPQ}{q}\right]_n\left[\frac{ad}{b}\frac{pqQ}{P};\frac{pqQ}{P}\right]_n}\times$$

in (3.1.2) and make use of (3.1.11) we get :

$$\times\frac{\left[\frac{ad^2}{bc}Q^2;Q^2\right]_n}{\left[\frac{bc}{d}\frac{bqP}{Q};\frac{pqP}{Q}\right]_n}+\frac{a^2d(1-d)(1-c/ad)(1-d/bc)(1-b/ad)}{(1-ad)(1-b/d)(c-d)(1-ad/bc)}.$$

Now, putting these values in (3.1.3) we get the following transformation :

$$\sum_{n=0}^{\infty}\frac{(1-adp^nq^nP^nQ^n)\left(c-d\frac{P^nQ^n}{p^nq^n}\right)\left(1-\frac{b}{d}\frac{p^nP^n}{q^nQ^n}\right)}{(1-ad)(c-d)(1-b/d)}\times$$

$$\times\frac{\left(1-\frac{ad}{bc}\frac{p^nQ^n}{q^nP^n}\right)[a;p^2]_n[c;q^2]_n[b;P^2]_n}{(1-ad/bc)\left(d\frac{qPQ}{p};\frac{qPQ}{p}\right)_n\left(\frac{ad}{c}\frac{pPQ}{q};\frac{pPQ}{q}\right)_n}\times$$

$$\times\frac{[ad^2/bc;Q^2]_n(zq^2)^n}{\left(\frac{ad}{b}\frac{pqQ}{P};\frac{pqQ}{P}\right)_n\left(\frac{bc}{d}\frac{pqP}{Q};\frac{pqP}{Q}\right)_n}$$

$$=\frac{(1-a)(1-b)(1-c)(1-ad^2/bc)(1-z)}{(1-ad)(c-d)(1-b/d)(1-ad/bc)}\times$$

$$\sum_{n=0}^{\infty}\frac{[ap^2;p^2]_n[cq^2;q^2]_n[bP^2;P^2]_n}{\left[d\frac{qPQ}{p};\frac{qPQ}{p}\right]_n\left[\frac{ad}{c}\frac{pPQ}{q};\frac{pPQ}{q}\right]_n\left[\frac{ad}{b}\frac{pqQ}{P};\frac{pqQ}{P}\right]_n}\times$$

(v) Next, setting

$$\alpha_r=\frac{[\beta;p]_r[c;q]_r[y;P]_r[\beta cy/d^2;pP/q]_r\,q^r}{[dq;q]_r\left[\frac{\beta cp}{d};p\right]_r\left[\frac{\beta y}{d}\frac{pP}{q};\frac{pP}{q}\right]_r\left[\frac{cy}{d}P;P\right]_r}\times$$

$$\times\left\{\left(1-\frac{\beta cy}{d}p^rP^r\right)\left(1-\frac{y}{d}P^rq^{-r}\right)\left(1-\frac{\beta}{d}p^rq^{-r}\right)\right\}$$

in (3.1.2) and making use of (3.1.12) we get :

$$\times\frac{\left[\frac{ad^2}{bc}Q^2;Q^2\right]_n z^n}{\left[\frac{bc}{d}\frac{pqP}{Q};\frac{pqP}{Q}\right]_n}+\frac{da^2(1-d)(1-c/ad)(1-d/bc)(1-b/ad)}{(1-ad)(1-b/d)(c-d)(1-ad/bc)}. \quad (4)$$

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$$\beta_n = \frac{(1-d)(1-cy/d)(1-\beta y/d)(1-\beta c/d)}{(c-d)} - \frac{(1-\beta)(1-c)(1-y)(1-\beta cy/d^2)}{(c-d)} \times$$

$$\times \frac{[\beta p;p]_n[cq;q]_n[yP;P]_n\left[\frac{\beta cy}{d^2}\frac{pP}{q};\frac{pP}{q}\right]_n}{\left[\frac{\beta cp}{d};p\right]_n[dq;q]_n\left[\frac{cyP}{d};P\right]_n\left[\frac{\beta y}{d}\frac{pP}{q};\frac{pP}{q}\right]_n}$$

Now, putting these values in (3.1.3) we get the following interesting transformations :

$$\sum_{n=0}^{\infty}\frac{[\beta;p]_n[c;q]_n[y;P]_n[\beta cy/d^2;pP/q]_n(zq)^n}{[dq;q]_n\left[\frac{\beta cp}{d};p\right]_n\left[\frac{\beta y}{d}\frac{pP}{q};\frac{pP}{q}\right]_n\left[\frac{cy}{d}P;P\right]_n}\times$$

$$\times\left\{\left(1-\frac{\beta cy}{d}p^nP^n\right)\left(1-\frac{y}{d}P^nq^{-n}\right)\left(1-\frac{\beta}{d}p^nq^{-n}\right)\right\}$$

$$= \frac{(1-d)(1-cy/d)(1-\beta y/d)(1-\beta c/d)}{(c-d)} - \frac{(1-\beta)(1-c)(1-y)(1-\beta cy/d^2)(1-z)}{(c-d)}\times$$

$$\times\sum_{n=0}^{\infty}\frac{[\beta p;p]_n[cq;q]_n[yP;P]_n\left[\frac{\beta cy}{d^2}\frac{pP}{q};\frac{pP}{q}\right]_n}{\left[\frac{\beta cp}{d};p\right]_n[dq;q]_n\left[\frac{cyP}{d};P\right]_n\left[\frac{\beta y}{d}\frac{pP}{q};\frac{pP}{q}\right]_n} \quad (5)$$

(vi) If we take

$\alpha_r = \frac{[a,y;q]_r\, q^r}{[q,ayq;q]_r}$ in (3.1.2) and make use of (3.1.4) we get :

$$\beta_n = \frac{[aq,yq;q]_n}{[q,ayq;q]_n}.$$

Now, putting these values in (3.1.3) we have :

$${}_2\Phi_1\left[\begin{matrix}a,y;q;zq\\ ayq\end{matrix}\right] = (1-z)\,{}_2\varphi_1\left[\begin{matrix}aq,yq;q;z\\ ayq\end{matrix}\right] \quad (6)$$

(vii) Taking

$\alpha_r = \frac{[a,q\sqrt{a},-q\sqrt{a},e;q]_r\, e^{-r}}{[q,\sqrt{a},-\sqrt{a},aq/e;q]_r}$ in (3.1.2) and making use (3.1.5) we get :

$$\beta_n = \frac{[aq,eq;q]_n}{[q,aq/e;q]_n\, e^n}.$$

Putting thsese values in (3.1.3) we have the following transfroamtion, after some simplifications :

$${}_4\Phi_3\left[\begin{matrix}a,q\sqrt{a},-q\sqrt{a},e;q;z\\ \sqrt{a},-\sqrt{a},aq/e\end{matrix}\right]$$

$$=(1-ez)\,{}_2\varphi_1\left[\begin{matrix}aq,eq;q;z\\ aq/e\end{matrix}\right]. \quad (7)$$

(viii) Taking

$$\alpha_r = \frac{[a,q\sqrt{a},-q\sqrt{a},b,c,d;q]_r\, q^r}{[q,\sqrt{a},-\sqrt{a},aq/b,aq/c,aq/d;q]_r}$$

in (3.1.2) and making use of (3.1.6) we get :

$$\beta_n = \frac{[aq,bq,cq,dq;q]_n}{[q,aq/b,aq/c,aq/d;q]_n}$$

Putting these values in (3.1.3) we get :

$${}_6\Phi_5\left[\begin{matrix}a,q\sqrt{a},-q\sqrt{a},b,c,d\quad;q;zq\\ \sqrt{a},-\sqrt{a},aq/b,,aq/c,aq/d\end{matrix}\right]$$

$$=(1-z)\,{}_4\varphi_3\left[\begin{matrix}aq,bq,cq,dq\quad;q;z\\ aq/b,aq/c,aq/d\end{matrix}\right],\ a=bcd. \quad (8)$$

ix) Lastly, taking

$\alpha_r = \frac{[a,b;q]_r\, q^r}{[e,f;q]_r}$, where ef=abq^2, in (3.1.2) and making use of (3.1.7) we get :

$$\beta_n = \frac{(e-abq)(q-e)}{(aq-e)(e-bq)} - \frac{(1-a)(1-b)eq}{(aq-e)(e-bq)}\frac{[aq,bq;q]_n}{[e,abq^2/e;q]_n}$$

Putting these values in (3.1.3) we have the following transformation :

$${}_3\Phi_2\left[\begin{matrix}a,b,q;q;zq\\ e,abq^2/e\end{matrix}\right]$$

$$=\frac{(e-abq)(q-e)}{(aq-e)(e-bq)} - \frac{(1-z)eq(1-a)(1-b)}{(aq-e)(e-bq)}\,{}_3\Phi_2\left[\begin{matrix}aq,bq,q;q;z\\ e,abq^2/e\end{matrix}\right] \quad (9)$$

4.3 we shall establish the transformation formulae by making use of (4.1.3).
Taking

$u_r = \frac{1}{[q;q]_r}, v_r = \frac{[aq;q]_r}{q^{r^2/2}}$ in (4.1.1) and (4.1.2) we get:

$$\text{If} \quad \beta_n = \frac{[aq;q]_n}{[q;q]_n \, q^{n^2/2}} \sum_{r=0}^{n} \frac{(-1)^r q^{r/2} [q^{-n};q]_r [aq^{n+1};q]_r \, \alpha_r}{q^{r^2}} \quad (1)$$

$$\text{and} \quad \gamma_n = \frac{[aq;q]_{2n}}{q^{2n^2}} \sum_{r=0}^{\infty} \frac{[aq^{2n+1};q]_r}{[q;q]_r \, q^{r^2/2+2nr}} \delta_r, \quad (2)$$

$$\text{then} \quad \sum_{n=0}^{\infty} \alpha_n \gamma_n = \sum_{n=0}^{\infty} \beta_n \delta_n, \quad (3)$$

provided the series involving are convergent. We shall now use (4.3.1), (4.3.2) and (4.3.3) in order to establish the required transformations.

(i) Replacing a by x^2y^2 in (4.3.1) and (4.3.2) and then taking

$$\alpha_r = \frac{[x,-xq;q]_r \, q^{r^2+r/2} (-)^r}{[q,xyq,-xyq,x^2q;q]_r}$$

in (4.3.1) and making use of (4.2.5) we have

$$\beta_n = \frac{[x^2y^2q;q]_n \, x^n [x^2q^2;q^2]_m [y^2q^2;q^2]_m}{q^{n^2/2} [x^2q;q]_n [x^2y^2q^2;q^2]_m [q^2;q^2]_m},$$

Where m is the greatest integer <n/2

Again, taking

$$\delta_r = z^r q^{r^2/2} \text{ in } (4.3.2)$$

We get after some simplifications

$$\gamma_n = \frac{[x^2y^2zq;q]_\infty [x^2y^2q;q]_{2n} (-)^n q^{n/2}}{[z;q]_\infty \, q^{n^2} \, [x^2q^2zq;q]_n [q/z;q]_n} \quad (5)$$

References

[1] Agarwal, R. P., A note on certain partial sums of hypergeometric series, The Mathematics Students, Vol. XL, No.4 (1972), pp.389 - 391.

[2] Agarwal, R. P., Resonance of Ramanujan's Mathematics, Vol. II, New Age International (P) limited, New Delhi, (1996).

[3] Gasper, G. and Rahman, M., Basic Hypergeometric Series, Cambridge University Press, Cambridge 1991.

[4] Slater, L. J., Generalized Hypergeometric Functions, Cambridge University Press, Cambridge (1966)

[5] Y. L. Luke Mathematical Functions and Their Approximations Academic Press, New York (1975)

[6] J. L. Lavoie Some evaluations for the Some summation formulas for the series 3F2 (1) Math. Comp., 49 (1987), pp.269 - 274

[7] H. M. Srivastava A simple algorithm for the evaluation of a class of generalized hypergeometric series Stud. Appl. Math., 86 (1992), pp.79 - 86

[8] G. Gasper, M. Rahman Basic hypergeometric series Encyclopedia of Mathematics and Its Applications, Vol.35, Cambridge University Press, Cambridge (1990)

[9] W. N. Bailey Generalized Hypergeometric Series Cambridge University Press, Cambridge (1935)

[10] L. J. Slater Generalized Hypergeometric Functions Cambridge University Press, Cambridge (1966)

[11] R. Askey The q - gamma and q - beta functions Appl. Analysis, 8 (1978), pp.125 - 141

[12] D. S. Moak The q - gamma function for q > 1 Aequationes Math., 20 (1980), pp.278 - 285

[13] D. S. Moak The q - gamma function for x < 0 Aequationes Math., 21 (1980), pp.179 - 191

[14] Verma, V. K. Jain Some summation formulae for nonterminating basic hypergeometric series SIAM J. Math. Anal., 16 (1985), pp.647 - 655

[15] J. Thomae Beiträge zur Theorie der durch die Heinesche Reihe …J. Reine Angew. Math., 70 (1869), pp.258 - 281

[16] F. H. Jackson A generalization of the functions Γ (n) and xn Proc. Roy. Soc. London, 74 (1904), pp.64 - 72

[17] R. Askey Ramanujan's extensions of the gamma and beta functions Amer. Math. Monthly, 87 (1980), pp.346 - 359

[18] M. E. H. Ismail, L. Lorch, M. E. Muldoon Completely monotonic functions associated with the gamma function and its q - analoguesJ. Math. Anal. Appl., 116 (1988), pp.1 - 9

[19] V. L. Kocić A note on q - gamma function Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat., 1 (1990), pp.31 - 34

[20] O. I. Marichev, V. K. Tuan Some properties of the q - gamma function Dokl. Akad. Nauk BSSR, 26 (1982), pp.488 - 491

[21] O. I. Marichev, V. K. Tuan Some properties of the q - gamma function Dokl. Akad. Nauk BSSR, 26 (1982), p.571

[22] Rosengren, H.2004. Elliptic hypergeometric series on root systems. Adv. Math., 181, 417–447.

[23] Rosengren, H.2006. New transformations for elliptic hypergeometric series on the root system An. Ramanujan J., 12, 155–166.

[24] Rosengren, H.2007. Sums of triangular numbers from the Frobenius determinant. Adv. Math., 208, 935–961.

[25] Rosengren, H.2008. Sums of triangular numbers from elliptic pfaffians. Int. J. Number Theory, 4, 873–890.

[26] Rosengren, H.2008. Schur Q - polynomials, multiple hypergeometric series and enumeration of marked shifted tableaux. J. Combin. Theory Ser. A, 115, 376–406.