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Finite Difference Method Solution to Garlerkin's Finite Element Discretized Beam Equation

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Abstract: ***Garlerkin's Finite Element discretization of the beam equation of the form*** $u_{tt} + c^2 u_{xxxx} = f(x,t)$ ***where*** c^2 ***has the meaning of flexural rigidity per linear mass density,*** $u(x,t)$ ***is the beamdeflection and*** $f(x,t)$ ***is the external forcing term gave a system of ordinary differential equations [1]. I have investigated the approximation of the system of ordinarydifferential equations using finite difference method.***

Keywords: Finite Difference Method, central finite difference scheme,beam deflection equation

1. Introduction

Garlerkin's Finite Element discretization of the beam equation gave an equilibrium equation as:

$$\frac{\mu L}{2\pi}\begin{bmatrix} -0.5 & 0.4714 & 0.5 & 0.3771 & 0.1667 \\ 0.4714 & 0 & 0.8485 & 0.6667 & 0.3367 \\ 0.5 & 0.8485 & 0.1667 & 0.8081 & 0.5 \\ 0.3771 & 0.6667 & 0.8081 & 0 & 0.6285 \\ 0.1667 & 0.3367 & 0.5 & 0.6285 & -0.1 \end{bmatrix}\begin{bmatrix} \ddot{\alpha}_1 \\ \ddot{\alpha}_2 \\ \ddot{\alpha}_3 \\ \ddot{\alpha}_4 \\ \ddot{\alpha}_5 \end{bmatrix} + \frac{EI\pi^3}{2L^3}\begin{bmatrix} -0.5 & 1.885 & 4.5 & 6.033 & 4.167 \\ 1.885 & 0 & 30.54 & 42.67 & 33.67 \\ 4.5 & 30.54 & 13.5 & 116.37 & 112.5 \\ 6.033 & 42.67 & 116.37 & 0 & 251.41 \\ 4.167 & 33.67 & 112.5 & 251.41 & -62.5 \end{bmatrix}\begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \\ \alpha_5 \end{bmatrix} = \frac{\mu L}{\pi}\begin{bmatrix} -0.7071.f(x,t) \\ 0 \\ 0.2357 f(x,t) \\ 0.25 f(x,t) \\ 0.14142 f(x,t) \end{bmatrix} \quad (1)$$

which is a system of second order ordinary differential equations.

Recurrence scheme with central finite difference scheme

Consider the Taylor expansions shown below [7]:

$$x_{i+1} = x_i + h\dot{x}_i + \frac{h^2}{2}\ddot{x}_i + \frac{h^3}{6}\dddot{x}_i + \ldots\ldots\ldots \quad (2)$$

$$x_{i-1} = x_i - h\dot{x}_i + \frac{h^2}{2}\ddot{x}_i - \frac{h^3}{6}\dddot{x}_i + \ldots\ldots\ldots \quad (3)$$

where time interval $h = \Delta t$.

Adding equation (2) and equation (3) and ignoring higher order terms we obtain:

$$\ddot{x}_i = \frac{1}{h^2}\left(x_{i-1} - 2x_i + x_{i+1}\right) \quad (4)$$

which can be rearranged to;

$$x_{i+1} = 2x_i - x_{i-1} + h^2\ddot{x}_i \quad (5)$$

which is known as recurrence formula or scheme. We can rewrite the recurrence formula equation in the form;

$$u_{n+1} = 2u_n - u_{n-1} + h^2\ddot{u}_n \quad (6)$$

Approximation ofsolutions of Equation (1)

Equation (1) can be rewritten in the forms:

$$\frac{\mu L}{2\pi}\left[-0.5\ddot{\alpha}_1 + 0.4714\ddot{\alpha}_2 + 0.5\ddot{\alpha}_3 + 0.3771\ddot{\alpha}_4 + 0.1667\ddot{\alpha}_5\right] + \frac{\pi^3 EI}{2L^3}\left[-0.5\alpha_1 + 1.885\alpha_2 + 4.5\alpha_3 + 6.033\alpha_4 + 4.167\alpha_5\right]$$

$$= \frac{-\mu L f(x,t).0.7071}{\pi} \quad (7)$$

$$\frac{\mu L}{2\pi}\left[0.4714\ddot{\alpha}_1+0.8485\ddot{\alpha}_3+0.6667\ddot{\alpha}_4+0.3367\ddot{\alpha}_5\right]+\frac{\pi^3 EI}{2L^3}\left[1.885\alpha_1+30.54\alpha_3+42.67\alpha_4+33.67\alpha_5\right]=0 \quad (8)$$

$$\frac{\mu L}{2\pi}\left[0.5\ddot{\alpha}_1+0.8485\ddot{\alpha}_2+0.1667\ddot{\alpha}_3+0.8081\ddot{\alpha}_4+0.5\ddot{\alpha}_5\right]+\frac{\pi^3 EI}{2L^3}\left[4.5\alpha_1+30.54\alpha_2+13.5\alpha_3+116.37\alpha_4+112.5\alpha_5\right]=\frac{\mu L f(x,t).0.7071}{3\pi} \quad (9)$$

$$\frac{\mu L}{2\pi}\left[0.3771\ddot{\alpha}_1+0.6667\ddot{\alpha}_2+0.8081\ddot{\alpha}_3+0.6285\ddot{\alpha}_5\right]+\frac{\pi^3 EI}{2L^3}\left[6.033\alpha_1+42.67\alpha_2+116.37\alpha_3+251.41\alpha_5\right]=\frac{\mu L f(x,t)}{4\pi} \quad (10)$$

$$\frac{\mu L}{2\pi}\left[0.1667\ddot{\alpha}_1+0.3367\ddot{\alpha}_2+0.5\ddot{\alpha}_3+0.6285\ddot{\alpha}_4-0.1\ddot{\alpha}_5\right]+\frac{\pi^3 EI}{2L^3}\left[4.167\alpha_1+33.67\alpha_2+112.5\alpha_3+251.41\alpha_4-62.5\alpha_5\right]=\frac{\mu L f(x,t).0.7071}{5\pi} \quad (11)$$

From Equation (4) the approximate second order derivative by central difference is;

$$\ddot{\alpha}_i=\frac{1}{h^2}\left(\alpha_{i-1}-2\alpha_i+\alpha_{i+1}\right)$$

hence:

$$\ddot{\alpha}_1=\frac{2\alpha_1-\alpha_0-\alpha_2}{-h^2}$$

$$\ddot{\alpha}_2=\frac{2\alpha_2-\alpha_1-\alpha_3}{-h^2}$$

$$\ddot{\alpha}_3=\frac{2\alpha_3-\alpha_2-\alpha_4}{-h^2}$$

$$\ddot{\alpha}_4=\frac{2\alpha_4-\alpha_3-\alpha_5}{-h^2}$$

$$\ddot{\alpha}_5=\frac{2\alpha_5-\alpha_4-\alpha_6}{-h^2} \quad (12)$$

Substituting equation (12) into (7) to (11) we obtain:

$$-\frac{\mu L}{2\pi h^2}\left[-0.5(2\alpha_1-\alpha_0-\alpha_2)+0.4714(2\alpha_2-\alpha_1-\alpha_3)+0.5(2\alpha_3-\alpha_2-\alpha_4)+0.3771(2\alpha_4-\alpha_3-\alpha_5)+0.1667(2\alpha_5-\alpha_4-\alpha_6)\right]$$
$$+\frac{\pi^3 EI}{2L^3}\left[-0.5\alpha_1+1.885\alpha_2+4.5\alpha_3+6.033\alpha_4+4.167\alpha_5\right]=\frac{\mu L\times 0.7071\times f(x,t)}{\pi} \quad (13)$$

$$-\frac{\mu L}{2\pi h^2}\left[0.4714(2\alpha_1-\alpha_0-\alpha_2)+0.8485(2\alpha_3-\alpha_2-\alpha_4)+0.6667(2\alpha_4-\alpha_3-\alpha_5)+0.3367(2\alpha_5-\alpha_4-\alpha_6)\right]+$$
$$\frac{\pi^3 EI}{2L^3}\left[1.885\alpha_1+30.54\alpha_3+42.67\alpha_4+33.67\alpha_5\right]=0 \quad (14)$$

$$-\frac{\mu L}{2\pi h^2}\left[0.5(2\alpha_1-\alpha_0-\alpha_2)+0.8485(2\alpha_2-\alpha_1-\alpha_3)+0.1667(2\alpha_3-\alpha_2-\alpha_4)+0.8081(2\alpha_4-\alpha_3-\alpha_5)+0.5(2\alpha_5-\alpha_4-\alpha_6)\right]$$
$$+\frac{\pi^3 EI}{2L^3}\left[4.5\alpha_1+30.54\alpha_2+13.5\alpha_3+116.37\alpha_4+112.5\alpha_5\right]=\frac{\mu L\times f(x,t)\times 0.7071}{3\pi} \quad (15)$$

$$\frac{-\mu L}{2\pi h^2}\left[0.3771(2\alpha_1-\alpha_0-\alpha_2)+0.6667(2\alpha_2-\alpha_1-\alpha_3)+0.8081(2\alpha_3-\alpha_2-\alpha_4)+0.6285(2\alpha_5-\alpha_4-\alpha_6)\right]$$

$$+\frac{\pi^3 EI}{2L^3}\left[6.033\alpha_1+42.67\alpha_2+116.37\alpha_3+251.41\alpha_5\right]=\frac{\mu L f(x,t)}{4\pi} \quad (16)$$

$$-\frac{\mu L}{2\pi h^2}\left[0.1667(2\alpha_1-\alpha_0-\alpha_2)+0.3367(2\alpha_2-\alpha_1-\alpha_3)+0.5(2\alpha_3-\alpha_2-\alpha_4)+0.6285(2\alpha_4-\alpha_3-\alpha_5)-0.1(2\alpha_5-\alpha_4-\alpha_6)\right]$$

$$+\frac{\pi^3 EI}{2L^3}\left[4.167\alpha_1 + 33.67\alpha_2 + 112.5\alpha_3 + 251.4\alpha_4 - 62.5\alpha_5\right] = \frac{\mu \times f(x,t) \times 0.7071}{5\pi} \quad (17)$$

Given the following properties,

$$E = 1.5 \times 10^{-6} M^4,\ I = 2.00 \times 10^9 N/M^2,\ \mu = 21.6kg,\ L = 4m,\ \pi = 3.142$$

h= 2.0×10^{-2} ,Values of $\alpha_0 = \alpha_6 = 0$ as they are outside the domain $\Omega = [0, L]$,

Then:

$$m = \frac{-\mu L}{2\pi h^2}$$

$$n = \frac{\pi^3 EI}{2L^3}$$

$$\omega = \frac{\mu L}{\pi} \quad (18)$$

Calculated values are given as:

$$m = -34373.01, n = 762.99, \omega = 27.4984 \quad (19)$$

Substitutingthe values of Equations (18) and (19) into (13) to (17) then rearranging we obtain:

$$(-1.4714m - 0.5n)\alpha_1 + (0.9428m + 1.885n)\alpha_2 + (0.1515m + 4.5n)\alpha_3 + (0.0875m + 6.033n)\alpha_4 + (0.0437m + 4.167n)\alpha_5 = -0.7071 \times \omega \times f(x,t)$$

$$(0.9428m + 1.885n)\alpha_1 + (-1.3199m)\alpha_2 + (1.0303m + 30.54n)\alpha_3 + (0.1482m + 42.67n)\alpha_4 + (0.0067m + 33.67n)\alpha_5 = 0$$

$$(0.1515m + 4.5n)\alpha_1 + (1.0303m + 30.54n)\alpha_2 + (-1.3232m + 13.5n)\alpha_3 + (0.9495 + 116.37)\alpha_4 + (0.1919m + 112.5n)\alpha_5 = \frac{\omega \times 0.7071 \times f(x,t)}{3}$$

$$0.0875m + 6.033n)\alpha_1 + (0.1482m + 42.67n)\alpha_2 + (0.9495m + 116.37n)\alpha_3 + (-1.4366m)\alpha_4 + (1.257m + 251.4n)\alpha_5 = \frac{\omega \times f(x,t)}{4}$$

$$(-0.0033m + 4.167n)\alpha_1 + (0.0067m + 33.67n)\alpha_2 + (0.0348m + 112.5n)\alpha_3 + (0.857m + 251.4n)\alpha_4 + (-0.8285m - 62.5n)\alpha_5 = \frac{0.7071.\omega.f(x,t)}{5} \quad (20)$$

Substituting the values boundary conditions $\alpha_1(t) = \alpha_5(t) = 0$ into Equation (20) we obtain:

$$\begin{bmatrix} 45368.94 & -12112.79 & 27462.70 \\ -12112.79 & 55782.73 & 55964.64 \\ 27462.70 & 56117.24 & 49380.27 \end{bmatrix} \begin{bmatrix} \alpha_2(t) \\ \alpha_3(t) \\ \alpha_4(t) \end{bmatrix} = \begin{bmatrix} 0 \\ 648.14 \\ 687.46 \end{bmatrix}$$

(21)

which are linear algebraic equations in matrix form. Using mat lab we obtain solutions as:

$$\alpha_2(t) = 0.0012739s, \alpha_3(t) = 0.0097102s, \alpha_4(t) = 0.0021783s. \quad (22)$$

2. Results

The assumed solution of the deflection beam equation byGarlekin's

Finite Element Method was given as:

$$\hat{u}(x,t) = \sum_{i=1}^{5} \phi_i(x)\alpha_i(t) = \sum_{i=1}^{5} \alpha_i(t)\sin\left(\frac{i\pi x}{L}\right) \quad (23)$$

where

$\phi_i(x)$ is the shape function or basis function

and $\alpha_i(t)$ the Fourier coefficients. Substituting Fourier coefficients in Equation (22) into Equation (23)we obtain the approximate solution of the beam deflection is given by:

$$\hat{u}(x,t) = 0.0012739\sin 0.5\pi x + 0.0097102\sin 0.75\pi x + 0.0021783\sin \pi x \quad (24)$$

Graphical output of Equation (24)

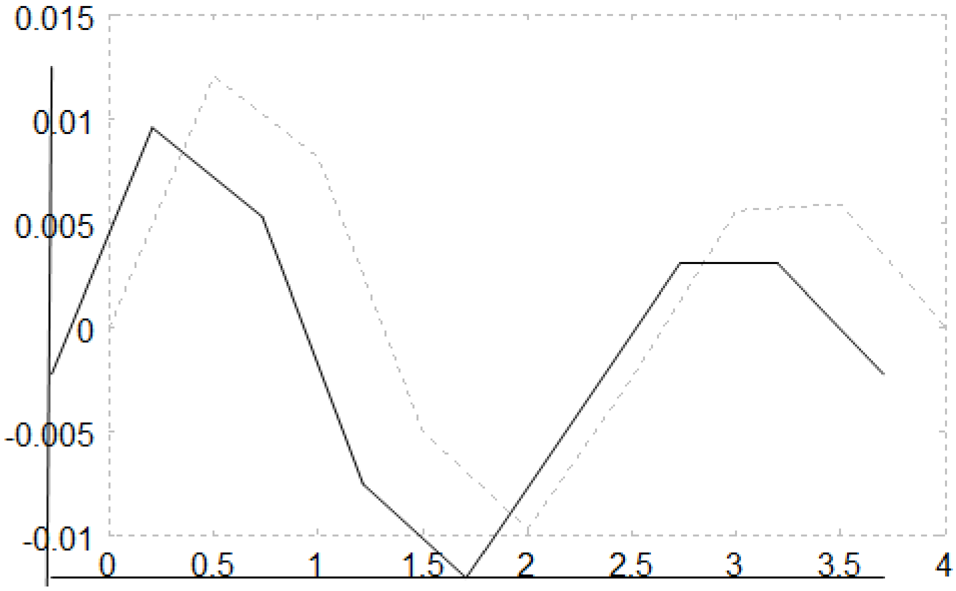

Figure 1: *Deflections* $\hat{u}(x,t) against - length(x) metres$

3. Conclusion

Above results show that beam equation subject to specific boundary conditions can be solved by Finite Element Method (FEM) and finite difference method in closed form solution.

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