Modified Navier-Stokes Equation for Incompressible Fluid Impacts

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Abstract: ***This study has come up with a numerical momentum-impulse scheme that modelsfluid impact on sea shores, off-shore structures using the Modified Navier–Stokes Equation. Finite volume method discretized Modified Navier-Stokes Equation. Non-slip boundaryconditions at Reynolds number of about 500 000 were formulated. Modified Navier-Stokes Equation in the x-z axis was coupled with continuity equation to obtain the pressure field. Corrected velocities were computed using SIMPLER algorithm approach. Using the corrected velocities fluid impulses were determined in each control volume interfaces at constant temperature of 298K, 1 atmosphere.***

Keywords: Modified Navier-Stokes Equation, Reynolds number, Fluid Impulse, Pressure-Impulse Theory, Pressure Field, Control Volume, Wall Shear Stress

1. Introduction

Owing to climate change, moving icebergs can be big enough to cause damage to offshore structures or a breaking sea wave can cause large and sudden fluid forces to be exerted on structures or sea going vessels. This study came up with with momentum-impulse theory that models fluid impact on offshore structures or sea going vessels. It formulated a numerical solution for a Modified Navier-Stokes Equation in the x and z directions of the Cartesian plane using Finite Volume Method (FVM). Studies and experiments conducted by researchers likeLamb (1932), Bagnold (1939),Chan (1994),Hattori (1994), Chan and Melville(1988), Wood and Peregrine (1998)Zenit and Hunt(1998), Cox and Cooker(2000), it was found that there was no analyticaljustification of pressure-impulse theory during short time impacts. Secondly pressure-impulse theory neglected non-linear terms in Navier-Stokes governing differential equation, therefore pressure-impulse theory did not adequately model impacts for incompressible fluids. Pressure –impulse theory neglected even convective term in Eulerdifferential Equation. However pressure-impulse theory models potential flow which satisfy Laplace equation at low Reynolds number. It's therefore necessary to come up with Modified Navier-Stokes Equation which can model incompressible fluid impacts at high Reynolds number characterized by turbulence. Modified Navier-Stokes Equation can describe fluid impacts at the boundary layerwhere most of the terms in Navier –Stokes Equation were neglected by pressure-impulse theory.

2. Governing Equations

Modified Navier-Stokes Equation in and z directions coupled with Continuity Equation in a Cartesian plane are given as follows:

x- direction

$$\frac{\partial \rho u}{\partial t} + \nabla.(\rho u\vec{v}) = -\frac{\partial P}{\partial x} + \nabla.(\Gamma \nabla \tau_{\omega}) \tag{1}$$

z- direction

$$\frac{\partial \rho w}{\partial t} + \nabla.(\rho w\vec{v}) = -\frac{\partial P}{\partial z} + \nabla.(\Gamma \nabla \tau_{\omega}) + \rho g_z \tag{2}$$

Continuity Equation

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \tag{3}$$

u - is velocity component of the fluid in the x-direction, w - is velocity component of the fluid in the z-direction, ρ - density of the fluid flowing, τ_{ω} - wall shear stress, Γ - wall shear stress diffusion coefficient.

2.1 Discretized governing differential equations using Finite Volume Method (FVM)

Discretized Modified Navier-Stokes Equation in the x-direction:

$$u_e = \tilde{u} + d_e(P_p - P_E)^{n+1}$$

$$\text{where } d_e = \frac{\Delta z}{a}, \tilde{u} = \frac{\sum_{nb} D^u_{nb}(\tau_\omega)_{nb} + b^u}{a} \tag{4}$$

u_e gives the velocity in the x-direction at (n+1) control volume face value.

Discretized ModifiedNavier-Stokes Equation in the z-direction:

$$w_n = \tilde{w}_n + d_n(P_p - P_N)^{n+1}$$

$$\text{where } d_n = \frac{\Delta x}{a}, \tilde{w}_n = \frac{\sum_{nb} D_{nb}(\tau_\omega)^w_{nb} + b^w}{a} \tag{5}$$

W_n is the velocity in the z-direction at (n+1) control volume face value.

Discretized continuity equation:

$$(u_{i+1} - u_i)\Delta z = -(w_{j+1} - w_j)\Delta x \tag{6}$$

On a staggered grid the horizontal and vertical velocities can be shown on the diagram below

Figure 1: 3×3 staggered mesh

In staggered grid discretization the scalars are stored at the centre of the control volume and velocity are centred at the faces of control volumes.

From discretized Continuity Equation u_{i+1}, u_i are velocitiesin the x-direction , w_{j+1}, w_j are velocities in the z-direction of the control volume and at the centre is Pressure(P) which actas a driving force for fluid velocities.

3. Pressure Equation

Susbstituting the horizontal velocity Eqn (4) and vertical velocity Eqn (5) into Continuity Equation gives the pressure equation in the form:

$$a_p P_p = a_e P_E + a_n P_N + b^p \tag{7}$$

where $a_n = d_n \Delta x, a_e = d_e \Delta z, a_p = d_e \Delta z + d_n \Delta x$

and

$$b^p = -\Delta z \sum_{nb} D^u_{nb} (\tau_\omega)^u_{nb} - \Delta z b^u + u_i \Delta z + w_i \Delta x - \Delta x \sum_{nb} D^w_{nb} (\tau_\omega)^w_{nb} - \Delta x b^w$$

Consider a fluid flow domain Ω below which has been divided into 16 rectangular control volumes

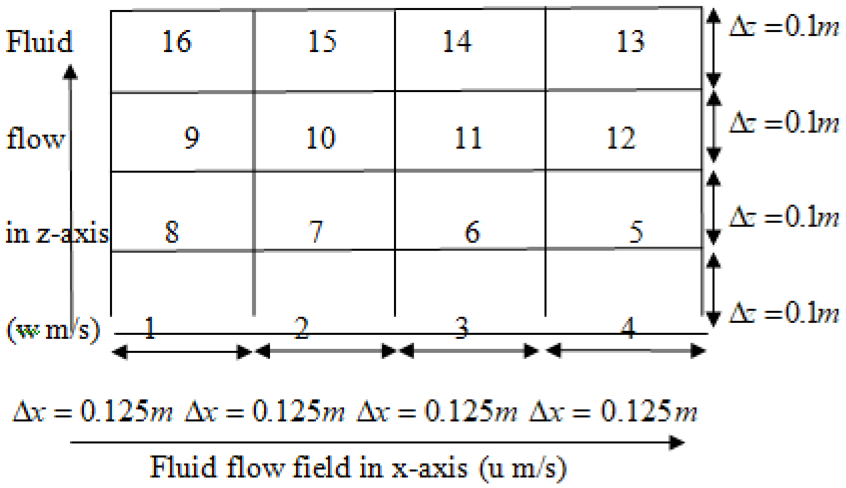

Figure 2: Discretized domain mesh in x-z directions

Reynolds number of 500 000, the speed of fluid flux in the x-axis and z-axis is given as:

$$\text{Re} = \frac{Lu}{v}$$

$$500000 = \frac{0.5 \times u}{9.368 \times 10^{-7}}$$

which gives $u \approx 1m/s$ as the speed of the fluid in the boundary of x-direction.

$$500000 = \frac{0.4 \times w}{9.368 \times 10^{-7}}$$

which gives $w \approx 1m/s$ as the speed of the fluid in the boundary of z-direction.

$$a = \rho \Delta x \Delta z = 1023.6 \times 0.124 \times 0.1 = 12.795$$

where $\rho = 1023.6 kg/m^3$ is the density of sea water

$$d_e = \frac{\Delta z}{a} = \frac{0.1}{12.795} = 0.007816$$

$$d_n = \frac{\Delta x}{a} = \frac{0.125}{12.795} = 0.0097$$

$$a_n = d_n \Delta x = 0.00977 \times 0.125 = 0.00122$$

$$a_e = d_e \Delta z = 0.007816 \times 0.1 = 0.0007816$$

$$a_p = d_e \Delta z + d_n \Delta x = 0.0020016$$

$$D_n = \frac{\Gamma_n \Delta x}{\delta z} = \frac{15.4 \times 0.125}{0.1} = 19.25 m^2/s$$

$$D_s = \frac{\Gamma_s \Delta x}{\delta z} = \frac{15.4 \times 0.125}{0.1} = 19.25 m^2/s$$

$$D_e = \frac{\Gamma_e \Delta z}{\delta x} = \frac{15.4 \times 0.1}{0.125} = 12.32 m^2/s$$

$$D_w = \frac{\Gamma_w \Delta z}{\delta x} = \frac{15.4 \times 0.1}{0.125} = 12.32 m^2/s$$

where $\Gamma = 15.4 m^2/s$ which is the diffusion coefficient of sea water at 25°C and1 atmosphere Richardson *et al*(1965)).

Approximation of pressure field in the 16 control volumes gives the following 16 algebraic equations:

$$P_1 = 0.6095P_8 + 0.3905P_2 + 620.5$$
$$P_2 = 0.6095P_7 + 0.3905P_3 + 609.7$$
$$P_3 = 0.6095P_6 + 0.3905P_4 + 613$$
$$P_4 = 0.6095P_5 + 607.4$$
$$P_5 = 0.6095P_{12} + 611.4$$
$$P_6 = 0.6095P_{11} + 0.3905P_5 + 611.7$$
$$P_7 = 0.6095P_{10} + 0.3905P_6 + 612.5$$
$$P_8 = 0.6095P_9 + 0.3905P_7 + 610.2$$
$$P_9 = 0.6095P_{16} + 0.3905P_{10} + 608.6$$
$$P_{10} = 0.6095P_{15} + 0.3905P_{11} + 611.39$$
$$P_{11} = 0.6095P_{14} + 0.3905P_{12} + 611.485$$
$$P_{12} = 0.6095P_{13} + 614.86$$

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$$P_{13} = 613.1245$$
$$P_{14} = 0.6095P_{13} + 611.211$$
$$P_{15} = 0.6095P_{14} + 611.17$$
$$P_{16} = 0.6095P_{15} + 608.76 \qquad (8)$$

System of algebraic equations Eqn (8) was solved using Gauss Seidel iteration

method. Partial derivative of the equations ($\frac{\partial P_1}{\partial P_8} < 1$, $\frac{\partial P_2}{\partial P_7} < 1$, $\frac{\partial P_3}{\partial P_6} < 1 \sim \frac{\partial P_{16}}{\partial P_{15}} < 1$), this shows that unique solutions for the system of equations exists. Eqn (8) also satisfy the Scarborough criterion for convergence.

4. Results

Iteration of equation (8) by Gauss Seidel method gave the following solutions:

Table 1: Iterated solutions Pressure Equation

	P_1	P_2	P_3	P_4	P_5	P_6	P_7	P_8	P_9	P_{10}	P_{11}	P_{12}	P_{13}	P_{14}	P_{15}	P_{16}
0	600	600	600	600	600	600	600	600	600	600	600	600	600	600	600	600
1	1221	1210	1213	973.1	977.1	1212	1213	1210	1209	1211	1211	981	613	845.5	845.5	843
2	2568	1822	1732	1203	1209	1732	1824	1820	1595	1600	1510	989	613	850.6	941.3	939
3	2442	2398	2138	1345	1214	2004	2027	2297	1806	1775	1516	989	613	850.6	943.3	976
4	2957	2680	2360	1347	1214	2010	2477	2502	1897	1778	1516	989	613	850.6	943.3	977
5	3192	3040	2357	1347	1214	2010	2481	2733	1899	1778	1516	989	613	850.6	943.3	977
6	3474	3042	2357	1347	1214	2010	2481	2736	1899	1778	1516	989	613	850.6	943.3	977
7	3476	3042	2357	1347	1214	2010	2481	2736	1899	1778	1516	989	613	850.6	943.3	977
8	3476	3042	2357	1347	1214	2010	2481	2736	1899	1778	1516	989	613	850.6	943.3	977

The approximated pressure field in each control volume is shown in the fig 28 below:

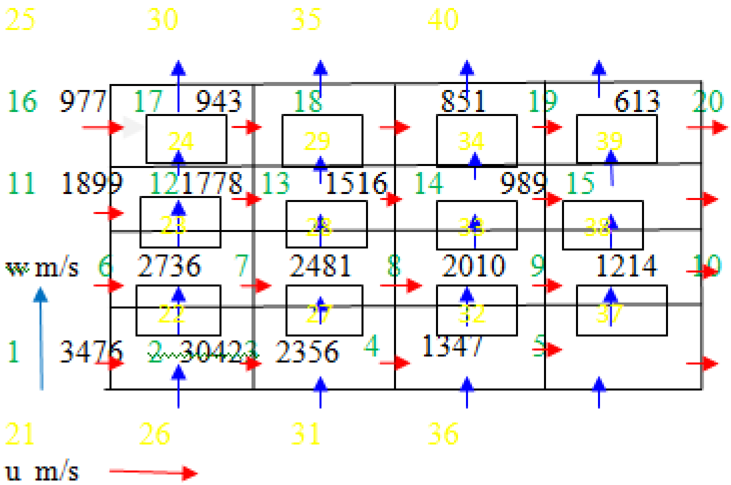

Figure 3: The Pressure field and calculated interface velocities

Using the pressure field in figure 3 we can find velocities between interfaces of the control volumes using the SIMPLER algorithm formulae from Pantanker S.V (1980) given by:

$$u_e = \tilde{u}_e + d_e(P_P - P_E)$$
$$w_n = \tilde{w}_n + d_n(P_P - P_N) \quad (9)$$

where u_e is the velocity horizontally across the control volumes in x- direction, $\tilde{u}_e$ is pseudo- velocities in the x-direction, w_n is the velocity vertically across the control volume in the z-direction, $\tilde{w}_n$ is pseudo- velocities in the z-direction. Pseudo-velocities are given by the formulae:

$$\tilde{u} = \frac{\sum_{nb} D^u_{nb}(\tau_\omega)_{nb} + b^u}{a}$$

$$\tilde{w}_n = \frac{\sum_{nb} D_{nb}(\tau_\omega)^w_{nb} + b^w}{a} \qquad (10)$$

Using formulae in Eqn (9) and pressure field in figure 3 corrected velocities across the control volumes are computed as below:

$$|u_2| = \tilde{u} + d_e(P_1 - P_2) = -0.2808 + 1.0946 + 0.007816...(3476 - 3042.) = 4.206.m/s$$
$$|w_{22}| = \tilde{w} + d_n(P_1 - P_8) = -0.2808 - 8.7246 + 0.00977...(3476 - 2736..) = 1.7756..m/s$$
$$|u_3| = \tilde{u} + d_e(P_2 - P_3) = -0.048806 + 1.098525 + 0.007816..(3042 - 2357.) = 6.4037.m/s$$
$$|w_{27}| = \tilde{w} + d_n(P_2 - P_7) = -0.2808 - 8.7747 + 0.00977.(3042 - 2481) = 3.5745m/s$$
$$|u_4| = \tilde{u} + d_e(P_3 - P_4) = -0.046714 + 1.0985 + 0.007816...(2357 - 1347.) = 8.9460..m/s$$
$$|w_{37}| = \tilde{w} + d_n(P_4 - P_5) = -0.0062528. - 8.7736 + 0.00977...(1347 - 1214.) = 7.4804..m/s$$
$$|w_{32}| = \tilde{w} + d_n(P_3 - P_6) = -0.01172 - 8.8152 + 0.00977...(2357 - 2010.) = 5.4367..m/s$$
$$|u_7| = \tilde{u} + d_e(P_8 - P_7) = -0.0408 + 1.0315 + 0.007816...(2736 - 2481.) = 2.9834..m/s$$
$$|w_{23}| = \tilde{w} + d_n(P_8 - P_9) = -0.05112 - 8.712 + 0.00977...(2736 - 1899.) = 0.5857..m/s$$
$$|u_8| = \tilde{u} + d_e(P_7 - P_6) = -0.04693 + 1.0315 + 0.007816...(2481 - 2010.) = 4.6631..m/s$$
$$|w_{28}| = \tilde{w} + d_n(P_7 - P_{10}) = -0.0368 - 8.7592 + 0.00977...(2481 - 1778.) = 1.9277..m/s$$
$$|u_9| = \tilde{u} + d_e(P_6 - P_5) = -0.047098 + 1.03517 + 0.007816...(2010 - 1214.) = 7.2096..m/s$$
$$|w_{33}| = \tilde{w} + d_n(P_6 - P_{11}) = -0.016272 - 8.7696 + 0.00977...(2010 - 1516.) = 3.9595.m/s$$
$$|w_{38}| = \tilde{w} + d_n(P_5 - P_{12}) = -0.001584 - 8.7736 + 0.00977...(1214 - 989.) = 6.5769..m/s$$
$$|u_{12}| = \tilde{u} + d_e(P_9 - P_{10}) = -0.0174 + 1.0189 + 0.007816...(1899 - 1778.) = 1.9472..m/s$$
$$|w_{24}| = \tilde{w} + d_n(P_9 - P_{16}) = -0.051096 - 8.696 + 0.00977...(1899 - 977.) = 0.2608..m/s$$
$$|u_{13}| = \tilde{u} + d_e(P_{10} - P_{11}) = -0.0211 + 1.0189 + 0.007816...(1778 - 1516.) = 3.0456..m/s$$
$$|w_{29}| = \tilde{w} + d_n(P_{10} - P_{15}) = -0.03312 - 8.7552 + 0.00977...(1778 - 943) = 0.6304..m/s$$
$$|u_{14}| = \tilde{u} + d_e(P_{11} - P_{12}) = -0.0211 + 1.033 + 0.007816...(1516 - 898.) = 5.8422..m/s$$
$$|w_{39}| = \tilde{w} + d_n(P_{12} - P_{13}) = -0.00952 - 8.792 + 0.00977..(898 - 613.) = 6.0171..m/s$$
$$|u_{17}| = \tilde{u} + d_e(P_{16} - P_{15}) = -0.0084 + 1.0135 + 0.007816...(977 - 943.) = 1.2708..m/s$$
$$|u_{18}| = \tilde{u} + d_e(P_{15} - P_{14}) = -0.0110 + 1.0276 + 0.007816...(943 - 851.) = 1.7357..m/s$$
$$|u_{19}| = \tilde{u} + d_e(P_{14} - P_{13}) = -0.0110 + 1.014 + 0.007816...(851 - 613.) = 2.8632..m/s$$
$$|w_{34}| = \tilde{w} + d_n(P_{11} - P_{14}) = -0.01632 - 8.7848 + 0.00977...(1516 - 851.) = 2.3041..m/s \quad (11)$$

Table 2: Fluid impulses in the x-direction of the control volumes

Control volume	Δx (metres) (Control volume Side length)	$IMPULSE = I = \{(\rho u)^{n+1} - (\rho u)^n\}\Delta Z$ Ns
1	0 - 0.125	$\rho\Delta z(u_2 - u_1) = 1023.6 \times 0.1(4.206 - 1) = 328.1662$
2	0.125 – 0.250	$\rho\Delta z(u_3 - u_2) = 1023.6 \times 0.1(6.4037 - 4.206) = 224.9566$
3	0.250 – 0.375	$\rho\Delta z(u_4 - u_3) = 1023.6 \times 0.1(8.9460 - 6.4037) = 260.2298$
4	0.375 – 0.500	-
5	0.375 – 0.500	-
6	0.250 – 0.375	$\rho\Delta z(u_9 - u_8) = 1023.6 \times 0.1(7.2096 - 4.6631) = 260.6597$
7	0.125 – 0.250	$\rho\Delta z(u_8 - u_7) = 1023.6 \times 0.1(4.6631 - 2.9834) = 171.9341$
8	0 - 0.125	$\rho\Delta z(u_7 - u_6) = 1023.6 \times 0.1(2.9834 - 1) = 203.0208$
9	0 - 0.125	$\rho\Delta z(u_{12} - u_{11}) = 1023.6 \times 0.1(1.9472 - 1) = 96.9554$
10	0.125 – 0.250	$\rho\Delta z(u_{13} - u_{12}) = 1023.6 \times 0.1(3.0456 - 1.9472) = 112.4322$
11	0.250 – 0.375	$\rho\Delta z(u_{14} - u_{13}) = 1023.6 \times 0.1(5.8422 - 3.0456) = 286.2599$

12	0.375 – 0.500	-
13	0.375 – 0.500	-
14	0.250 – 0.375	$\rho\Delta z(u_{19} - u_{18}) = 1023.6 \times 0.1(2.8632 - 1.7357) = 115.4109$
15	0.125 – 0.250	$\rho\Delta z(u_{18} - u_{17}) = 1023.6 \times 0.1(1.7357 - 1.2708) = 47.5872$
16	0.125 – 0.250	$\rho\Delta z(u_{17} - u_{16}) = 1023.6 \times 0.1(1.2692 - 1) = 27.5553$

Table 3: Fluid Impulses in the z-direction of the control volumes

Control volume	Δz (metres) Control volume side width	$IMPULSE = I = \{(\rho w)^{n+1} - (\rho w)^{n}\}\Delta x$ Ns
1	0 – 0.1	$\rho\Delta x(w_{22} - w_{21}) = 1023.6 \times 0.125(1.7756 - 1) = 99.2380$
2	0 – 0.1	$\rho\Delta x(w_{27} - w_{26}) = 1023.6 \times 0.125(3.5745 - 1) = 329.4073$
3	0 – 0.1	$\rho\Delta x(w_{32} - w_{31}) = 1023.6 \times 0.125(5.4367 - 1) = 567.6758$
4	0 – 0.1	$\rho\Delta x(w_{37} - w_{36}) = 1023.6 \times 0.125(7.4804 - 1) = 829.1672$
5	0.1 – 0.2	$\rho\Delta x(w_{38} - w_{37}) = 1023.6 \times 0.125(6.5769 - 7.4804) = -115.6028$
6	0.1 – 0.2	$\rho\Delta x(w_{33} - w_{32}) = 1023.6 \times 0.125(3.9595 - 5.4367) = -189.7081$
7	0.1 – 0.2	$\rho\Delta x(w_{28} - w_{27}) = 1023.6 \times 0.125(0.5857 - 1.7756) = -152.2477$
8	0.1 – 0.2	$\rho\Delta x(w_{23} - w_{22}) = 1023.6 \times 0.125(0.57879 - 1.5229) = -120.789$
9	0.2 – 0.3	$\rho\Delta x(w_{24} - w_{23}) = 1023.6 \times 0.125(0.25566 - 0.57879) = -41.345$
10	0.2 – 0.3	$\rho\Delta x(w_{29} - w_{28}) = 1023.6 \times 0.125(0.63066 - 1.92622) = -165.767$
11	0.2 – 0.3	$\rho\Delta x(w_{34} - w_{33}) = 1023.6 \times 0.125(2.3041 - 3.9595) = -211.8084$
12	0.2 – 0.3	$\rho\Delta x(w_{39} - w_{38}) = 1023.6 \times 0.125(6.0171 - 6.5769) = -71.6264$
13	0.3 – 0.4	-
14	0.3 – 0.4	-
15	0.3 – 0.4	-
16	0.3 – 0.4	-

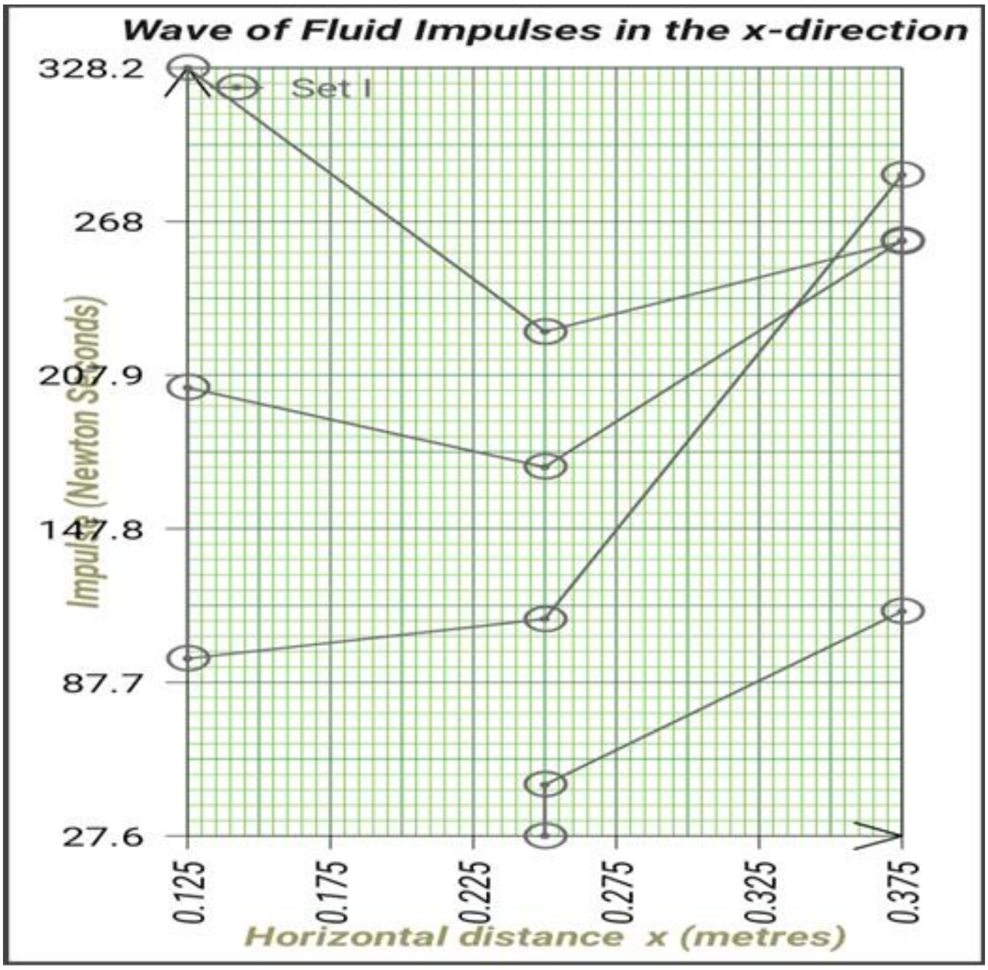

Figure 4: Impulses in the x-direction of the control volumes (Graphical output)

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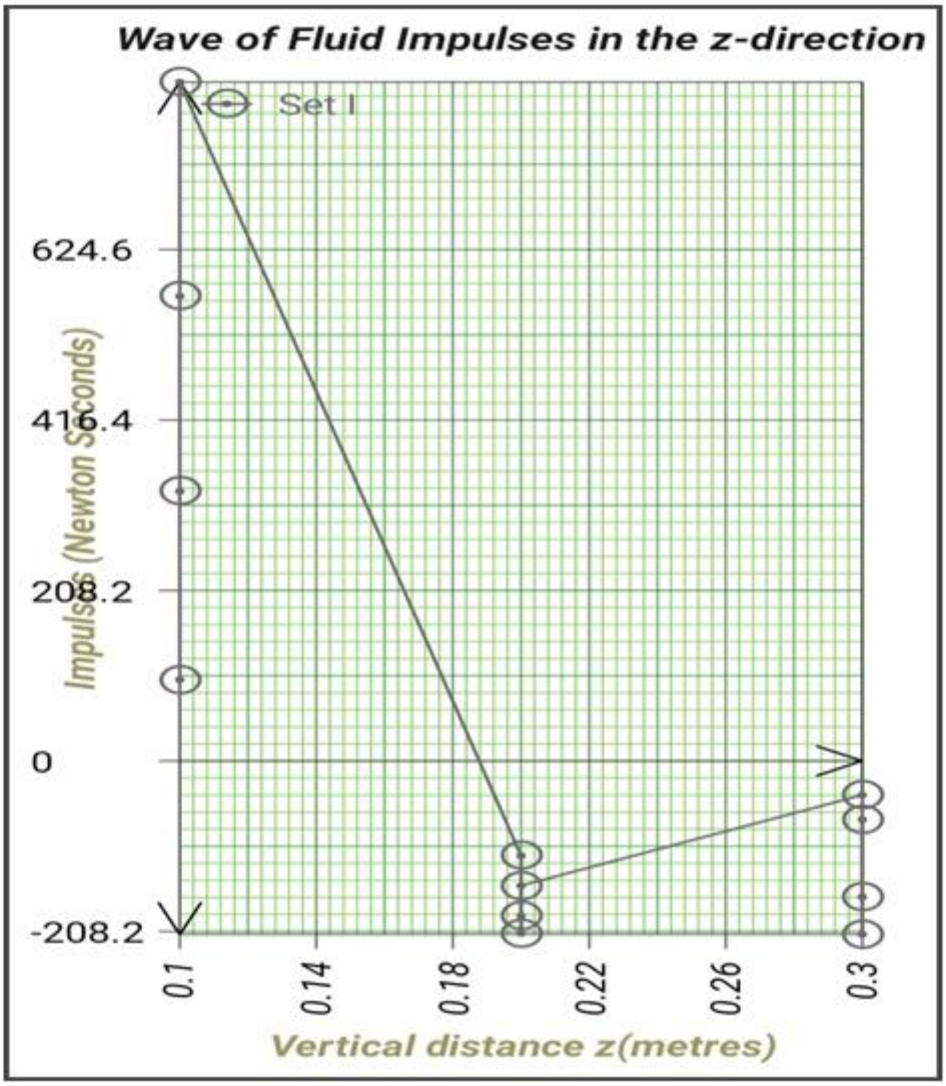

Figure 5: Impulses in z-direction of the control volumes (Graphical output)

5. Conclusion

The change in flux velocities in the control volume interfaces causes impulse. Impulsive effects are higher at deeper levels of sea water than at shallow levels as shown by the graphical outputs.

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