International Journal of Science and Research (IJSR)
ISSN: 2319-7064
SJIF (2019): 7.583

An Overview of Definitions of Riemann-Liouville's Fractional Derivative and Caputo's Fractional Derivative

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Abstract: ***In the present review we survey on Riemann-Liouville fractional derivative and Caputo's fractional derivative. These two definitions are most popular definitions of fractional derivative and plays vital role in the development of the theory of fractional calculus. Further we discuss the difference between Riemann-Liouville and Caputo formula for fractional derivatives.***

Keywords: Gamma function, Beta function, Fractional derivative

1. Introduction

Fractional calculus does not mean the calculus of fraction or the fraction of derivative and integration. Fractional calculus means the derivative and integration of arbitrary order. In fact fractional calculus is the extension of integer order calculus.

Fractional calculus was born in the letters of L'Hospital to Leibnitz in 1695. Leibnitz used the notation $\frac{D^n f(x)}{Dx^n}$ for nth order derivative of a function $f(x)$ in one of his publication. L'Hospital wrote a letter to Leibnitz and asked him "what would be the result if $n = 1/2$". Leibnitz replied him "an apparent paradox from which one day useful consequences will be drawn". In these words fractional calculus was born. From last two decades the study of fractional calculus has become a most popular field of research in Mathematics, Physics and Engineering. Most of the well-known mathematician like Euler, Fourier, Laplace, Lacroix, Leibnitz, and Abeletc. were attracted towards fractional calculus. Recently a tremendous work has been done on fractional calculus. The mathematicians who have contributed directly and indirectly in the development of fractional calculus are N.H.Abel(1823-1826), J. Liouville(1832-1873), Riemann(1847), Holmgren(1865-18670, A. K. Grunwald (1867-1872), A.V.Letnikov (1868-1872), H. Laurent (1884), P. A. Nekrassov(1888), A.Kurg (18890), J. Hadmard (1892), O. Heaviside (1892-1912), G. H. Hardy and J.E. Littlewoods (1917-1928), H. Weyl (1917), Buss (1929), P. Levy (1923), A. Marchaud (1927), H. Devis (1924-1927), Goldman((1949), Oldham and Spanier(1974),L.Debnath(1992),Miller Ross(1993), R.Gorenflo and F.Mainardi (2000), I. Podlubny (2003) and many more.

In the present review article first we shall see the special functions which are useful in fractional calculus. Then we shall see the most popular definitions of fractional derivatives viz. Riemann-Liouville fractional derivative and Caputo fractional derivative. At the end we shall discuss the difference between these two definitions of fractional derivative.

Functions useful in fractional calculus

1) The Gamma Function:

The gamma function on a complex plane is defined as,

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1}\, dt \quad , \qquad Re(z) > 0 \qquad (1)$$

The gamma function plays an important role in the fractional calculus.

We can easily verify that,

$$\Gamma(z+1) = z\Gamma(z)$$

In n is a positive integer then,

$$\Gamma(n+1) = n!$$

2) The Beta function:

The beta function defined on a complex plane is given by,

$$\mathrm{B}(z,w) = \int_0^1 t^{z-1}(1-t)^{w-1} dt \ , Re(z) > 0 \ and \ Re(w) > 0 \quad (2)$$

From the definition we observed that beta function is symmetric

$$i.e. \mathrm{B}(z,w) = \mathrm{B}(w,z)$$

The relation between the beta function and the gamma function given by

$$\mathrm{B}(z,w) = \frac{\Gamma(z)\Gamma(w)}{\Gamma(z+w)}$$

3) The Mittage-Leffler function

The Mittage-Leffler function is very useful in the fractional derivative and integration.It is the generalization of an exponential function.

The one parameter Mittage-Leffler function is given by,

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)} , \qquad \alpha > 0 \qquad (3)$$

The two parameter Mittage-Leffler function is given by,

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$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}, \qquad \alpha > 0 \ and \ \beta > 0 \qquad (4)$$

In the two parameter Mittage-Leffler function if we put $\alpha = \beta = 1$ then we get an exponential function.

$$E_{1,1}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k+1)} = \sum_{k=0}^{\infty} \frac{z^k}{k!} = e^z$$

Now we move towards the definitions of fractional derivative.

Riemann-Liouville definition of fractional derivative

Let α be any positive real number ant n be a natural number such that $n-1 < \alpha < n$. Let $f(t)$ be a continuous function in the interval$[a,T], T > a$. Then the Riemann-Liouville derivative of order α is given by,

$${}_aD_t^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_a^t (t-\tau)^{n-\alpha-1} f(\tau) d\tau \quad (5)$$

Here nth derivative is operated outside the integral sign.

Caputo definition of fractional derivative

Let α be any positive real number ant n be a natural number such that $n-1 < \alpha < n$. Let $f(t)$ be a continuous function in the interval$[a,T], T > a$. Then the Caputo's derivative of order α is given by,

$${}_a^CD_t^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t (t-\tau)^{n-\alpha-1} f^{(n)}(\tau) d\tau \quad (6)$$

Here nth derivative is operated inside the integral sign.

The following important consequence is obtained from Caputo definition of fractional derivative.

Theorem: If $n \in \mathbb{N} \ and \ \alpha \in \mathbb{R}$ such that $n-1 < \alpha < n$, then

$$\lim_{\alpha \to n} {}_0^CD_t^\alpha f(t) = f^{(n)}(t) \quad and \quad \lim_{\alpha \to n-1} {}_0^CD_t^\alpha f(t) = f^{(n-1)}(t) - f^{(n-1)}(0)$$

Proof: Let $n \in \mathbb{N} \ and \ \alpha \in \mathbb{R}$ such that $n-1 < \alpha < n$.

The Caputo fractional derivative of order α of a function $f(t)$ in the $(0,t]$is given by,

$${}_0^CD_t^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-\tau)^{n-\alpha-1} f^{(n)}(\tau) d\tau$$

Integrating by using integration by parts we get,

$${}_0^CD_t^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \left[\left\{ f^{(n)}(\tau) \cdot \frac{(t-\tau)^{n-\alpha}}{-(n-\alpha)} \right\}_{\tau=0}^{\tau=t} - \int_0^t \frac{(t-\tau)^{n-\alpha}}{-(n-\alpha)} f^{(n+1)}(\tau) d\tau \right]$$

$${}_0^CD_t^\alpha f(t) = \frac{1}{(\mathrm{n}-\alpha)\,\Gamma(n-\alpha)} \left[\left\{ 0 + f^{(n)}(0).\, t^{n-\alpha} \right\} + \int_0^t (t-\tau)^{n-\alpha} f^{(n+1)}(\tau) d\tau \right] \quad (7)$$

Using the result $(\mathrm{n}-\alpha)\,\Gamma(n-\alpha) = \Gamma(n-\alpha+1)$ and taking the limit as $\alpha \to n$, we get

$$\lim_{\alpha \to n} {}_0^CD_t^\alpha f(t) = \frac{1}{\Gamma(1)} \left[f^{(n)}(0) + \int_0^t f^{(n+1)}(\tau)\, d\tau \right]$$

$$\lim_{\alpha \to n} {}_0^CD_t^\alpha f(t) = f^{(n)}(t)$$

Now taking the limit as $\alpha \to n-1$ in equation (7) we get,

$$\begin{aligned} \lim_{\alpha \to n-1} {}_0^CD_t^\alpha f(t) &= \frac{1}{\Gamma(2)} \left[f^{(n)}(0)\, t + \int_0^t (t-\tau)\, f^{(n+1)}(\tau)\, d\tau \right] \\ &= f^{(n)}(0).t + \left(t\, f^{(n)}(\tau) \right)_{\tau=0}^{\tau=t} - \int_0^t \tau\, f^{(n+1)}(\tau) d\tau \\ &= f^{(n)}(0).t + t\, f^{(n)}(t) - t f^{(n)}(0) - \left[t f^{(n)}(t) - f^{(n-1)}(t) + f^{(n-1)}(0) \right] \end{aligned}$$

$$\lim_{\alpha \to n-1} {}_0^CD_t^\alpha f(t) = f^{(n-1)}(t) - f^{(n-1)}(0)$$

This says thatfor fractional differential equation with Caputo derivative the initial condition contain the limit values of integer order derivatives of unknown function.

Difference between Riemann-Liouville and Caputo fractional derivative

There are several differences between Riemann-Liouville and Caputo fractional derivative. Out of theses we shall see some notable differences.

1) The main difference between Riemann-Liouville's and Caputo's fractional derivative is in the form of initial conditions. Most of the applied problem needs the fractional derivatives under the utilization of initial condition with known physical interpretation.

Let us consider the following initial value problem which contains the limit values of Riemann-Liouville fractional derivative at lower terminal $t = a$

$$\begin{aligned} &\lim_{t \to a} {}_aD_t^{\alpha-1} f(t) = c_1 \\ &\lim_{t \to a} {}_aD_t^{\alpha-2} f(t) = c_2 \\ &\vdots \qquad \vdots \qquad \vdots \\ &\lim_{t \to a} {}_aD_t^{\alpha-n} f(t) = c_n \end{aligned}$$

$$where \ c_k, k = 1,2, \ldots..n \quad are \ known \ constants$$

The above initial value problem can be solved but the obtained solution does not have any known physical interpretation. On the other hand Caputo derivative allows us to take initial values of classical integer order derivative at lower terminal $t = a$ and the solution obtained is with known physical interpretation. This is the main advantage of Caputo approach.

2) The Caputo fractional derivative of a constant is zero whereas the Riemann-Liouville fractional derivative of a constant with a finite lower terminal is not zero.

The Riemann-Liouville's fractional derivative of a constant C with a finite lower terminal is given by,

$$ {}_aD_t^\alpha C = \frac{C\, t^{-\alpha}}{\Gamma(n-\alpha)} $$

In the definition of Riemann-Liouville fractional derivative if we take the value of lower terminal $a = -\infty$,Then derivative of a constant C becomes zero.

$$ i.e.\ {}_{-\infty}D_t^\alpha C = 0 $$

In the study of fractional ordered steady state dynamical process, the lower terminal $a \to -\infty$ means we have to set the initial time of physical process to $-\infty$.Under this assumption the two definitions coincides.

3) Another difference between Riemann-Liouville and Caputo fractional derivative is in terms of interchange of integer order derivative and fractional order derivative. The interchange of integer order derivative and fractional order derivative is allowed under different conditions.

For Riemann-Liouville derivative we have,

$$ {}_aD_t^k \left({}_aD_t^\alpha f(t) \right) = {}_aD_t^{\alpha+k} f(t) $$

$$ where\ k = 0,1,2, \ldots\ldots\ldots \quad and\ \alpha \in \mathbb{R}\ with\ n-1 < \alpha < n $$

The above interchange is allowed under the condition,

$$ f^{(j)}(0) = 0\,, \qquad j = n, n+1, n+2, \ldots\ldots..k $$

Whereas for Caputo derivative we have,

$$ {}_a^CD_t^\alpha \left({}_a^CD_t^k f(t) \right) = {}_a^CD_t^{\alpha+k} f(t) $$

$$ where\ k = 0,1,2, \ldots\ldots\ldots \quad and\ \alpha \in \mathbb{R}\ with\ n-1 < \alpha < n $$

The interchange is allowed under the condition,

$$ f^{(j)}(0) = 0\,, \qquad j = 0,1,2, \ldots\ldots k $$

4) In case of Riemann-Liouville fractional derivative it is sufficient that the function $f(t)$ is integrable in the interval $[a, t], t > a$, then integral on right hand side of equation (5) exists and it can be differentiated n times whereas in case of Caputo fractional derivative, the function $f(t)$ is $(n-1)$ times continuously differentiable and its n^{th} derivative is integrable in $[a, t], t > a$

Under the condition that the continuous function $f(t)$ is $(n-1)$ times differentiable and its n^{th} derivative is integrable in the given interval then the two definitions are equivalent.

2. Conclusion

In the present review article we have surveyed a quit general approach to the two definitions Riemann-Liouville and Caputo fractional derivative. The Riemann-Liouville definition of fractional derivatives plays an important role in the development of the theory of fractional calculus. It also has tremendous applications in pure mathematics.

Most of the applied problem needs fractional derivative with proper utilization of initial conditions with known physical interpretation especially in the theory of viscoelasticity and solid mechanics. In such cases Caputo approach is more applicable because in Caputo fractional derivative the initial conditions are same as that of integer ordered differential equation and these initial conditions have known physical interpretation of the problem.

The two definitions Riemann-Liouville fractional derivative and Caputo fractional derivative both have their own importance in the theory of fractional calculus and its applications.

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