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An Investigation and Study of Three-World Framework for Solving Algebraic Equations

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Abstract: ***Algebra is widely considered as a key to positive result in secondary mathematics subject. However, instruction remains mostly teacher-centred and in procedural terms of orientation, with limited chances for students to enhance algebraic understanding. In this scope of paper, the research is seriously considering the data from an investigation in which high school students shift from linear to quadratic equations. In this context, the students did not show the term of 'didactic cut' the subtleties grow from conceiving an equation as a balance to figure out the linear equations. Rather than Students utilised the term of 'procedural embodiments', shifting terms around with including 'rules' to gained the fairly accurate answer. To address the with quadratic equations, the high school students teach to find out the formula and seems satisfied not enough with the progress. Furthermore, the scheme of interpretation of this data require a high level of investment within comprehensive structure that places them in developing context. This study succeeds to implement the enhancement of structure of mathematics section regarding the fundamentally on human perceptions and action. Which can use to encourage student to pleasurable more, and avoiding similar problems in future by causing confusion and even mathematical anxiety. This subject of predicting the observed data and structure explains lie beyond the scope of our investigation, which can provide students with a sound theoretical and practical.***

Keywords: Theories of learning; Solving Equations; Three worlds of mathematics

1. Introduction

Improving student outcomes in algebra have been a global focusin recent decades[1,2]. In the U.S., research has shown that school algebra is considered a pivotal gatekeeper to higher-level mathematics and a predictor of later academic success[3]. Proficiency in algebra cannot be achieved without high-quality instruction that creates learning opportunities to help students develop conceptual understanding of algebraic ideas and fluency with algebraic procedures. Investigations into algebra classrooms worldwide have shown variability in emphasis and pedagogy, both between and within countries [4].

In order to improve students' learning opportunities, it is useful to understand specific aspects of teaching that have the potential to support students in learning algebraic ideas. In the U.S., efforts at instructional improvement have largely focused on ambitious mathematics teaching instruction that provides students opportunities to reason about mathematics, explain their thinking, and engage with mathematics in contextualized ways through authentic problems. For instance, the following studies [5] were conducted on developed a theoretical framework on which the teaching and learning of linear algebra in a technological environment with strong emphasis to geometry connections is based: "The core of this framework is three learning/teaching principles: The oncreteness Principle, the Necessity Principle, and the Generalizability Principle".

Studies of [6] are well documented, it is also well acknowledged that formulated the concreteness principle, as a fundamental approach for the teaching and learning of linear algebra. Previous studies [7] have emphasized founded in idea of conceptualentities. According to this principle, "for students to abstract a mathematical structure from a given model of that structure the elements of that model must be conceptual entities in the student's eyes; that is to say the student has mental procedures that can take these objects as inputs" [5]. Concreteness principle requires that "students build their understanding of a concept in a context that is concrete to them". The author recommends MATLAB as a tool that would help students visualize vectors and matrices as concrete mathematical objects, in accordance with the concreteness principle.

One such leverage point in algebra may be how procedures are taught. The degree to which algebra instruction should focus on concepts or procedures (or both) has been a longstanding debate, yet it is likely that the two types of knowledge are intertwined and develop in tandem [8,9].In algebra, procedures are prevalent, even with the current emphasis on teaching for understanding. The goal is not to abandon procedures but rather to teach procedures in ways that connect them with their conceptual underpinnings. For example, if students understand how and why a procedure works, they are better able to adapt procedures flexibly to new situations. Comparing and contrasting multiple procedures for solving the same problem promotes both procedural fluency and conceptual understand-ding. and affords students the opportunity to see and understand algebraic structure [10]. Providing students opportunities to connect procedures, representations (e.g., graphs, tables, and equations), and algebraic ideas supports students' learning of the embedded mathematical relationships and promotes a deeper understanding of expressions, equations, and the related procedures [11, 12].

Another approach is to consider how to support students in making sense of new algebraic ideas. Algebra is more abstract than the mathematics students have previously encountered, something that can cause difficulty for students [13]. Students can develop meaning for these abstractions when they connect symbolic representations for algebraic ideas and procedures to their concrete [14]. The author [15] recommend connecting familiar and concrete representations of mathematical ideas to their abstract and symbolic

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counterparts in order to promote both learning and transfer. Concreteness fading explicitly moving from concrete to more abstract representationshas also been shown to help students develop meaning for more abstract ideas [16].

2. Building a new framework for long term of Mathematics

While various approaches to the curriculum have led to "Math Wars" arguing between approaches to mathematics learning, we can now shift to a higher multi-contextual level where learning "the basics of arithmetic" can be related flexibly to the meaning of expressions. As children experience mathematical ideas in practical contexts, they will naturally pick up aspects related to each context. Making sense of different contexts to draw out common ideas is more complicated than having available simple principles that work in multiple contexts. This is part of a much broader framework for making long-term sense in mathematics as a whole. In How Humans Learn to Think Mathematically [17] formulated a framework for long-term mathematical thinking beginning from the child's perceptions and operations with the physical world and with others in society. One strand of development senses the properties of objects, initially physical, and then constructed mentally, which researcher termed conceptual embodiment.

Another strand focuses on the properties of operations that researcher termed operational symbolism. Both of these develop in sophistication from practical mathematics based on the coherence of properties that occur in practice in theoretical mathematics where the properties are defined and relationships are deduced one from another in what may be termed consequence [18]. At the turn of the twentieth century, a further strand developed based on properties defined using set theory or logic which researcher termed axiomatic formalism. For many mathematicians, formal mathematical proof starts with Euclidean geometry. But there is a huge difference between mathematics based on properties of pictures or on known calculations and mathematics based on formal definition and proof. Prior to the end of the nineteenth century, the study of mathematics and science based on naturally occurring phenomena was described as "natural philosophy". Researcher therefore distinguish "theoretical mathematics" based on "natural phenomena" from "axiomatic formal mathematics" based on set theory and logic (Figure 1.).

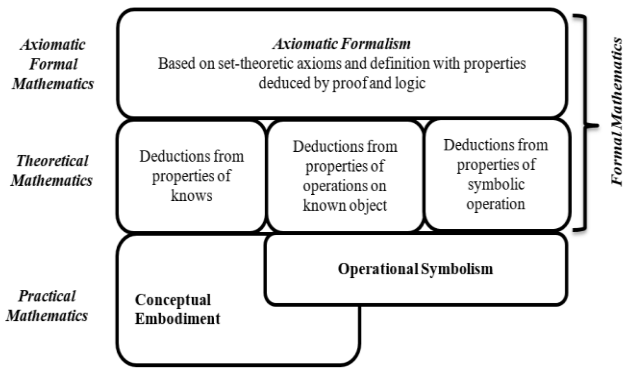

Figure 1: The long-term development of mathematical thinking

Figure 1. simplified view of the theoretical framework developed in [17], based on the new information available from neuroscience. Researcher termed the three main strands as "worlds of mathematics" because each world represents a fundamentally different way of thinking that evolves both in history and in the individual. Conceptual embodiment existed in many species and in human ancestors several hundred thousand years ago. Operational symbolism evolved in Homo sapiens in the last fifty thousand years, proliferating in various communities in Egypt, Babylon, India and China around five thousand years ago, becoming increasingly theoretical in Greek mathematics with the first flowering of mathematical proof two and a half thousand years ago.

Axiomatic formal mathematics has been around for little more than a century. Now new possibilities are emerging in our digital age enabling Homo sapiens to use new digital tools to enhance the possibilities of enactive interface, dynamic visualization, symbolic computation and emergence of new forms of artificial intelligence. In this ongoing evolution, the biological brain evolves slowly. There is no reason to suppose that the biological brain of the ancient Greeks is substantially different from our own. In contrast, the technical evolution of digital tools available to support the mathematical mind that have occurred within a generation is immense. Although researcher now knows that the biological brain is more complex than a simple duality between left and right brain, it still continues to support conceptual embodiment and operational symbolism with the forebrain taking an increasing role in integrating mathematical thinking in new forms of axiomatic formalism [18].

It is interesting to note that the diagram in Figure 1. nowhere explicitly mentions the role of language. Instinctively, when researcher originally thought about the framework, researcher saw mathematical thinking to be related to the complementary roles of visual imagination, sequential symbolic operation and later logical deduction, with verbal language being used to describe connections between different parts of the framework. The resulting two-dimensional picture gives only a partial idea of the broader complexity of the workings of the human brain. For example, it focuses on cognitive aspects that occur in the surface areas of the cortex and says little about the activity of the limbic system in the center of the brain that not only performs many cognitive tasks relating to short-term and long-term memory but also responds emotionally to supportive and problematic aspects of mathematical thinking.

3. Experimental data and theoretical structure for the solution of linear equations

According to theories of learning view, which can improve the phenomena all over time. it is important to noticed that formulated consistent ways that later need to take new data into consideration. The concept of idea is possible for student to comprehend. Thus, specific data in linear equations and the transition to quadratic equations can lead to place in a broader structure for cognitive enhancement that provides together some distinct strands of study within a

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single theory. The authors [19] bring some information about the background of the problem and propose that a formula such as 3x − 1 =5 with an terms on the left anda number on the right is more easier to find out symbolically than an formula such as 3x + 2 = x + 6. The reason behinds that the first canbe 'undone' arithmetically by reversing the function of 'multiply by 3 and subtract 1 to get 5' by add up$1-5$ to get $3x\ = 6$ and thendividing 6 by 3 to get the solution $x\ =\ 2$.

Meanwhile the equation $3x + 2 = x + 6$ cannot be solved by arithmetic undoing and requires algebraic operations to be performed to simplify the equation to give a solution. A more comprehensive description can be found in [20]. This phenomenon is known as 'the didactic cut'. It interfaces to the observation that students to notice the 'equals' sign as an function, growing out of experience in arithmetic where an formula of the structure$3 + 4 = 7$ is known as a dynamic function to perform the calculation, 'three plus four rather than requiring algebraic manipulation [B]. makes 7', Thus, that an formula such as $3x - 1 = 5$ is clear as an operation which can lead to possibly be solved. For example, research [21] suggests that classified an equation of the form 'expression = number' as an evaluation equation, because it involved the numerical evaluation of an algebraic expression where the input value of$x$ could be found by numerical 'undoing', and more general linear equations as manipulation equations, because they required algebraic manipulation for their solution.

It is important to bear in mind, if the solution of linear equations is considered in point of view of the conceptually embodied notion of a 'balance', the complicated of the equations is reversed. The equation 3x + 2 = x + 6 can easily be solved as a balance by imagining the xs to be identical unknown objects of the same weight and representing the equation with 3 xs and 2 units on the left and one x*and* 6 units on the right. It is then possible to remove 2 units from either side to retain the balance as 3x = x + 4, and then remove an x from both sides to obtain 2x = 4, leading to x = 2. In authorship the prophetic study [22] has entitlement 'the balance model: hindrance or encourage for the solving of linear equations with one uncertain. The study has mention that, since the moment a negative quantities or subtraction are concerned, then the embodiment turns into more sophisticated and hinders understanding. By way of illustration, the with 1 bring away if the value of $x$is not known. equation $3x - 1 = 5$ cannot be able to get reshown directly as a balance because the left-hand side $3x - 1$ cannot be able to imagined as $3x$.In this connection, this brings to light that the didactic cut and the balance model provide rise to very different orders of difficulty. In the didactic cut the equation 3x − 1 = 5 is easier to solve than the equation 3x + 2 = x + 6, but in the balance model the order of difficulty is reversed.

The data of study [23] presented an analysis of Brazilian students' work with linear equations that did not fit either the didactic cut or the balance model. Their teachers had used an 'expert-novice' view of teaching and had introduced the students to the methodology that they, as experts, found appropriate for solving equations, using the general principle of 'doing the same thing to both sides' to simplify the equation and move towards a solution. However, when interviewed after the course, students rarely used the general principle. They did not treat the equation as a balance to 'do the same thing to both sides', nor did they show any evidence of the didactic cut. Instead, they focused more on the specific actions that they performed to shift symbols around and 'move towards a solution' using two main tactics:

1) 'swop sides, swop signs' to get:
in which an equation $3x - 1 = 3 + x$ is operated upon by shifting the 1 to the right and the x to the left and changing signs

$$3x - x = 3 + 1$$
$$2x = 4.$$

2) 'swop sides and place underneath' in which the 2 associated with the expression $2x$ in the equation above is moved from one side of the equation to the
other, then placed underneath to give:

$$x = \frac{4}{2} = 2$$

In an attempt to use such rules, some students made mistakes, such as changing 2x = 4 to:

(a) $x = 4 - 2$ (b) $x = \frac{4}{-2}$ (c) $x = \frac{4}{2}$

In $(a) X = 4 - 2$ the 2 is passed over the other side and its sign is changed; (b) $x = \frac{4}{-2}$ correctly 'shifts the 2 over and puts it underneath' but also 'swops the sign'; (c) $x = \frac{4}{-2}$ shifts the 2 over and puts the 4 underneath. When questioned, no student mentioned the principle of 'doing the same thing to both sides', instead they developed what Lima and Tall called procedural embodiments which involved embodied actions on the symbols to 'pick them up' and 'move them to the other side' with an extra 'magic' principle such as 'change signs' or 'put it underneath' to 'get the right answer'. Procedural embodiments worked for some students but they also proved to be fragile and misremembered by many others, leading to the wide range of errors that are well known in the literature [24, 25]. The aim is not easy to achieved and catalogue errors. The study seeks to evolve a single theoretical framework that covers all three aspects: the didactic cut, the balance model and the problem with 'doing the same thing to both sides'. Such a theoretical framework can be relating to both cognitive development and the emotional effects of the learning experience.

4. The Development of Algebraic Reasoning

The issue-posing paper researcher identify takes in account the domain of algebra. The research has been carried out with attention to detail. The learning algebra requires mastery of abstract symbolic systems of representation, and require students to make integrate into between different representational forms. Moreover, students can be a critical transition from operating on known quantities based on symbols to represent unknown quantities. In this regard, manipulate, solve equations, and to represent functions in various formats [26, 27]. Students might view an algebraic equation as a string of operations rather than a statement of

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equality and might have difficulty operating on variables conceptualized as both fixed unknown quantities and quantities that change [28]. Students struggle to navigate the structural "grammar" of algebraic expressions, to symbolically represent their intuitive understanding of relationships, research suggests that embedding algebra in relevant contexts can allow students to draw upon their everyday knowledge and informal ways of reasoning to support their learning. In this way, personalization may be a particularly effective approach to allowing students to grasp the abstractions and multiple representations involved with learning algebra.

5. The three worlds of Mathematics

This is the field of study [17,29] that deals with the framework of three worlds of mathematics, which is an total theory of cognitive and meaningful increase in mathematics that has improved to build from the early enhancement of ideas in the child, by the years of schooling and on to the boundaries of study in formal mathematics. It is closely connected to a wide range of theoretical frameworks formulated by [30], theories of advanced mathematical thinking by [31]. Moreover, theories from cognitive science such as the embodied theory of Lakoff and his colleagues [32]. In this regard, the blending of cognitive structures formulated for example by Fauconnier and Turner [33] and other aspects such as the role of various levels of consciousness by [34]. These approaches have been influential in the field by [17]. On the other hand, the key objective of the theoretical framework is not to collate all these theories together with all their intricate details that differ in many ways, but to find the essential ideas that they have in common sense . As a rule, the learning of school mathematics involve that the student blends together.In this context, the proposed by the following authors Fauconnier and Turner [35], embodied perception and operation that can be lead to geometry. At the higher levels of school mathematics, methods of reasoning lead to Euclidean proof in geometry and symbolic proof– based on the 'rules of arithmetic' – in arithmetic and algebra. Moreover, in university can used mathematicians broadly build on their experience of natural phenomena to construct mathematical. The one hand arithmetic and algebra on the other. Bothof them can be blended together, for instance, through representation of relationships in the Cartesian plane, where perceptual ideas of dynamic change are related to operational techniques for computing change and growth in calculus models that can be used to reason about situations and compute solutions. Pure mathematicians take natural ideas and translate them into formal objects specified set-theoretically and deducing their properties using mathematical proof. Underlying this whole enhancement is the nature of the species Homo Sapiens where the child builds on initial sensory perception and action and evolves increasingly sophisticated forms of mathematical thinking using language and symbolism. The sensory side develops through exploring and interacting with the structures of target, recognizing properties, Applinglanguage to describe, define and deduce relationships in an increasingly sophisticated mental world of conceptual embodiment that includes geometry and other perceptual representations; it develops over the longer term from physical perception to increasingly subtle mental imagination using by experiments. This may be described using the four van Hiele levels that may usefully be subdivided into two distinct forms of thinking: the practical ideas of shape and space developed through recognition and description and the theoretical ideas of Euclidean geometry developed through definition and deduction using Euclidean proof [17].

6. Tasks with quadratic equations

In this task with quadratic equations, the data used to investigate the students' conceptions of quadratic equations came from two instruments, an equation solving task, with three linear equations and four quadratic equations:

$$3l^2 - l = 0,\ r^2 - r = 2,\quad a^2 - 2a - 3 = 0,\quad m^2 = 9,$$

together with a questionnaire that included two quadratic equations:

$$t^2 - 2t = 0 \quad (y - 3)(y - 2) = 0.$$

The questionnaire also included a request to respond to the solution of the final quadratic equation as given by an imaginary student 'John': Interviews with some selected students revealed additional personal information on how they interpreted the tasks and their thinking in seeking solutions.

7. Conceptual embodiment and the transition to operational symbolism

Students' responses bring little evidence of attempts to make use of conceptual embodiments of equations. Indeed, if we look at previous research involving both linear and quadratic equations, we find that such embodiments tend to have limitations beyond the simpler cases. The work, for example, has already shown how the conceptual embodiment of a linear equation as a balance proves to be supportive in simple cases but is problematic where negative quantities are involved. In relation to quadratic equations, an interesting visual approach arose from the time of the Babylonians, and extended in Arabic mathematics in terms of physically 'completing the square'. Based on this idea, Radford and Guérette [36] designed 'a teaching sequence whose purpose is to lead the students to reinvent the formula that solves the general quadratic equation'. An example is given in Figure 2.

The length of rectangle is 10 units; its width is unknown. The study place square on one of the sides of the rectangle as shown in the figure. Together the two shapes have an area of 39 square units.

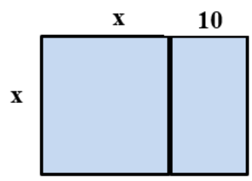

Figure 2: The Babylonian geometric model

The pieces were cut out of cardboard and the solution could be found by cutting the rectangle vertically in half (Figure 3a), rearranging the pieces to move one half rectangle round to the bottom (Figure 3b), then realizing that what is missing to 'complete the square' is the corner square with sides 5×5. Filling this in to get a total area of $39 + 25 = 64$ units (Figure 3c), we find the larger square has side 8 units and so, taking off the 5 units leaves $x = 3$. Students were

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then encouraged to think of a number of similar examples and derive a symbolic solution to equations of the form $x2 + bx = c$ to find the general solution. Figure4. presents attempting to cut off two rectangles of size 5 × x.

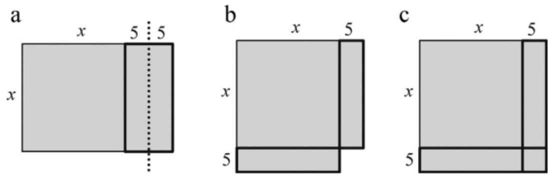

Figure 3: (a) Cut the 10 $\times$ x rectangle, (b) rearrange the pieces, (c) complete the square

$$x = \sqrt{c + \left(\frac{b}{2}\right)^2} - \left(\frac{b}{2}\right).$$

They were shown how $ax^2 + bx = c$ could be rewritten as x^2 + (b/a)x = (c/a), and substituting $b/a\,for$ band $c/a\,for\,c$ gives the general solution of $ax^2 + bx = c$ as:

$$x = \sqrt{c + \left(\frac{b}{2a}\right)^2} - \left(\frac{b}{2a}\right).$$

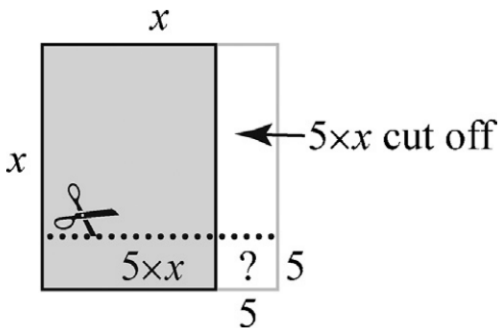

Figure 4: Attempting to cut off two rectangles of size 5 × x.

The next step suggested is to replace c by $-c$ to obtain the solution of $ax^2 + bx + c = 0$ as:

$$x = \sqrt{-c + \left(\frac{b}{2a}\right)^2} - \left(\frac{b}{2a}\right).$$

This formula is equivalent to the well-known formula.

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

where, in order to obtain all the numerical solutions, one also needs to consider the negative square root $of\ b^2 - 4ac$. This leads us to the formula:

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

The authors suggest that this is a good way to introduce the quadratic formula for students because it relates geometry and algebra, aiming 'to provide a useful context to help the students develop a meaning for symbols'. They note that many students were able to solve the initial tasks but 'need some time to abandon the geometrical context themselves to the numerical formulae', commenting on the complexity of the semiotic structures, without any explicit reasons for the difficulties. The three-world framework clarifies the details.

The representation of variables geometrically as lengths requires the adding rectangles 5 $\times$ x, this involves cutting them away. Having cut off one rectangle from the right-hand side of the square, quantities to be positive. If the same method is applied to an equation of the form $x^2 - bx = c$ such as $x^2 - 10x = 64$, instead of as in Fig. 4, the lower right 10 × 5 square has already been removed, so it is no longer possible to cut away the full rectangle size 5 × x along the bottom.

8. Conclusion

The development of algebra is part of the whole growth of mathematical thinking which is formulated as blending embodiment and symbolism in school mathematics, leading to embodied and symbolic forms of reasoning, which are later transformed into an axiomatic formal world of set-theoretic definition and proof in university pure mathematics. The three-world framework formul-ates the cognitive and affective development of mathematical thinking over a lifetime from a newborn child to the full spectrum of adult mathematical thinking. It includes the effects of supportive met-before that enable generalizations in new contexts and problematic met-before that impede progress, with a growing awareness of the crystalline structure of mathematical concepts that enable them to be grasped and manipulated as mental entities with flexible meaningful links between them.

The particular study of the solution of linear and quadratic equations occurs in operational symbolism with some support from embodied representations. The forms of reasoning appropriate to school algebra involve more formal use of embodiment and symbolism without any reference to the third world of axiomatic formalism. The reasoning in the solution of algebraic equations builds symbolically on the operations of generalized arithmetic, shifting from evaluation equations to equations requiring more general symbolic manipulation that give rise to the problematic aspects of the didactic cut. This may be blended with various conceptual embodiments such as seeing the solution of equations as the intersection of graphs, imagining the equation as a physical balance or cutting up squares in the case of quadratic equations.

Methods that work with physical quantities such as the equation as a balance, or the representation of x^2 as a physical square – become problematic when negative quantities are introduced. The introduction of more general strategies, such as 'doing the same thing to both sides' prove to be problematic for students who interpret the generalities in terms of procedural symbol-shifting. The perceptual divide reveals a spectrum of performance between those who remain limited to learning step-by-step procedures and those with the flexibility of being able to grasp the crystalline structure of mathematical concepts.

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