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Innovating Extension of "Almost Pythagorean Triples" in Prime Numbers and Possible Uses in Science

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Abstract: ***A Pythagorean triple (PT) consists of three natural numbers a, b, and c, which follow the mathematical relation:$a^2 + b^2 = c^2$, and is commonly written as (a, b, c). Pythagorean triples are useful in various fields of Science and Technology, especially in Mathematics. In recent past, some new triples (x, y, z) of 3 natural numbers were found tofollow the mathematical relation: $x^2 + y^2 = z^2+1$, (very close to the PT relation), and have been given the name "Almost Pythagorean Triples" (APTs). I have found several Almost Pythagorean Triplesin prime numbers and in my present work, I have shown that their relation has very small percentage of deviation (PD%) from PT relation. It has been shown that multiples of an APT (MAPTs) show same amount of percentage of deviation(from PT relation) as the corresponding APT. APTs and MAPTs can have several uses and applications in Mathematics and other fields of Science & Technology.***

Keywords: Pythagorean triple, Almost Pythagorean Triples, prime numbers, percentage of deviation

1. Introduction

The famous "Pythagorean theorem" gives the mathematical relation among three sides of a right triangle, stating that the area of the square whose side is the hypotenuse (the side opposite the right angle) is equal to the sum of the areas of the squares on the other two sides [1]-[3]. Thus, the relation can be expressed as a mathematical equation (often called the "Pythagorean equation") for a right triangle of side-lengths a, b, and c, in the form: $a^2 + b^2 = c^2$, where c is the longest side opposite to the right angle, called "hypotenuse". The Pythagorean theorem, first proved by Pythagoras of Samos during 6th century B.C., has been a very important fundamental relation in Euclidean geometry among the three sides of a right triangle in a plane. This theorem has been utilized in various fields of Mathematics, like geometry, mensuration, trigonometry, vector algebra, etc. and different branches of Science and Technology. There are many proofs available for Pythagorean theorem, from Pythagoras' own proof in 6th century B.C., through Euclid's proof, Thabit ibn Qurra's proof in 9th century, the Indian mathematician Bhaskara's proof in 12th century, James Grafield's proof in 19th century to modern proofs of this theorem [1].The famous book written by E. S. Loomis, entitled "The Pythagorean Proposition" (whose first edition was published in 1928, and second edition in 1940), provided a major collection of proofs of Pythagoras theorem [1]. This book has collected as many as 370 different proofs of Pythagorean Theorem.

Let us now discuss briefly the concept of a "**Pythagorean triple**" (**PT**). APythagorean triple consists of three natural numbers (i.e., positive integers) *a*, *b*, and *c*, such that $a^2 + b^2 = c^2$, [4]-[6],and is commonly written as (*a*, *b*, *c*). Some well-known examples are (3, 4, 5), (5, 12,13), (8,15,17), etc. Thus, Pythagorean triples describe the three integer side lengths of a right triangle. However, right triangles with non-integer sides do not form Pythagorean triples, e. g., the triangle with sides $a = b = 1$ and $c = \sqrt{2}$ is a right triangle, since $1^2 + 1^2 = (\sqrt{2})^2$, but (1, 1, √2) is not a Pythagorean triple because √2 is not an integer. Pythagorean triples have been known since ancient times. The oldest known record came from a Babylonian clay tablet (ancient Cuneiform tablet, dating from about 1800 BC), known as "Plimpton 322" (named after George Arthur Plimpton), which was discovered by Edgar James Banks shortly after 1900, and was sold to George Arthur Plimpton in 1922, for $10[7],[8]. This Plimpton Collection, one of the world's most famous ancient mathematical artefacts, is preserved in the Rare Book and Manuscript Library of Columbia University. Plimpton 322 is just one of several thousand mathematical documents surviving from ancient Iraq (also called Mesopotamia), and in its current state, it comprises a four-column, fifteen-row table of Pythagorean triples (in sexagesimal number system), written in cuneiform (wedge-shaped) script on a clay tablet measuring about 13 by 9 by 2 cm,[7]. Pythagorean triples were also known in India, the earliest Baudhāyana-Sulbasutra contains five such triples[6], [9].

A **primitive Pythagorean triple** (PPT) is a Pythagorean triple (a, b, c), in which a, b and c are **co-prime, i.e., highest common factor** (HCF or GCD) **of a, b, and c is 1**[5], [6], [9], [10]; thus they do not have any common factor other than 1. The most common example of PPT is (3,4,5).Kak and Prabhu(2014) [9] have discussed about primitive Pythagorean triples forCryptographic applications. They identified certain sequences and named those as "Baudhāyana sequences", after the name of Baudhāyana, the author of one of the earliest Sulbasutras (documents containing some of the earliest Indian mathematics), who used Pythagorean triples several centuries before Pythagoras (i.e., dating back to the sixth century before Christ).

There are 16 Primitive Pythagorean Triples (a, b, c) with c less than 100,[6], as shown in **Table-1**, where we find that $(a^2 + b^2) = c^2$.

Table 1: Primitive Pythagorean Triples (a, b, c) with c <100

S. N.	Primitive Pythagorean Triple (a, b, c)	a^2	b^2	$a^2 + b^2$	c^2
1.	(3,4,5)	9	16	25	25
2.	(5,12,13)	25	144	169	169
3.	(8,15,17)	64	225	289	289
4.	(7,24,25)	49	576	625	625
5.	(20,21,29)	400	441	841	841
6.	(12,35,37)	144	1225	1369	1369
7.	(9,40,41)	81	1600	1681	1681
8.	(28,45,53)	784	2025	2809	2809
9.	(11,60,61)	121	3600	3721	3721
10.	(16,63,65)	256	3969	4225	4225
11.	(33,56,65)	1089	3136	4225	4225
12.	(48,55,73)	2304	3025	5329	5329
13.	(13,84,85)	169	7056	7225	7225
14.	(36,77,85)	1296	5929	7225	7225
15.	(39,80,89)	1521	6400	7921	7921
16.	(65,72,97)	4225	5184	9409	9409

Given a Primitive Pythagorean triple (a, b, c), its multiple (na, nb, nc) is also a Pythagorean triple,where n is a natural number, this is because of the fact that, $\{(na)^2 + (nb)^2\} = (nc)^2$.**Table-2** shows some multiples of PPT (3, 4, 5), all of these follow the PT relation. It may be noted that for n=1, it is the PPT itself.

Table 2: Some Multiples of the Primitive Pythagorean triple (3, 4, 5)

S. n	Pythagorean Triple (na, nb, nc)	$(na)^2$	$(nb)^2$	$(na)^2 + (nb)^2$	$(nc)^2$
1	(3, 4, 5)	9	16	25	25
2	(6, 8, 10)	36	64	100	100
3	(9, 12, 15)	81	144	225	225
4	(12, 16, 20)	144	256	400	400
5	(15, 20,25)	225	400	625	625
6	(18, 24, 30)	324	576	900	900

Similarly multiples of other PPTs also can be shown to be Pythagorean triples.

1) Almost Pythagorean triples (APTs) in Prime numbers:

Orrin Frink (1987) [11] introduced some new triples (x, y, z) of 3 natural numbers, which show the mathematical relation: $x^2 + y^2 = z^2+1$, and named them as "Almost Pythagorean Triples", as the above relation is very close to the Pythagorean Triple (PT)relation: $a^2 + b^2 = c^2$. While squaring & adding terms of the sequence 5,10,15,20,25,30, 35, …, he noticed that $(10^2 + 15^2) = (18^2 + 1)$; $(20^2 + 25^2) = (32^2 + 1)$; $(25^2 + 35^2) = (43^2 + 1)$, etc. So, he suggested for solving the Diophantine equation:$x^2 + y^2 = z^2+1$, and called its positive integer solutions (x, y, z) of the above equation as "Almost Pythagorean Triples" (APTs). The first six APTs, in increasing values of "z", were found to be (5, 5, 7), (4,7, 8),(8, 9, 12), (7, 11, 13), (11, 13, 17) and (10, 15, 18).

I have found that **several prime number triples** follow the above mathematical relation of **"Almost Pythagorean Triples"**. For extending the above new concept of APTs [11]for special case of prime numbers, let me represent these triples as **(a, p, t),** which will befollowingthe mathematical equation:

$$(a^2 + p^2) = t^2+1 \qquad \dots (1)$$

It may be noted that a prime number is a natural number, which is divisible by 1 and itself only. In case, all the three numbers in the APT are distinct prime numbers, they are co-prime also, since they will have 1 as the only common divisor. Hence, most of the APTs in prime numbers are also Primitive Almost-Pythagorean Triples (a, p, t), following the equation-(1).

Table-3 gives listing of 12 APTs in prime numbers. The list can be extended anytime, if someone is interested in this. I wish to show that these APTs have very close relation with Pythagorean Triple (PT) relation. For this, let me find the Percentage of deviation of APT (a, p, t) from PT relation using the following formula:

Percentage of deviation of APT (PD in %)from PT relation= $[\{(a^2 + p^2) - t^2\}/ t^2]*100 = [1/ t^2]*100$ = $\mathbf{[100/ t^2]}$ … **(2)**

Table 3: Almost-Pythagorean Triples (APTs) in Prime numbers

APT in Prime numbers (a, p, t)	$a^2 + p^2$	t^2	PD in %*	t^2+1
(5,5,7)	50	49	2.04	50
(7,11,13)	170	169	0.59	170
(11,13,17)	290	289	0.35	290
(13,19,23)	530	529	0.19	530
(23,29,37)	1370	1369	0.07	1370
(13,41,43)	1850	1849	0.05	1850
(29,37,47)	2210	2209	0.05	2210
(31,43,53)	2810	2809	0.04	2810
(41,53,67)	4490	4489	0.02	4490
(43,59,73)	5330	5329	0.02	5330
(43,71,83)	6890	6889	0.01	6890
(61,83,103)	10610	10609	0.01	10610

[*PD (in %), calculated correct up to second decimal place]

Table-3 also shows Percentage of deviation of the abovementioned 12 Almost-Pythagorean Triples in Prime numbers (PD% from PT relation), calculated correct up to second decimal places, using the above formula (2).As seen from Table-3, we find that APTs have very smallPercentage of deviation from PT relation, less than 0.1% for most of the cases. The PD% decreases sharply as the numerical values of (a, p, t) increase, e.g., for APTs in Table-3, PD% decreases from 2.04% for APT (5, 5, 7) to ~0.01% for APT(61,83,103). Also, another interesting finding from Table-3 is that, the unit's digit of "t" (the third number of the APT) is either 7 or 3; and the unit's digits in the numerical values of $(a^2 + p^2)$ and (t^2+1) are zero (0).

The finding of these Almost-Pythagorean Triples in Prime numbers is very relevant in this year (2021), as it is believed that **this year is going to be a year of prime numbers**, as the number 2021 is product of two consecutive prime numbers 43 and 47. My cousin Dinabandhu forwarded me a message which had a mathematical meaning: "For a mathematics lover, 2021 is product of two prime numbers, 43 and 47, (as 43*47 = 2021). Interestingly, it can also be expressed as the product (a-b)*(a+b), as 2021 is(45-2)*(45+2) = 43*47…So, **2021 appears to havea partner in Prime.** The number 2021 is both the concatenation of consecutive integers (20 & 21) and the product of two consecutive primes 43*47.Also, multiplication of 2021 by its reverse number 1202, is a palindrome (i.e., same, when read from both sides, front & opposite sides), as2021*1202 = 2429242, which will be same 2429242 when read from the opposite side". Many Mathematics lovers may like the message.

2) Multiples of Almost-Pythagorean Triples:

If (a, p, t) represents an APT, then (ma, mp, mt) with "m" being a natural number, can be called Multiples of Almost-Pythagorean Triples (MAPT).

For simplicity, let **A = ma , P = mp , and T = mt**, where m = 1, 2, 3, …

Then the **MAPT (A, P, T)** will satisfy the following formula: $(A^2+ P^2) = (T^2 + m^2)$**… (3)**; obviously, for m=1, it will be the APT (a, p, t) itself.

For MAPTs (A, P, T), we can define their Percentage of deviation from Pythagorean Triple (PT) relation, as follows: **Percentage of deviation** from PT relation = $[\{(A^2 + P^2) - T^2\}/ T^2]*100 = [\{ m^2 \}/ T^2]*100 = \{100* m^2 \}/ T^2$ **… (4)**

In **Table-4**, I have taken an APT in 3 prime numbers (13, 19, 23) and have listed various multiples of this APT, all of which follow equation (3), and also calculated PD %, using formula (4).

Table 4: Some Multiples (A, P, T) of the Almost - Pythagorean Triple (13, 19, 23)

m	MAPT (A, P, T)	$(A^2 +P^2)$	$(T)^2$	PD in % *	$(T^2 +m^2)$
1	(13,19,23)	530	529	0.19	530
2	(26,38,46)	2120	2116	0.19	2120
3	(39,57,69)	4770	4761	0.19	4770
4	(52,76,92)	8480	8464	0.19	8480
5	(65,95,115)	13250	13225	0.19	13250

[*PD (in %), calculated correct up to second decimal place]

From **Table-4**, it is found that all such MAPTs of APT(13,19, 23) show equal values of PD (0.19 %). If we extend this Table with higher values of "m", we shall get the same amount of Percentage of deviation from PT relation.Also, we can try with any other APT and we shall find the multiples (MAPT) will have same amount of PD %, as the corresponding APT. Thus, MAPTs can be equally useful as APTs.

3) Usefulness of approximate relations, quasi-conditions, semi-methods, etc. in Science

Specialists and experts may point out that why should we study about some approximate or almost relations, when there are so many exact and perfect relations available in Science & Technology. For this, let us have some discussions here. In various branches of Science and Technology, "Quasi" word is often utilized for describing an approximation or some approximate conditions or approximate values/states. In Earth & Atmospheric Sciences, this word has been utilized for different approximate conditions, like quasi-geostrophic &quasi-hydrostatic atmosphere. The great Meteorologist and pioneer of successful numerical weather prediction(NWP), Dr.Jule Gregory Charney(1948) [12]considered horizontal velocity of wind to be "quasi-geostrophic" and atmospheric pressure to be "quasi-hydrostatic", while modelling the motion of large scale atmospheric disturbances [12]. Similarly, Baldwin et.al., (2001) [13] studied about the "quasi-biennial oscillation (QBO)", which dominates the variability of the equatorial stratosphere (∼16–50 km) and is easily seen as downward propagating easterly and westerly wind regimes, with a variable period averaging approximately 28 months (i.e., little more than 2 years' periodicity). From a fluid dynamical perspective, the QBO is a fascinating example of a coherent, oscillating mean flow that is driven by propagating waves with periods unrelated to that of the resulting oscillation [13]. Although the QBO is a tropical phenomenon, it affects the stratospheric flow from pole to pole by modulating the effects of extra-tropical waves. Indeed, study of the QBO is inseparable from the study of atmospheric wave motions that drive it and are modulated by it. Thus research on QBO is very important, even though it is a quasi-periodic oscillationof the equatorial zonal wind in the tropical stratosphere. In Mathematics, there have been discussions about "Quasi-Pythagorean Triples" for an Oblique Triangle, by Kay Dundas (1977) [14].

A.J. Robert (1982) [15] developed a"semi-lagrangian and semi-implicit" numerical integration scheme for the primitive meteorological equations for weather prediction. I utilized a dynamic model with semi-Lagrangian semi-Implicit time integration scheme, for studying the impact of digital filtering Initialization on the model performance for short range weather prediction over Indian region [16]. There are many other examples of uses of such semi-techniques. Thus, study of quasi, semi, approximate or almost relations can be useful in certain branches of Science & Technology. In the following section, I shall give some examples, how APTs and MAPTs can be useful.

4) Some useful applications of APT & MAPT

Let us take the APT (29, 37, 47) and check how can it be useful.

(A) In mensuration, we calculate area of triangle by various methods. Now for calculation of area of a triangle of sides 29 cm, 37 cm, and 47 cm,we can assume that it is almost a right angled triangle, as (29, 37, 47) is an APT, the largest side 47 resembles the hypotenuse, and the area is calculated simplyas half of the product of other two sides, i.e., (1/2)*29*37 = **536. 5 sq. cm**

Otherwise, if we calculate it by Heron's formula, first we need to calculate the semi-perimeter value, s =(1/2)*(a+p +t) = (1/2)*(29+37+47) = 56.5

Then, s*(s-a)*(s-p)*(s-t) = 56.5*27.5*19.5*9.5 = 287832.1875. So, the area of the triangle will be square root of 287832.1875, which comes out to be **536.49994 sq.cm**, if we go correct up to 5 decimal points (and it will be 536.5 sq. cm, if we go correct up to 1 decimal point only).

Someone can argue that it is not an exact right-angled triangle, hence the second method gives the accurate result, while the first method will have some error. It is quite correct, but if we compare the results, we find that both results are almost same and they are exactly same if we go for the value correct up to first decimal point. Even going up to 5 decimal points, the percentage error for the first method will be ~0.00001%, which is quite negligible for any practical purpose. But the benefit of the first method is that it takes much less time, much less computations and in modern day objective question system, it is very useful as the student gets very small time for answering each question. Thus, knowledge of APTs can be useful for school students as well as their teachers.

(B) In trigonometry, we can calculate the angles by values of the sides.For the triangle of sides 29 cm, 37 cm, and 47 cm, we can tell that angle opposite to 47 cm side is nearly 90°, as (29, 37, 47) forms an APT and 47 is the highest value (so that largest side of 47cm resembles the hypotenuse). The other two angles can be calculated simply through Trigonometrical ratios (sin, cos, tan). On the other hand, for calculation of angles, if we use cosine formulation of properties of triangles, we need to calculate many things including squares of the sides. So, the first method is much easier and faster, with very small percentage error in the result.

If the sides of the triangle are multiples of APT (MAPT), then also we can use the easier method, as described in (A) and (B).

(C) In analytic geometry, the equation of APT represents a hyperboloid of revolution of one sheet, on a doubly ruled surface [11].

(D) Concepts of APT and MAPT can be useful in number theory and possibly it can lead to future evolution of these concepts.

Thus, knowledge of APTs and MAPTs can be very useful in Mathematics. As many branches of Science & Technology often utilize Mathematical concepts, therefore the applications can be manifold. I think that these concepts can also be utilized in artificial intelligence (AI) algorithms, as relations of APTs and MAPTs have certain patterns, which a machine (i.e., computer) can learn, just as human brains learn new things and later utilize those.

2. Conclusions

In this paper, I have discussed about "Almost Pythagorean Triples" (APTs) in a very simple and popular manner and highlighted my finding of several APTs in prime numbers. I have shown that APTs show very small percentage of deviation(PD)from Pythagorean Triple (PT) relation, even less than 0.1% for most of the APTs, and hence these can have scientific applications with very small margin of error. Multiples of APTs (MAPTs) show same amount of PD %, as the corresponding APT and hence can be equally utilized. Through certain examples, I have argued that these concepts can be useful in Mathematics as well as various branches of Science & Technology. I hope that these concepts will be well taken by the scientists and researchers.

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