

Extension of Definition of First Order Derivative (First Principle) to Obtain New Definition of Conformable Fractional Derivative

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Abstract: In this paper we define new definition conformable fractional derivative and verify its validity for Linearity property, product rule, Quotient rule, and verify derivative of some Standard function. The definition satisfies the previous results for ordinary derivative and derivative of some standard function. This definition coincide with the classical definition of first order derivative.

Keywords: fractional derivative, new definition of conformable fractional derivative

1. Introduction

Fractional derivative is as old as calculus. L'Hospital in 1695 asked what does it mean if $\frac{d^n f}{dx^n}$. If $n = \frac{1}{2}$. Since then, many researchers tried to put a definition of a fractional derivative. Most of them used an integral form for the fractional derivative.

Two of which are the most popular ones.

- 1) Riemann liouville definition. For $\alpha \in [n - 1, n)$ the α derivative of f is

$$D_a^\alpha = \frac{1}{\Gamma(n - \alpha)} \frac{d^n}{dt^n} \int_a^t \frac{f(x)}{(t - x)^{\alpha - n + 1}} dx$$

- 2) Coputo definition. For $\alpha \in [n - 1, n)$ the α derivative of f is

$$D_a^\alpha = \frac{1}{\Gamma(n - \alpha)} \int_a^t \frac{f^n(x)}{(t - x)^{\alpha - n + 1}} dx$$

2. Methodology

Now, all definitions including (i) and (ii) above satisfy the property that the fractional derivative is linear. This is the only property satisfied from the first derivative by all of the definitions.

However, the other properties of fractional derivative such as product rule of derivative and derivative of constant is zero and quotient or division rule for derivative, derivative of chain rule following are some of the rules that not satisfied by other definitions:

- 1) The Riemann–Liouville derivative definition does not satisfy $D_a^\alpha(1) = 0$ but Caputo derivative satisfies $D_a^\alpha(1) = 0$, if α is not a natural number.
- 2) All fractional derivatives do not satisfy the formula of the derivative of the product of two functions:

- 3) All fractional derivatives do not satisfy of the formula derivative of the quotient or division of two functions:

- 4) All fractional derivatives do not satisfy the chain rule:

$$D_a^\alpha(f \circ g) = f^\alpha(g(t))g^\alpha(t)$$

- 5) All fractional derivatives do not satisfy: $D^\alpha D^\beta f = D^{\alpha+\beta} f$ general.

3. Models and Analysis

- 1) Let $f: [0, \infty) \rightarrow \mathbb{R}$ and $t > 0$. definition of the derivative of first principle of f at t is $\frac{df}{dt} = \lim_{\epsilon \rightarrow 0} \frac{f(t+\epsilon) - f(t)}{\epsilon}$.

We can give the a similar definition for the fractional derivative of order α , where $0 < \alpha \leq 1$ and for any general α for $\alpha \in (n, n + 1]$ where $n \in \mathbb{N}$.

Definition 3.1. Given a function $f: [0, \infty) \rightarrow \mathbb{R}$. Then the “conformable fractional derivative” of f of order $\alpha \in (0, 1]$ is defined by

$$T_\alpha f(t) = \lim_{\epsilon \rightarrow 0} \frac{f(t_0 + \epsilon^\alpha t^{1-\alpha}) - f(t_0)}{\epsilon^\alpha}$$

This definition satisfy all the above five properties of T_1 of derivative.

For $\alpha = 1$

$$\begin{aligned} T_1 f(t) &= \lim_{\epsilon \rightarrow 0} \frac{f(t + \epsilon^1 t^{1-1}) - f(t)}{\epsilon^1} \\ &= \lim_{\epsilon \rightarrow 0} \frac{f(t + \epsilon) - f(t)}{\epsilon} \end{aligned}$$

Reduces to classical definition of derivative

Let us write T_α to denote the operator which is called the fractional derivative of order α . For $\alpha = 1$, T_1 satisfies the following properties:

- (i) $T_1(af \pm bg) = aT_1(f) \pm bT_1(g)$, for all $a, b \in \mathbb{R}$ and f, g in the domain of T_1 .
- (ii) $T_1(t^p) = pt^{p-1}$

(iii) $T_1(fg) = fT_1(g) + gT_1(f)$

(iv) $T_1\left(\frac{f}{g}\right) = \frac{gT_1(f) - fT_1(g)}{g^2}$

(v) $T_1(\lambda) = 0$, for all constant functions $f(t) = \lambda$.

2) Now, we present our new definition, which is the simplest and most natural extension of derivative definition of first principle. and efficient definition of fractional derivative of order $\alpha \in (0, 1]$. the definition can be generalized to include any α .

4. Results and Discussion

For all $t > 0, \alpha \in (0, 1)$. If f is α -differentiable in some $(0, a), a > 0$, and $\lim_{t \rightarrow 0^+} f^\alpha(t)$ exists then define $\lim_{t \rightarrow 0^+} f^\alpha(t) = f^\alpha 0$

$f^\alpha(t), T_\alpha f(t)$ are same notation for the conformable fractional derivatives of f of order α, f is α -differentiable. if the new conformable fractional derivative of f of order α exists from the definition above following theorem can obtain.

Theorem 3.1. *If a function $f: [0, \infty) \rightarrow R$ is α -differentiable at $t_0 > 0, \alpha \in (0, 1]$, then f is continuous at t_0 .*

Proof:

Since $f(t_0 + \varepsilon^\alpha t^{1-\alpha}) - f(t_0) = \frac{f(t_0 + \varepsilon^\alpha t^{1-\alpha}) - f(t_0)}{\varepsilon^\alpha} \cdot \varepsilon^\alpha$ then

$$\lim_{\varepsilon \rightarrow 0} f(t_0 + \varepsilon^\alpha t^{1-\alpha}) - f(t_0) = \lim_{\varepsilon \rightarrow 0} \frac{f(t_0 + \varepsilon^\alpha t^{1-\alpha}) - f(t_0)}{\varepsilon^\alpha} \cdot \lim_{\varepsilon \rightarrow 0} \varepsilon^\alpha$$

$$\lim_{\varepsilon \rightarrow 0} f(t_0 + \varepsilon^\alpha t^{1-\alpha}) - f(t_0) = f^\alpha(t_0) \cdot 0$$

$$\lim_{\varepsilon \rightarrow 0} f(t_0 + \varepsilon^\alpha t^{1-\alpha}) - f(t_0) = 0$$

(iii) $T_\alpha(t^p) = pt^{p-\alpha}$ for all $p \in R$ this is same as for first order derivative.

$$T_\alpha(t^p) = \lim_{\varepsilon \rightarrow 0} \frac{(t + \varepsilon^\alpha t^{1-\alpha})^p - t^p}{\varepsilon^{1-\alpha}}$$

$$= \lim_{\varepsilon \rightarrow 0} \frac{t^p + pt^{p-1}\varepsilon^\alpha t^{1-\alpha} + \frac{P(P-1)}{2} t^{p-2} \varepsilon^{\alpha^2} t^{(1-\alpha)^2} + \dots + -t^p}{\varepsilon^\alpha}$$

$$= \lim_{\varepsilon \rightarrow 0} \frac{pt^{p-1}\varepsilon^\alpha t^{1-\alpha} + \frac{P(P-1)}{2} t^{p-2} \varepsilon^{\alpha^2} t^{(1-\alpha)^2} + \dots}{\varepsilon^\alpha}$$

$$= \lim_{\varepsilon \rightarrow 0} pt^{p-1} t^{1-\alpha} + \frac{P(P-1)}{2} t^{p-2} \varepsilon^{\alpha^2} t^{(1-\alpha)^2} + \dots$$

$$= pt^{p-\alpha}$$

(iv) $T_\alpha(fg) = fT_\alpha(g) + gT_\alpha(f)$ i.e derivative of product is same as product of derivative for new conformable fractional derivative.

i) $T_\alpha(af + bg) = \lim_{\varepsilon \rightarrow 0} \frac{af(t_0 + \varepsilon^\alpha t^{1-\alpha}) + bg(t_0 + \varepsilon^\alpha t^{1-\alpha}) - \{af(t_0) + bg(t_0)\}}{\varepsilon^\alpha}$

$$= \lim_{\varepsilon \rightarrow 0} \frac{af(t_0 + \varepsilon^\alpha t^{1-\alpha}) - af(t_0) - \{bg(t_0 + \varepsilon^\alpha t^{1-\alpha}) - bg(t_0)\}}{\varepsilon^\alpha}$$

$$= \lim_{\varepsilon \rightarrow 0} \frac{a\{f(t_0 + \varepsilon^\alpha t^{1-\alpha}) - f(t_0)\} + b\{g(t_0 + \varepsilon^\alpha t^{1-\alpha}) - g(t_0)\}}{\varepsilon^\alpha}$$

$$= a \lim_{\varepsilon \rightarrow 0} \frac{f(t_0 + \varepsilon^\alpha t^{1-\alpha}) - f(t_0)}{\varepsilon^\alpha} + b \lim_{\varepsilon \rightarrow 0} \frac{g(t_0 + \varepsilon^\alpha t^{1-\alpha}) - g(t_0)}{\varepsilon^\alpha}$$

$= aT_\alpha f + b.T_\alpha g$.

Which implies that

$$\lim_{\varepsilon \rightarrow 0} f(t_0 + \varepsilon^\alpha t^{1-\alpha}) = f(t_0)$$

Hence f is continuous.

Theorem 3.2: *Let $\alpha \in (0, 1]$ and f, g be α -differentiable at a point $t > 0$. Then*

(i) $T_\alpha(af + bg) = aT_\alpha(f) + bT_\alpha(g)$, for all $a, b \in R$ and f, g in the domain of T_1 .

i) $T_\alpha(af + bg) = \lim_{\varepsilon \rightarrow 0} \frac{af(t_0 + \varepsilon^\alpha t^{1-\alpha}) + bg(t_0 + \varepsilon^\alpha t^{1-\alpha}) - \{af(t_0) + bg(t_0)\}}{\varepsilon^\alpha}$

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$$= a \lim_{\varepsilon \rightarrow 0} \frac{f(t_0 + \varepsilon^\alpha t^{1-\alpha}) - f(t_0)}{\varepsilon^\alpha} + b \lim_{\varepsilon \rightarrow 0} \frac{g(t_0 + \varepsilon^\alpha t^{1-\alpha}) - g(t_0)}{\varepsilon^\alpha}$$

(ii) $T_\alpha(af - bg) = aT_\alpha(f) - bT_\alpha(g)$, for all $a, b \in R$ and f, g in the domain of T_1 .

i) $T_\alpha(af - bg) = \lim_{\varepsilon \rightarrow 0} \frac{af(t_0 + \varepsilon^\alpha t^{1-\alpha}) - bg(t_0 + \varepsilon^\alpha t^{1-\alpha}) - \{af(t_0) - bg(t_0)\}}{\varepsilon^\alpha}$

$$= \lim_{\varepsilon \rightarrow 0} \frac{af(t_0 + \varepsilon^\alpha t^{1-\alpha}) - af(t_0) - \{bg(t_0 + \varepsilon^\alpha t^{1-\alpha}) - bg(t_0)\}}{\varepsilon^\alpha}$$

$$= \lim_{\varepsilon \rightarrow 0} \frac{a\{f(t_0 + \varepsilon^\alpha t^{1-\alpha}) - f(t_0)\} - \{b\{g(t_0 + \varepsilon^\alpha t^{1-\alpha}) - g(t_0)\}}}{\varepsilon^\alpha}$$

$$= a \lim_{\varepsilon \rightarrow 0} \frac{f(t_0 + \varepsilon^\alpha t^{1-\alpha}) - f(t_0)}{\varepsilon^\alpha} - b \lim_{\varepsilon \rightarrow 0} \frac{g(t_0 + \varepsilon^\alpha t^{1-\alpha}) - g(t_0)}{\varepsilon^\alpha}$$

(v)

$$T_\alpha(f/g) = \frac{gT_\alpha(f) - fT_\alpha(g)}{g^2}$$

this can follow from definition.

Let $h = \varepsilon^\alpha t^{1-\alpha} = \varepsilon^\alpha t^{1-\alpha}$ implies $\varepsilon^\alpha = ht^{\alpha-1}$ as $\varepsilon \rightarrow 0, h \rightarrow 0$

(vi) $T_\alpha(\lambda) = 0$, for all constant functions $f(t) = \lambda$. i.e. derivative for constant is zero.

vii) If in addition, f is differentiable then $T_\alpha(f)(t) = t^{1-\alpha} \frac{d}{dt} f(t)$

$$T_\alpha(f)(t) = t^{1-\alpha} \frac{d}{dt} f(t)$$

$$\begin{aligned} &= \lim_{h \rightarrow 0} \frac{f(t+h) - f(t)}{ht^{\alpha-1}} \\ &= \lim_{h \rightarrow 0} \frac{f(t+h) - f(t)}{ht^{\alpha-1}} \\ &= t^{1-\alpha} \frac{d}{dt} f(t). \end{aligned}$$

$$\lim_{\varepsilon \rightarrow 0} \frac{f(t + \varepsilon^\alpha t^{1-\alpha}) - f(t)}{\varepsilon^\alpha}$$

Fractional derivatives of certain functions from new definition for conformable fractional derivative.

1. $T_\alpha(t^p) = pt^{p-\alpha}$ for all $p \in \mathbb{R}$

$$\begin{aligned} T_\alpha(t^p) &= \lim_{\varepsilon \rightarrow 0} \frac{(t + \varepsilon^\alpha t^{1-\alpha})^p - t^p}{\varepsilon^{1-\alpha}} \\ &= \lim_{\varepsilon \rightarrow 0} \frac{t^p + pt^{p-1}\varepsilon^\alpha t^{1-\alpha} + \frac{P(P-1)}{2} t^{p-2} \varepsilon^{\alpha^2} t^{(1-\alpha)^2} + \dots + -t^p}{\varepsilon^\alpha} \\ &= \lim_{\varepsilon \rightarrow 0} \frac{pt^{p-1}\varepsilon^\alpha t^{1-\alpha} + \frac{P(P-1)}{2} t^{p-2} \varepsilon^{\alpha^2} t^{(1-\alpha)^2} + \dots}{\varepsilon^\alpha} \\ &= \lim_{\varepsilon \rightarrow 0} pt^{p-1} t^{1-\alpha} + \frac{\frac{P(P-1)}{2} t^{p-2} \varepsilon^{\alpha^2} t^{(1-\alpha)^2} + \dots}{\varepsilon^\alpha} \\ &= pt^{p-\alpha}. \end{aligned}$$

2. $T_\alpha(1) = 0$ follows from 1 by putting $1 = t^0$

3. $T_\alpha(e^{cx}) = cx^{1-\alpha} e^{cx}, c \in \mathbb{R}$

$$T_\alpha(e^{cx}) = \lim_{\varepsilon \rightarrow 0} \frac{e^{(cx + c\varepsilon^\alpha t^{1-\alpha})} - e^{cx}}{\varepsilon^\alpha}$$

$$T_\alpha(e^{cx}) = \lim_{\varepsilon \rightarrow 0} \frac{e^{ct} e^{c\varepsilon^\alpha t^{1-\alpha}} - e^{ct}}{\varepsilon^{1-\alpha}}$$

$$\begin{aligned} &= \cos bx \lim_{\varepsilon \rightarrow 0} \frac{\sin \frac{b\varepsilon^\alpha x^{1-\alpha}}{2}}{\frac{b\varepsilon^\alpha x^{1-\alpha}}{2}} \cdot bx^{1-\alpha} \\ &= \cos bx \cdot bx^{1-\alpha} \end{aligned}$$

In similar manner as above $T_\alpha(\cos bx) = -bx^{1-\alpha} \sin bx, b \in \mathbb{R}$ can be proved.

Here $h = \varepsilon^\alpha t^{1-\alpha} \Rightarrow \varepsilon^\alpha = h \cdot t^{\alpha-1}$

Putting the values

$$T_\alpha(e^{ct}) = \lim_{\varepsilon \rightarrow 0} \frac{e^{ct}(e^{ch} - 1)}{h \cdot t^{\alpha-1}}$$

$$T_\alpha(e^{cx}) = \lim_{h \rightarrow 0} \frac{e^{ct}(e^{ch} - 1) \cdot c}{c \cdot h \cdot t^{\alpha-1}}$$

$$T_\alpha(e^{cx}) = \frac{e^{ct} c}{t^{\alpha-1}} \lim_{h \rightarrow 0} \frac{(e^{ch} - 1)}{c \cdot h}$$

as $\varepsilon \rightarrow 0, \Rightarrow h \rightarrow 0$

$$T_\alpha(e^{ct}) = \frac{e^{ct} c}{t^{\alpha-1}} \cdot 1$$

$$T_\alpha(e^{ct}) = ct^{1-\alpha} e^{ct}, c \in \mathbb{R}.$$

4) $T_\alpha(\sin bx) = bx^{1-\alpha} \cos bx, b \in \mathbb{R}$
By using definition.

$$= \lim_{\varepsilon \rightarrow 0} \frac{\sin b(x + \varepsilon^\alpha x^{1-\alpha}) - \sin bx}{\varepsilon^\alpha}$$

$$= \lim_{\varepsilon \rightarrow 0} \frac{2 \cos b \left(x + \frac{\varepsilon^\alpha x^{1-\alpha}}{2} \right) \sin \frac{b\varepsilon^\alpha x^{1-\alpha}}{2}}{\varepsilon^{1-\alpha}}$$

However, it is worth noting the following conformable fractional derivatives of certain functions:

4. $T_\alpha(e^{\frac{1}{\alpha} t^\alpha}) = (e^{\frac{1}{\alpha} t^\alpha})$
5. $T_\alpha(\sin \frac{1}{\alpha} t^\alpha) = (\sin \frac{1}{\alpha} t^\alpha)$
6. $T_\alpha(\cos \frac{1}{\alpha} t^\alpha) = (\cos \frac{1}{\alpha} t^\alpha)$

Although the most important case for the range α is $(0,1)$ but if $\alpha \in (n, n+1]$ for some natural number n then the definition is given by following.

Definition 3.2 Let if $\alpha \in (n, n+1]$, and f be an n -differentiable at t , where $t > 0$ then the new definition of conformable fractional derivative of f of order α is defined as

$$\begin{aligned} T_\alpha f(t) &= \lim_{\varepsilon \rightarrow 0} \frac{f^{([\alpha]-1)}(t + \varepsilon^{[\alpha]} t^{([\alpha]-\alpha)}) - f^{([\alpha]-1)}(t)}{\varepsilon^{[\alpha]}} \end{aligned}$$

Where $[\alpha]$ is the smallest integer greater than or equal to α .

Remark 3.1: one can easily prove that

$$T_\alpha f(t) = t^{([\alpha]-\alpha)} f^{([\alpha]}(t)$$

Where $\alpha \in (n, n+1]$ and f is $n+1$ differentiable at $t > 0$

$$T_{\alpha}f(t) = \lim_{\varepsilon \rightarrow 0} \frac{f^{([\alpha]-1)}(t + \varepsilon^{[\alpha]}t^{([\alpha]-\alpha)} - f^{([\alpha]-1)}(t))}{\varepsilon^{[\alpha]}}$$

Put $h = \varepsilon^{[\alpha]}t^{([\alpha]-\alpha)}$ then $\varepsilon^{[\alpha]} = h \cdot t^{(\alpha-[\alpha])}$ as $\varepsilon \rightarrow 0$ then $h \rightarrow 0$

$$T_{\alpha}f(t) = \lim_{\varepsilon \rightarrow 0} \frac{f^{([\alpha]-1)}(t + h) - f^{([\alpha]-1)}(t)}{h \cdot t^{(\alpha-[\alpha])}}$$

$$T_{\alpha}f(t) = t^{([\alpha]-\alpha)} \lim_{\varepsilon \rightarrow 0} \frac{f^{([\alpha]-1)}(t + h) - f^{([\alpha]-1)}(t)}{h}$$

$$T_{\alpha}f(t) = t^{([\alpha]-\alpha)} f^{[\alpha]}(t)$$

5. Conclusion

We introduced new definition of conformable fractional derivative.

We conclude that this definition coincide with the classical definition of derivative.

We introduce definition For $\alpha \in (0,1]$ and generalize for any $\in (n, n + 1]$.

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