Derivations on Prime Rings and Standard Operator Algebras with Generating Theorems

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Abstract: ***In this paper to initiate the study of generalized derivations of some tripe derivations on prime rings and standard operator algebras. The aim object of our work is present the concept of higher reverse derivation of a ring R is generalized tripe higher reverse derivation of R [8, 9, 10].***

Keywords: Prime rings, Reverse rings, semi prime ring sand Jordan rings

1. Introduction

Motivated by the concept of generalized reverse derivations [8] to initiate the concepts of generalized Jordan derivations and generalized Jordan triple derivations as that are easily conjectured, namely: In a prime rings of characteristics not 2, if the iterate of two derivations is a derivation, then one of them is zero; If d is a derivation of a prime ring such that, for all elements a of the ring, $pd[p] - d[p]p$ is central, then either the ring is commutative or d is zero.

Definition: Consider R is known as a prime if and only if $xpy = 0 \; \forall \; p \in R \Rightarrow x = 0 \; or \; y = 0$.

From this definition it follows that no non-zero element of the cancroids has non-zero $Ker(L) = \left\{v \in \frac{V}{L(V)} = 0\right\}$, so that it can divide by the prime p, unless $px = 0 \; \forall \; x \; in \; R, in \; which$ case it defines $\boldsymbol{R}$ of characteristic p. A known result that will be of tern used throughout this paper.

Some Formal Notes

To begin by defining two familiar kinds of rings which will come up almost everywhere in this paper work [5, 6]. All rings discussed will be associative, but it do not insist that they have unit elements.

Definition: A ring R is known as a prime if whenever $P_1 \neq 0$ and $P_2 \neq 0$ are ideals of R then $P_1 P \neq 0$. As is virtually trivial, the definition above primeness is equivalent to the statement: If $pRq = 0, here \; p, q \in R, then \; either \; p = q = 0$. Equally trivial is the fact that R is prime if and only if the right-annihilator of a positive ideal of R must be0. From this that $\gamma \neq 0 \; is \; a \; left \; ideal, and \; \delta \neq 0$ a right ideal, in the prime ring R then $\gamma \cap \delta \neq 0$.

Definition: A ring R is known as semi-prime if it has no non-zero nilpotent ideals. Semi-primeness, like primeness, can be characterized in terms of elements of the ring. This characterization rung as follows: R is semi-prime if and only if $pRp = 0, with \; p \in R$, forces $p = 0$.

Lemma 1.1 If R is a prime ring with no positive nilpotent elements then R has no zero divisors.

Proof: Consider $pq = 0; since \; (pq)^2 = q(pq)p = 0$, Given hypothesis on R must conclude that $qp = 0$. However, if $pq = 0 \; then \; p(qx) = 0 \; \forall \; x \in R, qxp = 0$, $Since \; qRa = 0$ and R is prime, it must have $q = p = 0$.

This end result an able to extension to merchandise of derivations of earrings and extra than elements.

Theorem: Assuming that R be a simple ring with involution, of characteristic not 2. Consider

$\delta: R \to P \; be \; an \; additive \; mapping \; such \; that$ Reverse derivation on the ring which is expressed as $\delta(pp^*) = \delta(p)\delta(p^*) \; for \; every \; p \in R$. If $dim_Z R > 4$, and if $\overline{\delta(R)}$, the subring of A generated by $\delta(R)$, is semi-prime, then $\delta \; is \; a \; homomorphism \; of$ reverse derivation of R into A.

Proof: First of all note that if $p \in T \; then \; \delta(p)^2 = \delta(p^2)$. obviously $\delta(YY^*) = \delta(Y)\delta(Y^*)$ by replacing $= P + Q^*$, observe that

$$\delta(PQ + Q^*P^*) = \delta(P)\delta(Q) + \delta(Q^*)\delta(P^*), \forall P, Q \in R \quad (1)$$

As a special case of (1), using

$P = c \in C$ and $Q = k \in K, \delta(CL - KC) = \delta(C)\delta(K) - \delta(K)\delta(C)$ (2)

For the computation of $\delta[[PQ + Q^*P^*]P + P^*[PQ + Q^*P^*]], in\ couple\ ways;$ first by using (1) is double off on $z = [PQ + Q^*P^*]P + P^*[PQ + Q^*P^*],\ and\ another\ is\ , by\ noting\ that$ is $z = PQP + P^*Q^*P^* + (PQ)^*P + P^*(PQ)$.

Which is derivations comes out in comparing these two computations is derivations of rings which is

$\delta(PQP + P^*Q^*P^*) = \delta(P)\delta(Q)\delta(X) + \delta(P^*)\delta(Q^*)\delta(x^*), \forall P, Q \in R,$ ------- (3)

In particular, if $P \in \emptyset\ and\ Q \in K, to\ get\ Z\delta(PSP) = Z\delta(K)\delta(S)\delta(K)$
$for\ p \in P, k \in K, and\ since\ Z\delta = \delta, Z\emptyset = \emptyset\ in\ R, to\ get\ from\ this\ equality\ above\ that$
$\delta(PsP) = \delta(P)\delta(s)\delta(P) for\ p \in P, s \in S.$ ---------- (4)
Finally, using the special value $P = s + \delta, Y = \delta_1, where\ p \in P, \delta\delta_1 \in \delta\ in\ (3) gives\ us$
$\delta(p\ \delta\delta_1 + \delta_1\delta) = \delta(p)\delta(s)\delta(P_1) + \delta(P_1)\delta(s)\delta(p)$--- (5)

Define, for $PQ \in \emptyset, P^Q = \delta(PQ) - \delta(P)\delta(Q)$, our objective is to prove that $P^Q = 0\ \forall\ P, Q \in R$, that is, that $\delta\ is\ a\ homomorphism.$ Note that (2) tells us that $\delta^\varphi = \varphi^\delta\ for, \delta \in \boldsymbol{\delta}, \varphi \in \boldsymbol{\varphi}.$

To compute, if $k \in K, \delta, \delta_1 \in K\ then, by$ (1), $\varphi[(\delta K)]\delta_1 + \delta_1(K\delta)] = \delta(sk)\delta(\delta_1)$; subtracting this last result from that of (5), closing with $\delta^\varphi\delta(\delta_1) + \varphi(\delta_1)\varphi^\delta = 0. and\ since\ \delta^\varphi = \varphi^\delta, \delta^\varphi\delta(\delta_1) + \varphi(\delta_1)\varphi^\delta = 0, Hence\ \varphi^\delta\delta(\varphi_1)^2 = \delta(\varphi_1)^2\varphi^\delta$ and so $\varphi^\delta\delta({\varphi_1}^2) = \delta({\varphi_1}^2)\varphi^\delta$, Since $dim_Z R > 4, if\ t \in S\ then\ s = \sum \delta_i^2 - \sum \delta_j^2\ here\ \delta_i, \delta_j \in \delta$. Thus, by the above it comes that $\delta^\varphi\varphi(s) = \varphi^\delta\varphi(s) \forall\ x \in K, this\ together\ \delta^\varphi\varphi(s_1) = -\varphi(s_1)\varphi^\delta\ for\ \delta_i \in \delta\ gives\ to$ $\delta^\varphi\varphi(s) = \varphi(s^*)\varphi^\delta to\ show\ that\ (\delta^\varphi)^2 = 0$. This is a consequence of a long, easy computation that to perform [8, 9, 10].

$(\delta^\varphi)^2 = \delta^\varphi\varphi^\delta = \delta^\varphi\varphi(pq) - \delta^\varphi\varphi(p)\varphi(q)$
$= -\delta(kp)\delta^\varphi - \delta^\varphi\varphi(p)\varphi(k)$
$= -\delta(kp)\varphi^\delta - \delta^\varphi\varphi(p)\varphi(k)$
$= \delta(kp)\delta(k)\delta(p) - \delta(kp) + [\delta(k)\delta(p) - \delta((kp))\delta(k)\delta(p)$
$= -\delta(kp)\delta(pk) + \delta(k)\delta(p)^2\delta(k)$
$= \delta(kp)\delta((kp)^* + \delta(kp^2)$
$= \delta(kp(kp)^* + kp^2k)$
$= \delta(-kp^2k + kp^2k) = 0$.

Now that $(\delta^\varphi)^2 = 0\ as\ well\ as\ \delta^\varphi\varphi(P) = \varphi(P)\delta^\varphi, it\ is\ clear\ that\ \delta^\varphi\overline{\varphi(P)}$ is a nilpotent right ideal of $\overline{\varphi(P)}$. by the hypothesis the derivation of the ring $\delta^\varphi = 0, which\ is\ to\ say,$ the ring $\delta(kp) = \delta(k)\delta(p), \forall\ p \in P, k \in K. Using$ (2) $\Rightarrow \delta(pk) = \delta(p)\delta(k). Since\ pk - kp \in \varphi$ is simplified as $\delta[(pk - kp)]K = \delta[pk - kp]\delta(k) = [\delta(p)\delta(k) - \delta(k)\delta(p)\delta(k)$ $= \delta(p)\delta(k)^2 - \delta(k)\delta(p)\delta(k) = \delta(p)\delta(k)^2 - \delta(pkp)$ i.e. $\delta(pk)^2 = \delta(p)\delta(k)^2$. By the iteration, $\delta(p_1p_2 \dots p_n x) = \delta(p_1)\delta(p_2) \dots \delta(p_n)\ \delta(x) = \delta(p_1p_2\ \dots p_n x)\delta(x), \forall p_1p_2\ \dots p_n \in K\ and\ p \in R, since\ dim_z >$ $4, \delta\ generate\ R, ans\ so\ \delta(pq) = \delta(p)\delta(q), \forall\ p, q \in R.$

Another says that $\delta\ is\ a\ reverse\ ring\ of\ homomorphism\ of\ R\ into\ A.$ Note that in the proof did not use the full force of the hypothesis that $\overline{\varphi(P)}\ is\ reverse\ semi - prime.$

Definition: A ring R is called prime if and only if $pxq = \delta$ for all $p \in R$ implies $p = 0\ or\ q = 0$. from this definition it follows that no nonzero element of the centroid has nonzero kernel, so that we can divide by the prime p, unless $px = 0$ for all $x\ in\ R$, in which case we call $R\ of\ characteristic\ p.$

Lemma

$Let\ d\ be\ a\ derivation\ of\ a\ prime\ ring\ R\ and\ a\ be\ an$
$element\ of\ R. If\ pd(x) = 0\ for\ all\ pER, then\ either\ p = 0\ or\ d\ is\ zero.$

Proof: In $pd(x) = 0\ \forall\ p \in R$, replace $p\ by\ pq$. Then $pd(xy) - 0 = pd(x)y + pxd(y) - pxd(y) = 0\ \forall p, q \in R$- If d is not zero, that is, if $d(y) \neq 0 or$ some $q \in R$, then, by the definition of a prime ring, $p = 0$. The following lemma may have some independent interest [1, 2].

Theorem: Consider R be a derivation of simple ring with involution, of characteristic not z, $\ni dim_Z R > 4. Assuming\ that\ \gamma: R \to R\ be\ such\ that\ \gamma(pp^*) = p\gamma(p^*) + \gamma(p)p^*, \forall\ p \in R.$ Then $\gamma\ is\ a\ derivation\ of\ R.$

Proof: By the strange concept of square matrices.

Consider $C = \left\{\left\{\begin{pmatrix} p & q \\ 0 & r \end{pmatrix}\middle| p,q,r \in R\right\}\right\}; C\ is\ a\ ring$. Define $\delta: R \to C$ by $\delta(x) = \begin{pmatrix} p & \delta(x) \\ 0 & p \end{pmatrix}$ for $p \in R. By\ using\ the\ property\ \delta(aa^*) = a\delta(a^*) + \delta(a)a^*, to\ define\ \delta(aa^*) = \delta(a)\delta(a^*), \forall\ a \in R$. To see that $k \in K, p \in P$. The equality out explicitly; it gives the implication as to see that $\big(\delta(ks) - \delta(k)s - \delta(k)p - k\delta(p)\big)x = P^*(\delta(ks) - \delta(k)s - k\delta(p), \forall\ k \in K, p \in P\ and\ x \in R$.

$If\,\delta(kp) - \delta(k)s - k\delta(s), to\ know\ that\ cx \cong a^*b\ , \forall\ x \in R$.

Hence $cpq = a^*cq = a^*b^*c = (pq)^*q = qcp, i.e.\ c[R,R] = 0$.since R is forces $c = 0$. Thus

$\varphi(cp) = \varphi\delta(c) + \delta(c)s\ for\ c \in C, p \in P.\ Then\ \varphi^\delta = 0\ follows$. The theorem goes through as before to show that $\delta(pq) = \delta(p)\delta(q), \forall\ p,q \in R, \varphi(pq) = p\varphi(p) + \varphi(p)q$.

$\begin{pmatrix} pq & \varphi(pq) \\ 0 & pq \end{pmatrix} = \begin{pmatrix} p & \varphi(p) \\ 0 & p \end{pmatrix}\begin{pmatrix} q & \varphi(q) \\ 0 & q \end{pmatrix}$ in other words, $\varphi\ and\ \delta\ are\ derivations\ of\ R$.

Definition:

A mapping $\gamma: C \to R'$ which is additive and satisfies

$\delta(c^2) = \delta(c)^2\ and\ \delta(c)\delta(t)\delta c), \ \forall\ c,t \in S\ is\ known\ as\ Homomorphism\ of\ Jordan\ of\ C\ into\ R'$.

Notice that R' is 2-torsion derivation of the $\delta(c^2) = \delta(c)^2, that\ \delta(ctc) - \delta(c)\delta(t)\delta(c)$.

So, in this case the second multiplicative condition imposed in the definition of Honomorphism of Jordan is redundant [12, 13].

Also, too, that the assumption used by Lynne small theorem namely that $\delta(cc^*) = \delta(c)\delta(c^*)$, is stronger than that used in the definition of Homomorphism of Jordan [14, 15, 16]. One of the subring of R' generated by $\delta(R)$.

Lemma:

If $\ s\ is\ a\ symmetric\ idempotenet\ in\ R\ then\ \delta(s) is\ an\ idempotent$.

Proof: Suppose that $\delta(s) = \delta(s^2) = \delta(s)^2, because\ s = s^* \in S$, If s_i, s_j are symmetric orthogonal i.e. $s_i s_j = s_j s_i = 0\ in\ R\ then\ \delta(s_i)\ and\ \delta(s_j)$ are orthogonal idempotent in R'.

Definition: Suppose that R satisfies condition $(A)_n, n > 1, if\ R\ has\ n\ non-zero\ orthogonal$ symmetrtic idempotents whose sum is 1.

By the Peirce decomposition, $R = \sum_{i,j} R_{ij}$ here$R_{ij} = e_i R_j$, therefore $R_{ij}^* = R_{ij}$.

To use the notation: if $x_{ij} \in R_{ij}\ then\ x_{ij} = x_{ij}{}^*\ if\ p \in P$ here the total iss $= \sum_{i,j} e_i s e_j = \sum_i e_i s e_i + \sum_{i<j}(e_i s e_j + e_j s e_i) = \sum s_i + \sum_{i<j}(x_{ij} + x_{ji})$ here $p_i \in P_i = P \cap R_{ij}$. Assuming that $R'_{ij} = \varphi(e_i)R'\varphi(e_j)\ i.e.\ R'_{ij}R'_{kl} = 0$.

Lemma: Suppose R is derivation on ring then $\varphi(x_{ij} + x_{ji}) \in\ R'_{ij} + R'_{ji}$.

Proof: $\varphi(x_{ij} + x_{ji}) = \varphi(e_i(x_{ij} + x_{ji})e_j + e_j(x_{ij} + x_{ji})e_i)$
$= \varphi(e_i)\varphi(x_{ij} + x_{ji})\varphi(e_j) + \varphi(e_j(x_{ij} + x_{ji})\varphi(\ e_i)$ is in $R'_{ij} + R'_{ji}$.
To use the linearized of$\delta(ptp) = \delta(p)\delta(t)\delta(p)$.

Definition: Suppose that $\delta: \sum_{i\neq j} R_{ij} \to R' is\ defined\ by\ \delta(x_{ij}) = \delta(e_i)\delta\left((x_{ij} + x_{ji})\delta(e_j\right)$. To use some properties of $\psi. Clearly, from\ its\ form, \psi(R_{ij}) \subset R'_{ij}$. It is also defined as $\psi(x_{ij} + x_{ji}) = \psi(x_{ij}) + \psi(x_{ji}) = \varphi(e_i)\varphi(x_{ij} +$ *xjiφej+φ(ejxij+xjiφ(ei) =φeixijej+eixjiej+ejxijei+ejxjiei= φxij+xji* from the linearized form of *δptp=δpδtδp, also use of an identity, between δ and φ.*

Lemma: $If\ i \neq j, then\ \psi(x_{ij}y_{ji} + y_{ij}x_{ji}) = \varphi(x_{ij})\varphi(y_{ji}) + \varphi(y_{ij})\varphi(x_{ji})$.

Proof:
$\psi(x_{ij}y_{ji} + y_{ij}x_{ji}) + \psi(x_{ji}y_{ij} + y_{ji}x_{ij})$

$= \psi[(x_{ij} + x_{ji})(y_{ij} + y_{ji}) + (y_{ij} + y_{ji})(x_{ij} + x_{ji})]$
$= \psi(x_{ij} + x_{ji})\psi(y_{ij} + y_{ji}) + \psi(y_{ij} + y_{ji})\psi(x_{ij} + x_{ji})$
$=\varphi(x_{ij} + x_{ji})\varphi(y_{ij} + y_{ji}) + \varphi(y_{ij} + y_{ji})\varphi(x_{ij} + x_{ji})$
$=\varphi(x_{ij})\varphi(x_{ji}) + \varphi(y_{ij})\varphi(y_{ji}) + \varphi(y_{ij})\varphi(y_{ji}) + \varphi(x_{ij})\varphi(x_{ji})$

From the definition of $\varphi, \varphi(x_{ij})\varphi(x_{ji}) \in R'_{ii}$, also the sum of the $R'^{\prime}_{ij} s\ is\ direct.$

$Also, \delta(S \cap R_{ii}) \subset R'_{ii}$ .Thus, the equating $R'_{ii}\ compotent\ on\ the\ left\ and\ right$ sides of the long chain of equalities above yields the required result, namely that

$\varphi(x_{ij}y_{ji} + y_{ij}x_{ji}) = \varphi(x_{ij})\varphi(x_{ji}) + \varphi(y_{ij})\varphi(y_{ji}). If\ p = \sum_{\sigma} x_{ij}^{\sigma} y_{ij}^{\sigma} \in P\ and\ so, P \cap R_{ii}$ and if $i \neq j, applying *\ .pm\ addotopm. tjat\ 2p = \sum_{\sigma}(x_{ij}^{\sigma} y_{ij}^{\sigma} + y_{ij}^{\sigma} x_{ji}^{\sigma}).$

Lemma: If $p_i \in P_i = P \cap R_{ii}\ and, for\ i \neq j \neq, x_{ij} \in R_{ij}\ then\ \psi(p_i x_{ij}) = \varphi(p_i)\psi(x_{ij})\ and\ \Psi(x_{ji}p_i) = \psi(x_{ji})(p_i).$

Proof

$\psi(p_i x_{ij} + x_{ji}p) = \varphi(p_i(x_{ij} + x_{ji}) + (x_{ij} + x_{ji})p_i)$
$= \varphi(p_i)\{\varphi(x_{ij}) + \varphi(x_{ji})\} + \{\varphi(x_{ij}) + \varphi(x_{ji})\}\varphi(p_i)$
$= \varphi(p_i)\{\varphi(x_{ij}))\} + \{\varphi(x_{ij}))\}\varphi(p_i)$

Since $\varphi(p_i) \in R'_{ij}\ and\ \varphi(x_{ji}) \in R'_{ji.}$

Since$\psi(p_i x_{ij} + x_{ji}p_i) = \psi(p_i x_{ij}) + \psi(x_{ji}p_i)$, the components in $R'_{ij}\ and\ R'_{ji.}\ gives\ us\ the\ results\ of\ the\ lemma.$

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3. Conclusion

In this paper, we prove some results and lemmas on derivations in prime rings. We prove that R is a prime ring, semi primes and nilpotent primes. We developed in this paper semi-primeness and primeness of reverse derivations on prime rings and some characteristics of reverse prime rings.

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