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Modification of LCR Series Equations

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Abstract: ***Here I am going to provide a modification for LCR series circuit calculations which have been found to be in agreement with experimental observations. According to LCR series equations, at resonance resistance offered by resistor is the only impedance acting on the circuit.ie, inductive and capacitive elements are absent here. But it is also observed that at resonance we could measure the potential drop across inductor and capacitor. So definitely there is some impedance offered by these circuit elements. The existing LCR series equation fails to explain this. The modified equation can account for resonant frequency, potential drop across inductor and capacitor during resonance and also why the frequency response curves are broader to the right of resonant frequency than to its left. The analysis reveals where we went wrong earlier. The modified equations help us to develop circuits which can be operated on wide range of frequencies.***

Keywords: Phase changes in an inductor, Shape of frequency response curve, Impedance at resonance, Resonant frequency, Similarity between capacitor and inductor

1. Introduction

While working with LCR series equations, some questions puzzled me which the present equations failed to answer. It inspired me to go for some corrections in the way some concepts are currently understood. Here I present the modified LCR series equations which have been found to be in agreement with experimental observations. The modified equations can explain some circuit phenomena which the currently used LCR equations failed to do, as I mentioned in the 'abstract' section. These equations can have many applications as they shift our understanding of some circuit operations. Experimental observations have also been given to support the theory.

2. Detail Explanation

Consider a circuit containing an inductance L, capacitance C and a resistance R connected in series.

Let the applied voltage be

$$E = E_0 \sin \omega t$$

At any instant let I be the current through the circuit and q be the charge on the capacitor. Due to the applied voltage the potential drop developed across L, C and R are

$$-L\frac{dI}{dt}, \frac{q}{C} \text{ and } RI$$

respectively.(Normally we take $L\frac{dI}{dt}$ for potential across inductance.But since we do not know whether $L\frac{dI}{dt}$ or -$L\frac{dI}{dt}$ equals E= $E_0 \sin \omega t$ in an inductance only circuit, let us here take -$L\frac{dI}{dt}$. If this gives us experimentally observed results, the potential across inductance should be $-L\frac{dI}{dt}$

$$-L\frac{dI}{dt} + \frac{q}{C} + RI = E$$

$$-L\frac{dI}{dt} + \frac{q}{C} + RI = E_0 \sin \omega t \qquad (1)$$

As current oscillates with the same frequency as the applied voltage, the general solution of equation (1) may be written as

$$I = A \sin \omega t + B \cos \omega t \qquad (2)$$

Differentiating equation (2) we have

$$\frac{dI}{dt} = A\omega\, Cos\, \omega t + B\omega\, Sin\, \omega t$$

Putting I= $\frac{dq}{dt}$ in equation (2) and integrating, we get

$$q = \frac{-A}{\omega} \cos \omega t + \frac{B}{\omega} \sin \omega t$$

Substituting $\frac{dI}{dt}$ and q in equation (1) we have

-$LA\omega\, Cos\, \omega t + LB\omega\, Sin\, \omega t - \frac{-A}{C\omega} \cos \omega t + \frac{B}{C\omega} \sin \omega t + RA \sin \omega t + RB \cos \omega t = E_0 \sin \omega t$ (3)

When t= 0,$\sin \omega t = 0$, $Cos\, \omega t = 1$, equation (3) becomes

$$-LA\omega - \frac{A}{C\omega} + RB = 0$$

Or

$$B = \frac{A}{R}\left(L\omega + \frac{1}{C\omega}\right) \qquad (4)$$

When t= $\frac{T}{4}$, $\sin \omega t = 1$, $Cos\, \omega t = 0$, equation (3) becomes

$$LB\omega + \frac{B}{C\omega} + RA = E_0$$

Or

$$RA + B\left(L\omega + \frac{1}{C\omega}\right) = E_0 \qquad (4)$$

Putting the value of B from equation (4)

$$A = \frac{E_0 R}{R^2 + (L\omega + \frac{1}{C\omega})^2}$$

Putting A in equation (4)

$$B = \frac{E_0(L\omega + \frac{1}{C\omega})}{R^2 + (L\omega + \frac{1}{C\omega})^2}$$

Substituting the values of A and B in equation (2) we get

$$I = A = \frac{E_0 R}{R^2 + (L\omega + \frac{1}{C\omega})^2}\sin\omega t + \frac{E_0(L\omega + \frac{1}{C\omega})}{R^2 + (L\omega + \frac{1}{C\omega})^2}Cos\omega t$$

$$I = \frac{E_0}{\sqrt{R^2+(L\omega+\frac{1}{c\omega})^2}}\left(\frac{R}{\sqrt{R^2+(L\omega+\frac{1}{c\omega})^2}}\sin\omega t + \frac{(L\omega+\frac{1}{c\omega})}{\sqrt{R^2+(L\omega+\frac{1}{c\omega})^2}}\mathrm{Cos}\,\omega t\right)$$

Put

$$\frac{R}{\sqrt{R^2 + (L\omega + \frac{1}{c\omega})^2}} = Cos\phi \qquad (5)$$

and

$$\frac{(L\omega + \frac{1}{\omega})}{\sqrt{R^2 + (L\omega + \frac{1}{C\omega})^2}} = Sin\phi \qquad (6)$$

Then

$$I = \frac{E_0}{\sqrt{R^2 + (L\omega + \frac{1}{c\omega})^2}}(Sin\omega tCos\phi + Cos\omega tSin\phi)$$

$$I = \frac{E_0}{\sqrt{R^2 + (L\omega + \frac{1}{c\omega})^2}}\ Sin\ (\omega t + \phi)$$

Ie, current leads the voltage by a phase angle ϕ

When $Sin\ (\omega t + \phi)$= 1, I= I_0

$$I_0 = \frac{E_0}{\sqrt{R^2 + (L\omega + \frac{1}{c\omega})^2}}$$

And

$$tan\ \phi = \frac{(L\omega + \frac{1}{C\omega})}{R}$$

Since all terms $L\omega, C\omega\ and\ R$ are positive, ϕ is also positive.

$$\frac{E_0}{I_0} = \sqrt{R^2 + (L\omega + \frac{1}{C\omega})^2} = \sqrt{R^2 + (X_L + X_C)^2}$$

$\frac{E_0}{I_0}$ which has the dimensions of resistance shows that the term $\sqrt{R^2 + (L\omega + \frac{1}{C\omega})^2}$ is the effective resistance offered by the LCR series circuit.

Resonance

For current to be maximum, $I_0 = \frac{E_0}{\sqrt{R^2+(L\omega+\frac{1}{c\omega})^2}}$, the term $R^2 + (L\omega + \frac{1}{C\omega})^2$ should be minimum.

Differentiating $R^2 + (L\omega + \frac{1}{C\omega})^2$ with respect to ω and equating to zero we get

$$L - \frac{1}{C\ (\omega_0)^2} = 0$$

$$\omega_0^2 = \frac{1}{LC}$$

Frequency at resonance $\upsilon_0 = \frac{1}{2\pi\sqrt{LC}}$

This shows that the frequency at which the 'effective resistance of LCR circuit' becomes minimum is still the same as in the conventional method. However here the impedance terms Lω and $\frac{1}{C\omega}$ do not vanish at resonance unlike in the old equation of $\sqrt{R^2 + (L\omega + \frac{1}{C\omega})^2}$ which gives proof why we are able to measure voltage drop across inductor and capacitor at resonance.

Selectivity and Quality Factor

Let $\omega_1\ and\ \omega_2$ be the angular frequencies corresponding to frequencies υ_1 and υ_2 at which current is $\frac{1}{\sqrt{2}}$ times the maximum value I_m (current at resonance).

This happens when (Lω + $\frac{1}{C\omega}$) becomes equal to resistance R. (This is a valid approximation since R^2 is much greater than (Lω + $\frac{1}{C\omega})^2$ at resonance).

Thus

$$L\omega_1 + \frac{1}{C\omega_1} = R \qquad (7)$$

$$L\omega_2 + \frac{1}{C\omega_2} = R \qquad (8)$$

Subtracting (7) from (8) we get

$$L\ (\omega_2 - \omega_1) + \frac{1}{C}\left(\frac{1}{\omega_2} - \frac{1}{\omega_1}\right) = 0$$

$$L\ (\omega_2 - \omega_1) + \frac{1}{C}\left(\frac{1}{\omega_1} - \frac{1}{\omega_2}\right)$$

$$L\ (\omega_2 - \omega_1) = \frac{1}{C}\left(\frac{\omega_2 - \omega_1}{\omega_1\omega_2}\right)$$

$$\omega_1\omega_2 = \frac{1}{LC} \qquad (9)$$

Let $\omega_0$be the angular frequency corresponding to the resonant frequency.

Then $L\omega_0 = \frac{1}{C\omega_0}$

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$$\omega_0^2 = \frac{1}{LC} \tag{10}$$

Comparing (9) and (10) we get

$$\omega_0^2 = \omega_1\omega_2$$

Adding (7) and (8)

$$L\,(\omega_1 + \omega_2) + \frac{1}{C}\left(\frac{1}{\omega_1} + \frac{1}{\omega_2}\right) = 2R$$

$$(\omega_1 + \omega_2) + \frac{1}{LC}\left(\frac{\omega_1 + \omega_2}{\omega_1\omega_2}\right) = \frac{2R}{L}$$

But $\omega_0^2 = \omega_1\omega_2 = \frac{1}{LC}$ which gives

$$\omega_1 + \omega_2 = \frac{R}{L}$$

$$2\pi\upsilon_2 + 2\pi\upsilon_1 = \frac{R}{L}$$

$$\upsilon_2 + \upsilon_1 = \frac{R}{2\pi L}$$

$$\frac{\upsilon_2 + \upsilon_1}{\upsilon_0} = \frac{R}{2\pi\upsilon_0 L} = \frac{R}{\omega_0 L} = \frac{1}{Q}$$

i.e. ,

Quality factor Q= $\frac{\omega_0 L}{R} = \frac{\upsilon_0}{\upsilon_2 + \upsilon_1}$

The expression $\frac{\omega_0 L}{R} = \frac{\upsilon_0}{\upsilon_2 + \upsilon_1}$ is in perfect agreement with experimental observations and experimentally proved. The results are given below.

Observations of my experiment

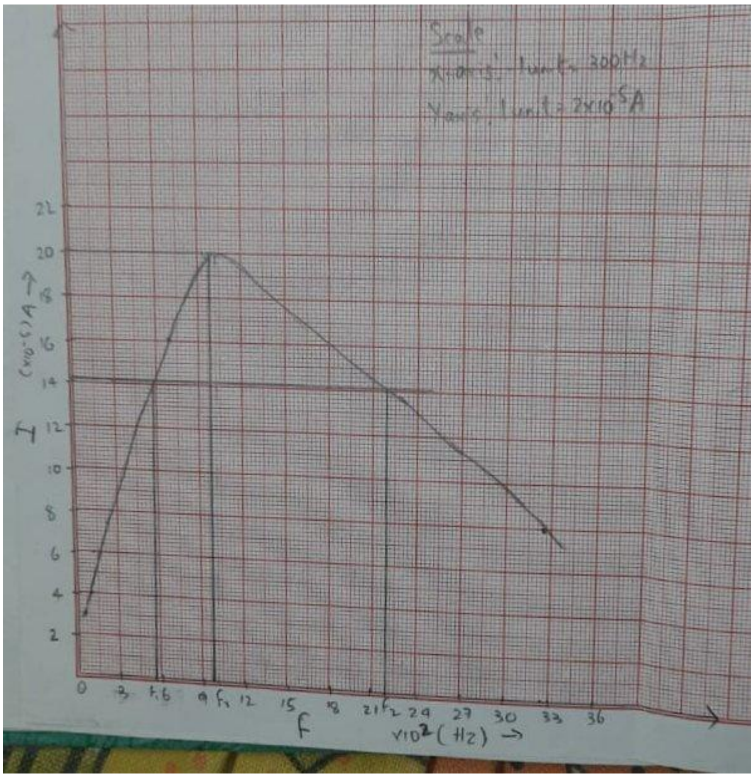
I applied a voltage of 3V peak to peak to the LCR series circuit and measured the voltage drop across resistance R. The experiment was repeated by varying the frequency of the input voltage. By knowing the voltage drop across resistance, current in the LCR series circuit was found for different frequencies. The observations and frequency response curve are as follows.

Inductance L= 0.115H, Capacitance C= 0.22μF, Resistance R= 2KΩ

* denotes multiplication

Frequency f(Hz)	Voltage across R, V_0 (Volts)	Current through the circuit, $I = \frac{V_0}{R}$ (Ampere)
56	0.06	$3 * 10^{-5}$
100	0.08	$4 * 10^{-5}$
110	0.12	$6 * 10^{-5}$
690	0.32	$16 * 10^{-5}$
930	0.4	$20 * 10^{-5}$
1000	0.4	$20 * 10^{-5}$
3250	0.16	$8 * 10^{-5}$

Frequency response curve



Comparison

Theoretical values

Resonant frequency, $\frac{1}{2\pi\sqrt{LC}}$ = 1001Hz

Quality factor, $\frac{L\omega_0}{R}$ = 0.36

Practical values:

From graph,

Resonant frequency=990Hz

Quality factor as per conventional equation $\frac{\upsilon_0}{\upsilon_2 - \upsilon_1}$ = 0.6

$(\upsilon_2 - \upsilon_1)$= Bandwidth

Quality factor as per the modified equation $\frac{\upsilon_0}{\upsilon_2 + \upsilon_1}$ =0.358

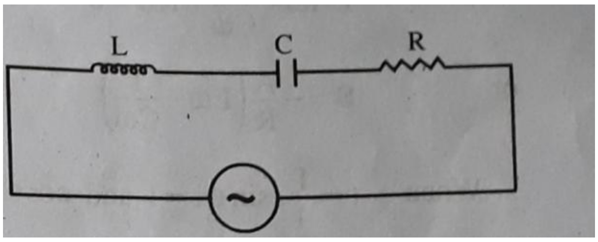
The experimental analysis proves that the modified equation is the correct equation for an LCR series circuit.

In graph we took a scale of 1 unit=300 Hz along X-axis and 1 unit =2*10^{-5}A along the Y-axis.

$\upsilon_2 = 2190HZ$ and $\upsilon_1 = 570HZ$

Normally we take $\frac{\upsilon_2 - \upsilon_1}{\upsilon_0}$ as the selectivity. Therefore a narrow bandwidth means more selective circuit (high quality factor). As the peak value of current increases in the frequency response curve the band width decreases. Similarly as the peak value of current decreases, the curve flattens (band width increases). According to $\frac{\upsilon_2 - \upsilon_1}{\upsilon_0}$, for the band width to increase either υ_1 should decrease or υ_2 should increase. But it is observed that the curve is always broader to the right of the peak current value. This happens because the actual expression of selectivity is $\frac{\upsilon_2 + \upsilon_1}{\upsilon_0}$, as derived before. So for the curve to get flattened and quality factor to decrease υ_2 should have more increase than decrease of υ_1. Otherwise the term $\upsilon_2 + \upsilon_1$ decreases with decrease in peak current which is not possible. So the term υ_2 has more increase than decrease of υ_1 as peak current value decreases. This accounts for why frequency response curves are broader to the right of peak current value.

So the above discussion shows that in an inductive circuit we have to equate $-L\frac{dI}{dt}= E_0 \sin\omega t$. This leads to a very interesting observation.



LCR series circuit

AC circuit containing inductance only

Consider a circuit containing an inductance L only. Let the applied voltage be

$$E= E_0 \sin\omega t$$

Whenever an alternating emf is applied to an inductance, due to self induction of the coil an induced emf is developed in the coil. As there is no ohmic voltage drop, the applied voltage has to overcome the self induced emf. Hence

$$-L\frac{dI}{dt} = E_0 sin\omega t$$
$$dI = -\frac{E_0}{L} sin\omega tdt$$

Integrating,

$I = \frac{E_0}{L\omega} Cos\omega t$

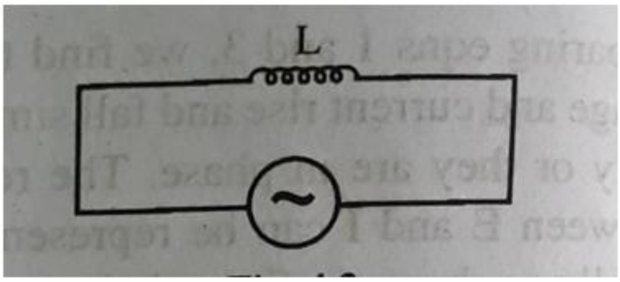
$$I = \frac{E_0}{L\omega} Sin\left(\omega t + \frac{\pi}{2}\right) = 1 \qquad (11)$$

I will be maximum when $Sin\left(\omega t + \frac{\pi}{2}\right) = 1$

$$I_0 = \frac{E_0}{L_w} \text{ is the maximum current.}$$

The inductive impedance $X_L = L\omega$.

Also from equation (11) the current in an inductive circuit leads the applied voltage by a phase angle of $\frac{\pi}{2}$. This is equivalent to the phase difference between current and applied voltage in a capacitor only circuit. Therefore inductor and capacitor can be used to produce the same phase difference. Since capacitive impedance $X_C = \frac{1}{C\omega,\ XC}$ increases with decrease in ω. But inductive impedance $X_L = L\omega, X_L$increases with increase in ω. So instead of using capacitor or inductor alone to produce phase changes, when we use a combination of both the circuit can be well operated in a variety of frequency ranges. This is one of the promising features of this discovery.



Inductance only circuit

3. Conclusion

The match between the theory and experimental observations holds the theory good. This theory could well explain many circuit phenomena. The modification is not against, but in congruence with the existing knowledge. From this work, it is understood that capacitor and inductor produce the same phase change in current with respect to the applied voltage. Since capacitor works well over high frequencies and inductor over low frequencies, their combinations can be used to drive low load applications from high voltage source with the required phase changes in current. Also in a series circuit these combinations are helpful to design circuits which work well over a wide range of frequencies with required phase changes in current. Cost effective circuits may be developed using the knowledge. This new discovery may also lead to unforeseen applications, which is obvious with scientific progress.

NB: The same approach can be used in parallel LCR circuits. According to present equation, at resonance, current through the circuit is zero. But in practice we do observe some current even at resonance. The modified approach can account for this fact while the conventional equation fails to do so. The LCR circuits are part of most curricula and can bring light to the observed discrepancy.

4. Acknowledgements

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