RGη-Closed Sets in Topological Spaces

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Abstract: ***In this paper, a new class of sets called regular generalized η-closed (briefly rgη-closed) sets is introduced and its properties are studied. The relationships among closed, α-closed, s-closed, η-closed, rgη-closed and their generalized closed sets are investigated. Several examples are provided to illustrate the behavior of these new class of sets.***

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1. Introduction

Many investigations related to generalized closed sets have been published in various forms of closed sets. In 1937, Stone [**12**] introduced the notion of regular open sets. In 1963, Levine [**7**] introduced the concept of semi-open sets. In 1965, Njastad [**11**] introduced the concept of α-open sets. In 1968, the notion of π-open sets were introduced by Zaitsev [**16**] which are weaker form of regular open sets in topological spaces. In 1970, Levine [**8**] initiated the study of so called generalized closed (briefly g-closed) sets. In 1994, Maki et al. [**9**, **10**] introduced the notion of αg-closed sets. In 2000, Dontchev and Noiri [**4**] introduced the notion of πg-closed sets. In 2007, Arockiarani and Janaki [**2**] introduced the notion of πgα-closed sets in topological spaces. In 2019, Subbulakshmi, Sumathi, Indirani [**14, 15**] introduced and investigated the notion of η-open and gη-closed sets. In 2019, Kumar and Sharma [**5**] introduced and investigated the notion of η-T_k (k = 0, 1, 2) and η-R_k (k = 0, 12) axioms in topological spaces. Recently, Kumar [**6**] introduced and investigated the notion of πgη-closed sets.

2. Preliminaries

Throughout this paper, spaces (X, ℑ), (Y, σ), and (Z, γ) (or simply X, Y and Z) always mean topological spaces on which no separation axioms are assumed unless explicitly stated. Let A be a subset of a space X. The closure of A and interior of A are denoted by cl(A) and int(A) respectively. A subset A is said to be **regular open** [12] (resp. **regular closed [12]**) if A = int(cl(A)) (resp. A = cl(int(A)). The finite union of regular open sets is said to be **π-open [16]**. The complement of a π-open set is said to be **π-closed [16]**.

Definition 2.1. A subset A of a topological space (X, ℑ) is said to be
(i) **s-open** [7] if A ⊂ cl(int(A)).
(ii) **α-open** [**11**] if A ⊂ int(cl(int(A))).
(iii) **η-open** [**14**] if A ⊂ in(cl(int(A))) ∪ cl(int(A)).
(iv) **η-closed** [**14**] if A ⊃ cl(int(cl(A))) ∪ int(cl(A)).

The complement of a s-open (resp. α-open, η-open**)** set is called **s-closed** (resp. **α-closed, η-closed)**. The intersection of all s-closed (resp. α-closed, η-closed) sets containing A, is called s-**closure** (resp. **α-closure, η-closure**) of A, and is denoted by **s-cl(A)** (resp. **α-cl(A), η-cl(A))**. The **η-interior** of A, denoted by **η-int(A)** is defined as union of all η-open sets contained in A. We denote the family of all η-open (resp. η-closed) sets of a topological space by **η-O(X)** (resp. **η-C(X)**).

Definition 2.2. A subset A of a space (X, ℑ) is said to be
(1) **generalized closed** (briefly **g-closed**) **[8]** if cl(A) ⊂ U whenever A ⊂ U and U ∈ ℑ.
(2) **πg-closed** [**4**] if cl(A) ⊂ U whenever A ⊂ U and U is π-open in X.
(3) r**g-closed** [**4**] if cl(A) ⊂ U whenever A ⊂ U and U is regular open in X.
(4) **α-generalized closed** (briefly **αg-closed**) [**9, 10**] if α-cl(A) ⊂ U whenever A ⊂ U and U ∈ ℑ.
(5) **πgα-closed** [**2**] if α-cl(A) ⊂ U whenever A ⊂ U and U is π-open in X.
(6) **gαr-closed** [**13**] if α-cl(A) ⊂ U whenever A ⊂ U and U is regular open in X.
(7) **generalized semi-closed** (briefly **gs-closed**) [**1**] if s-cl(A) ⊂ U whenever A ⊂ U and U ∈ ℑ.
(8) **πgs-closed** [**3**] if s-cl(A) ⊂ U whenever A ⊂ U and U is π-open in X.
(9) r**gs-closed** if s-cl(A) ⊂ U whenever A ⊂ U and U is regular open in X.
(10) **generalized η-closed** (briefly **gη-closed**) **[15]** if **η**-cl(A) ⊂ U whenever A ⊂ U and U ∈ ℑ.
(11) **πgη-closed** (briefly **gη-closed**) **[6]** if **η**-cl(A) ⊂ U whenever A ⊂ U and is π-open in X.
(12) **g-open** (resp**. πg-open**, **rg-open**, **αg-open, πgα-open, gαr-open**, **gs-open, πgs-open, rgs-open**, **gη-open, πgη-open**) set if the complement of A is g-closed (resp. πg-closed, rg-closed, αg-closed, πgα-closed, gαr-closed, gs-closed, πgs-closed, rgs-closed, gη-closed, πgη-closed).

3. Regular Generalized η-closed Sets

Definition 3.1. A subset A of a space (X, ℑ) is said to be **regular generalized η-closed** (briefly r**gη-closed**) if **η**-cl(A) ⊂ U whenever A ⊂ U and U is regular open in X. The family of all rgη-closed subsets of X will be denoted by

rgη-C(X).

Theorem 3.2. Every closed set is rgη-closed.

Proof. Let A be a closed set in X. Let U be a regular open set in X such that $A \subset U$. Since A is closed, that is, cl(A) = A, cl(A) $\subset$ U. But we have η-cl(A) $\subset$ cl(A) $\subset$ U. Therefore η-cl(A) $\subset$ U. Hence A is rgη-closed in X.

Theorem 3.3. For a topological space X the followings hold:

(1) Every g-closed set is rgη-closed.
(2) Every πg -closed set is rgη-closed.
(3) Every rg -closed set is rgη-closed.
(4) Every α-closed set is rgη-closed.
(5) Every αg-closed set is rgη-closed.
(6) Every πgα-closed set is rgη-closed.
(7) Every gαr-closed set is rgη-closed.
(8) Every s-closed set is rgη-closed.
(9) Every gs-closed set is rgη-closed.
(10) Every πgs-closed set is rgη-closed.
(11) Every rgs-closed set is rgη-closed.
(12) Every η-closed set is rgη-closed.
(13) Every gη-closed set is rgη-closed.
(14) Every πgη-closed set is rgη-closed.

Proof.

(1) Let A be a g-closed set in X. Let U be a regular open set in X such that $A \subset U$. Since every regular open set is open and since A is g-closed, that is, cl(A) $\subset$ U. But we have η-cl(A) $\subset$ cl(A) $\subset$ U. Therefore η-cl(A) $\subset$ U. Hence A is rgη-closed in X.

(2) Let A be a πg-closed set in X. Let U be a regular open set in X such that $A \subset U$. Since every regular open set is π-open and since A is πg-closed, that is, cl(A) $\subset$ U. But we have η-cl(A) $\subset$ cl(A) $\subset$ U. Therefore η-cl(A) $\subset$ U. Hence A is rgη-closed in X.

(3) Let A be a rg-closed set in X. Let U be a regular open set in X such that $A \subset U$. Since A is rg-closed, that is, cl(A) $\subset$ U. But we have η-cl(A) $\subset$ cl(A) $\subset$ U. Therefore η-cl(A) $\subset$ U. Hence A is rgη-closed in X.

(4) Let A be a α-closed set in X. Let U be a regular open set in X such that $A \subset U$. Since A is α-closed, that is, α-cl(A) = A, α-cl(A) $\subset$ U. But we have η-cl(A) $\subset$ α-cl(A) $\subset$ U. Therefore η-cl(A) $\subset$ U. Hence A is rgη-closed in X.

(5) Let A be a αg-closed set in X. Let U be a regular open set in X such that $A \subset U$. Since every regular open set is open and since A is αg-closed, that is, α-cl(A) $\subset$ U. But we have η-cl(A) $\subset$ α-cl(A) $\subset$ U. Therefore η-cl(A) $\subset$ U. Hence A is rgη-closed in X.

(6) Let A be a πgα-closed set in X. Let U be a regular open set in X such that $A \subset U$. Since every regular open set is π-open and since A is πgα-closed, that is, α-cl(A) $\subset$ U. But we have η-cl(A) $\subset$ α-cl(A) $\subset$ U. Therefore η-cl(A) $\subset$ U. Hence A is rgη-closed in X.

(7) Let A be a gαr-closed set in X. Let U be a regular open set in X such that $A \subset U$. Since A is gαr-closed, that is, α-cl(A) $\subset$ U. But we have η-cl(A) $\subset$ α-cl(A) $\subset$ U. Therefore η-cl(A) $\subset$ U. Hence A is rgη-closed in X.

(8) Let A be a s-closed set in X. Let U be a regular open set in X such that $A \subset U$. Since A is s-closed, that is, s-cl(A) = A, s-cl(A) $\subset$ U. But we have η-cl(A) $\subset$ s-cl(A) $\subset$ U. Therefore η-cl(A) $\subset$ U. Hence A is rgη-closed in X.

(9) Let A be a gs-closed set in X. Let U be a regular open set in X such that $A \subset U$. Since every regular open set is open and since A is gs-closed, that is, s-cl(A) $\subset$ U. But we have η-cl(A) $\subset$ s-cl(A) $\subset$ U. Therefore η-cl(A) $\subset$ U. Hence A is rgη-closed in X.

(10) Let A be a πgs-closed set in X. Let U be a regular open set in X such that $A \subset U$. Since every regular open set is π-open and since A is πgs-closed, that is, s-cl(A) $\subset$ U. But we have η-cl(A) $\subset$ s-cl(A) $\subset$ U. Therefore η-cl(A) $\subset$ U. Hence A is rgη-closed in X.

(11) Let A be a rgs-closed set in X. Let U be a regular open set in X such that $A \subset U$. Since A is rgs-closed, that is, s-cl(A) $\subset$ U. But we have η-cl(A) $\subset$ s-cl(A) $\subset$ U. Therefore η-cl(A) $\subset$ U. Hence A is rgη-closed in X.

(12) Let A be a η-closed set in X. Let U be a regular open set in X such that $A \subset U$. Since A is η-closed, that is, η-cl(A) = A, η-cl(A) $\subset$ U. But we have η-cl(A) $\subset$ U. Therefore η-cl(A) $\subset$ U. Hence A is rgη-closed in X.

(13) Let A be a gη-closed set in X. Let U be a regular open set in X such that $A \subset U$. Since every regular open set is open and since A is gη-closed, that is, η-cl(A) $\subset$ U. But we have η-cl(A) $\subset$ U. Therefore η-cl(A) $\subset$ U. Hence A is rgη-closed in X.

(14) Let A be a πgη-closed set in X. Let U be a regular open set in X such that $A \subset U$. Since every regular open set is π-open and since A is πgη-closed, that is, η-cl(A) $\subset$ U. But we have η-cl(A) $\subset$ U. Therefore η-cl(A) $\subset$ U. Hence A is rgη-closed in X.

Remark 3.4: From the above definitions, theorems and known results the relationship between rgη-closed sets and some other existing generalized closed sets are implemented in the following figure:

$$\begin{array}{ccccccc}
\text{closed} & \Rightarrow & \text{g-closed} & \Rightarrow & \pi\text{g-closed} & \Rightarrow & \text{rg-closed} \\
\Downarrow & & \Downarrow & & \Downarrow & & \Downarrow \\
\alpha\text{-closed} & \Rightarrow & \alpha\text{g-closed} & \Rightarrow & \pi\text{g}\alpha\text{-closed} & \Rightarrow & \text{g}\alpha\text{r-closed} \\
\Downarrow & & \Downarrow & & \Downarrow & & \Downarrow \\
\text{s-closed} & \Rightarrow & \text{gs-closed} & \Rightarrow & \pi\text{gs-closed} & \Rightarrow & \text{rgs-closed} \\
\Downarrow & & \Downarrow & & \Downarrow & & \Downarrow \\
\eta\text{-closed} & \Rightarrow & \text{g}\eta\text{-closed} & \Rightarrow & \pi\text{g}\eta\text{-closed} & \Rightarrow & \text{rg}\eta\text{-closed}
\end{array}$$

Where none of the implications is reversible as can be seen from the following examples:

Example 3.5: Let X = {a, b, c, d} and $\Im$ = {ϕ, {a}, {b}, {a, b}, {a, b, c}, {a, b, d}, X}. Then A = {a, b, c} and B = {a, b, d} are πg-closed as well as πgη-closed sets. A and B are also rgη-closed sets but not closed.

Example 3.6: Let X = {a, b, c, d} and $\Im$ = {ϕ, {a}, {d}, {a, d}, {c, d}, {a, c, d}, X}. Then A = {c} is πgα-closed as well as πgη-closed. It is also rgη-closed set. But it is neither closed nor g-closed. It is not πg-closed.

Example 3.7: Let X = {a, b, c, d} and $\Im$ = {ϕ, {c}, {d}, {c, d}, {b, c, d}, X}. Then A = {b} is g-closed, αg-closed, gη-closed, πgα-closed, πgη-closed. It is also rgη-closed set. But it is not closed.

Example 3.8: Let X = {a, b, c, d} and $\Im$ = {ϕ, {a}, {c}, {a, b}, {a, c}, {a, d}, {a, b, c}, {a, b, d}, {a, c, d}, X}. Then A = {a, b} is πgα-closed as well as πgη-closed. It is also rgη-closed set. But it is neither closed nor αg-closed set.

Example 3.9: Let X = {a, b, c} and $\Im$ = {ϕ, {a}, {c}, {a, c}, X}. Then A = {c} is η-closed as well as πgη-closed. It is also rgη-closed set. But it not α-closed.

Example 3.10: Let X = {a, b, c} and $\Im$ = {ϕ, {a}, {b, c}, X}. Then A = {a, b} is gη-closed as well as πgη-closed. It is also rgη-closed set. But it is not closed.

Example 3.11: Let X = {0, 1} and $\Im$ = {ϕ, {0}, X}. The topological space (X, $\Im$) is called the Sierpinski space. Then the set A = {0} rg-closed as well as rgη-closed sets but not closed.

Example 3.12. Let X = {a, b, c, d, e} and $\Im$ = {ϕ, {a, b}, {c, d}, {a, b, c, d}, X}. Then

1) **η-closed sets** are ϕ, {e}, {a, b}, {c, d}, {a, b, e}, {c, d, e} X.
2) **gη-closed sets** are ϕ, {a}, {b}, {c}, {d}, {e}, {a, b}, {a, e}, {b, e}, {c, d}, {c, e}, {d, e}, {a, b, e}, {a, c, e}, {a, d, e}, {b, c, e}, {b, d, e}, {c, d, e}, {a, b, c, e}, {a, b, d, e}, {a, c, d, e}, {b, c, d, e}, X.
3) **πgη-closed sets** are ϕ, {a}, {b}, {c}, {d}, {e}, {a, b}, {a, e}, {b, e}, {c, d}, {c, e}, {d, e}, {a, b, e}, {a, c, e}, {a, d, e}, {b, c, e}, {b, d, e}, {c, d, e}, {a, b, c, e}, {a, b, d, e}, {a, c, d, e}, {b, c, d, e}, X.
4) **rgη-closed sets** are ϕ, {a}, {b}, {c}, {d}, {e}, {a, b}, {a, c}, {a, d}, {a, e}, {b, c}, {b, d}, {b, e}, {c, d}, {c, e}, {d, e}, {a, b, c}, {a, b, d}, {a, b, e}, {a, c, d}, {a, c, e}, {a, d, e}, {b, c, d}, {b, c, e}, {b, d, e}, {c, d, e}, {a, b, c, d}, {a, b, c, e}, {a, b, d, e}, {a, c, d, e}, {b, c, d, e}, X.

4. Characteristics of rgη-Closed Sets

Theorem 4.1: If A is regular open and rgη-closed, then A is η-closed and hence clopen.

Proof: If A is regular open and πgη-closed, then η-cl(A) $\subset$ A. This implies A is η-closed. Hence A is clopen, since every η-closed (regular) open set is (regular) closed.

Theorem 4.2: If A and B are rgη-closed sets in X then A $\cup$ B is an rgη-closed set in X.

Proof: Let A and B be rgη-closed sets in X and U be any regular open set containing A and B. Therefore η-cl (A) $\subset$ U, η-cl(B) $\subset$ U. Since A $\subset$ U, B $\subset$ U then A $\cup$ B $\subset$ U. Hence η-cl(A $\cup$ B) = η-cl(A) $\cup$ η-cl(B) $\subset$ U. Therefore A $\cup$ B is rgη-closed set in X.

Theorem 4.3: A set A is rgη-closed set iff η-cl(A) – A contains no non-empty regular closed set.

Proof: Necessity: Let F be a regular closed set in X such that F $\subset$ η-cl(A) – A. Then A $\subset$ X – F. Since A is an rgη-closed set and X – F is regular open then η-cl(A) $\subset$ X – F. (i.e F $\subset$ X – η-cl(A)). Then F $\subset$ (X – η-cl(A)) $\cap$ η-cl(A) – A. Therefore F = ϕ.

Sufficency: Let us assume that η-cl(A) – A contains no non empty regular closed set. Let A $\subset$ U and U be regular-open. Suppose that η-cl(A) is not contained in U, then η-cl(A) $\cap$ U^c is non empty regular closed set of η-cl(A) – A which is a contradiction. Therefore η-cl(A) $\subset$ U. Hence A is an rgη-closed set.

Theorem 4.4: The intersection of any two subsets of rgη-closed sets in X is a rgη-closed set in X.

Proof: Let A and B be two subsets of a rgη-closed set. Assume A, B $\subset$ U, where U is regular-open. Then η-cl(A) $\subset$ U, η-cl(B) $\subset$ U. Therefore η-cl(A $\cap$ B) $\subset$ U. Since A and B are rgη- closed sets, A $\cap$ B is a rgη-closed set.

Theorem 4.5: If A is an gη-closed set in X and A$\subset$ B $\subset$ η-cl(A), then B is a rgη-closed set in X .

Proof: Since B $\subset$ η-cl(A), we have η-cl(B) $\subset$ η-cl(A) then η-cl(B) – B $\subset$ η-cl(A) – A. By **Theorem 3.2**, ηcl(A) – A contains no non empty regular closed set. Hence η-cl(B) – B contains no non empty regular closed set. Therefore B is a rgη-closed set in X.

Theorem 4.6: Let A $\subset$ Y $\subset$ X be a rgη-closed set in X. Then A is a rgη-closed set relative to Y .

Proof: Give that A $\subset$ Y $\subset$ X and A is a rgη-closed set in X. To prove that A is a rgη-closed set relative to Y, let us assume that A $\subset$ Y $\cap$ U, where U is regular open in X. Since A is an rgη-closed set, A $\subset$ U implies η-cl(A) $\subset$ U. It follows that Y $\cap$ η-cl(A) $\subset$ Y $\cap$ U. That is A is a rgη-closed set relative to Y.

Theorem 4.7: If A is both regular open and rgη-closed set in X then A is a regular closed set.

Proof: Since A is a regular open and rgη-closed set in X, η-cl(A) $\subset$ U. But A $\subset$ η-cl(A). Therefore A = η-cl(A). Therefore A is a regular closed set.

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Theorem 4.8: For x ∈ X, the set X – {x} is rgη-closed or regular open.

Proof: Suppose that X – {x} is not regular open, then X is the only regular open set containing X – {x}. (i.e η-cl(X – {x}) ⊂ X). Then X – {x} is a rgη-closed set in X.

Definition 4.9: Let (X, ℑ) be a topological space, A ⊂ X and x ∈ X. Then x is said to be a **η-limit point** of A iff every η-open set containing x contains a point of A different from x, and the set of all η-limit points of A is said to be the η- derived set of A and is denoted by $\mathbf{D_\eta(A)}$.

Usual derived set of A is denoted by D(A).

The proof of the following result is analogous to the well known ones.

Lemma 4.10: Let (X, ℑ) be a topological space and A ⊂ X. Then η-cl(A) = A ∪ D_η(A).

Theorem 4.11: Let A and B be rgη-closed sets in (X, ℑ) such that D(A) ⊂ D_η(A) and D(B) ⊂ D_η (B) . Then A ∪ B is rgη-closed.

Proof: For any set E ⊂ (X, ℑ), D_η(E) ⊂ D(E). Therefore D_η(A) = D(A) and D_η (B) = D(B). That is cl(A) = η-cl(A) and cl(B) = η-cl(B).
Let A ∪ B ⊂ U where U is regular open. Then A ⊂ U and B ⊂ U. Since A and B rgη-closed η-cl(A) ⊂ U and η-cl(B) ⊂ U. Now, cl(A ∪ B) = cl(A) ∪ cl(B) = η-cl(A) ∪ η-cl(B) ⊂ U. But η-cl(A ∪ B) ⊂ cl(A ∪ B). So η-cl(A ∪ B) ⊂ U and hence A ∪ B is rgη-closed.

Theorem 4.12: If A is rgη-closed and B is any set A ⊂ B ⊂ η-cl(A), then B is rgη-closed.
Proof: Let B ⊂ U where U is regular open. Then A ⊂ B implies A ⊂ U. Since A is πgη-closed, η-cl(A) ⊂ U. B ⊂ η-cl(A) implies η-cl(B) ⊂ η-cl(A). Thus η-cl(B) ⊂ U and shows that B is rgη-closed.

Regular Generalized η-open Sets and Regular Generalized η-Neighbourhoods

In this section new class of sets called regular generalized η-open (briefly rgη-open) sets and regular generalized η-neighborhoods (briefy rgη-nhd) in topological spaces are introduced and we study some of their properties.

Definition 5.1: Let (X, ℑ) be a topological space. A subset A of X is called **regular generalized η-open** (briefly **rgη-open**) iff its complement is rgη-closed set. We denote the family of all rgη-open sets of a topological space by **rgη-O(X)**.

Theorem 5.2: If A and B are rgη-open sets in a space X, then A ∪ B is also a rgη-open set in X.

Proof: If A and B are rgη-open sets in a space X, then A^c and B^c are rgη-closed sets in X. By **Theorem 3.1** $A^c \cup B^c$ is a rgη-closed set in X (i.e $A^c \cup B^c = (A \cap B)^c$ is a rgη-closed set in X). Therefore A ∪ B is a rgη-open set in X.

Remark 5.3: The union of two rgη-open sets in X is generally not a rgη-open set in X.

Example 5.4: Let X = {a, b, c} with ℑ = {ϕ, X, {a}, {c}, {a, c}, {b, c}}. The set A= {a} and B= {b} are rgη-open sets but A ∪ B = {a, b} is not a rgη-open set in X.

Remark 5.5: If A and B are rgη-open sets in X, then A ∩ B is not a rgη-open set in X.

Example 5.6: Let X = {a, b, c} with ℑ = {ϕ, X, {a}, {c}, {a, c}, {b, c}}. The set A= {a, c} and B = {b, c} are rgη-open sets but A ∩ B = {c} is not a rgη- open set in X.

Theorem 5.7: If int(B) ⊂ B ⊂ A and A is rgη-open set in X, then B is rg-open in X .

Proof: Suppose that int(B) ⊂ B ⊂ A and A is rgη-open in X then $A^c \subset B^c \subset A^c$. Since A^c is a rgη-closed set in X by **Theorem 5.2**, B is a rgη-open sets in X.

Definition 5.8: Let x be a point in a topological space X. A subset N of X is said to be a **rgη-nhd** of x iff there exists a rgη-open set G such that x ∈ G ⊂ N.

Definition 5.9: A subset N of space X is called a **rgη-nhd** of A ⊂ X iff there exists a rgη-open set G such that A ⊂ G ⊂ N.

Theorem 5.10: Every nhd N of x ∈ X is a rgη-nhd of x.

Proof: Let N be a nhd point of x ∈ X. To prove that N is a rgη-nhd of x, by definition of nhd, there exists an open set G such that x ∈ G ⊂ N. Hence N is a rgη-nhd of x.

Remark 5.11: In general, a rgη-nhd of x ∈ X need not be a nhd of x ∈ X as seen fron the following example.

Example 5.12: Let X = {a, b, c} with ℑ = {X, ϕ, {a}, {c}, {a, c}, {b, c}}. Then rgη-O(X) = {X, ϕ, {a}, {b}, {c} , {a, c}, {b, c}}. The set {a, c} is rgη-nhd of point b, since the rgη-open set {b} is such that b ∈ {b} ⊂ {a, b}. However, the set {a, b} is not a nhd of the point b, since no open set G exists such that b ∈ G ⊂ {a, c}.

Remark 5.13: The rgη-nhd N of x ∈ X need not be a rgη-open in X.

Example 5.14: Let X = {a, b, c} with ℑ = {X, ϕ, {a}, {c}, {a, c}, {b, c}}. Then rgη-O(X) = {X, ϕ, {a}, {b}, {c}, {a, c}, {b, c}}. The set {c} is a rgη-open set, but it is a rgη-nhd of {c}. Since {c} is a rgη-open set such that c ∈ {c} ⊂ {b, c}.

Theorem 5.15: If a subset N of a space X is rgη-open, then N is rgη-nhd of each of all its points.

Proof: Suppose N is rgη-open. Let $x \in N$ be an arbitrary point. We claim that N is a rgη-nhd of x. Since N is a rgη-open set and $x \in N \subset N$, it follows that N is a rgη-nhd of all of its points.

Remark 5.16: In general, a rgη-nhd of $x \in X$ need not be a nhd of $x \in X$ as seen from the following example.

Example 5.17: Let X = {a, b, c} with $\Im$ ={X, ϕ, {a}, {c}, {a, c}, {b, c}}. Then rgη-O(X) = {X, ϕ, {a}, {b}, {c}, {a, c}, {b, c}}. The set {a, b} is a rgη-nhd of b, since the rgη-open set {b} is such that $b \in \{b\} \subset \{a, b\}$. Also the set {a, b} is a rgη-nhd point of {b}, since the rgη-open set {b} is such that $b \in \{b\} \subset \{a, b\}$. (i.e. {a, b} is a rgη-nhd of each of its points). However the set {a, b} is not a rgη-open set in X.

Theorem 5.18: Let X be a topological space. If F is a rgη-closed subset of X and $x \in F^c$ then there exists a rgη-nhd N of x such that $N \cap F = \phi$.

Proof: Let X be an rgη-closed subset of X and $x \in F^c$. Then F^c is a rgη-open set of X. So by **Theorem 5.15** F^c contains a rgη-nhd of each of its points. Hence there exists a rgη-nhd N of x such that $N \cap F^c$. (i.e $N \cap F = \phi$).

5. Conclusion

The regular generalized η-closed set is defined and proved that the class forms a topology. The rgη-closed set can be used to derive a new decomposition of unity, closed map and open map, homeomorphism, closure and interior and new separation axioms. This idea can be extended to bitopological and fuzzy topological spaces.

References

[1] S. P. Arya and T. M. Nour, Characterizations of s-normal spaces, Indian J. Pure Appl. Math., 21(1990), 717-719.
[2] Arockiarani and C. Janaki, πgα-closed set and Quasi α-normal spaces, Acta Ciencia Indica Vol. **XXXIII** M. no. **2**, **(**2007), 657-666.
[3] A. Aslim, A. Caksu Guler and T. Noiri, On πgs-closed sets in topological spaces, Acta Math. Hungar., **112**(2006), 275-283.
[4] J. Dontchev and T. Noiri, Quasi-normal spaces and πg-closed sets, Acta Math. Hungar. 89 (2000), no. 3, 211-219.
[5] H. Kumar and M. C. Sharma, η-separation axioms in topological spaces, Jour. of Emerging Tech. and Innov. Res. Vol. 8, Issue 5, (2021), 516-524.
[6] H. Kumar, πgη-closed sets and some related topics, Jour. of Emerging Tech. and Innov. Res. Vol. 8, Issue 6, (2021), 367-374.
[7] N. Levine, Semi open sets and semi continuity in topological spaces, Amer. Math. Monthly **70**(1963), 36-41.
[8] N. Levine, Generalized closed sets in topology, Rend. Circ. Mat. Palermo (2), **19**(1970), 89-96.
[9] H. Maki, K. Balachandran, and R. Devi, Associated topologies of generalized α-closed sets and α-generalized closed sets, Mem. Fac. Sci. Kochi Univ. Ser. A, Math. 15 (1994), 51–63.
[10] H. Maki, R. Devi and K. Balachandran, Generalized α-closed sets in topology, Bull. Fukuoka Univ. Ed., Part-III, 42(1993), 13-21.
[11] O. Njastad, On some class of nearly open sets, Pacific. J. Math. , **15**(1965), 961-970.
[12] M. H. Stone, Application of the theory of boolean rings to general topology, Trans. Amer. Math. Soc. 41 (1937), 375-481.
[13] S. Sekar and G. Kumar, On gαr-closed sets in topological spaces, Int. Jour. of Pure and Appl. Maths. Vol. 108, **4**(2016), 791-800.
[14] D. Subbulakshmi, K. Sumathi, K. Indirani, η-open sets in topological spaces, International Jour. of Innovative and Exploring Engineering, Vol. 8, Issue 10S, 2019, 276-282.
[15] D. Subbulakshmi, K. Sumathi, K. Indirani, gη-continuous in topological spaces, International Jour. of Innovative and Exploring Engineering, Vol. 8, Issue 3, 2019, 677-282.
[16] V. Zaitsev, On certain classes of topological spaces and their bicompactifications, Dokl. Akad. Nauk SSSR, **178**(1968), 778-779.