Quasiaffine Inverses of Linear Operators in Hilbert Spaces

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Abstract: Let H denote a complex Hilbert space and B(H) denote the Banach algebra of bounded linear operators on H. Given operators $A, B, X \in B(H)$ ***, we define*** $R(A,B) : B(H) \to B(H)$ ***by*** $R(A,B)X = AXB - X$ ***and*** $C(A,B) : B(H) \to B(H)$ ***by*** $C(A,B)X = AX - XB$. ***In this paper, we investigate properties of the operators*** $A, B \in B(H)$ ***satisfying*** $R(A,B)X = 0$ ***or*** $R(B,A)Y = 0$ ***or both where X and Y are one-one or have a dense range or both. In particular, the case*** $R(A,B)X = 0 = R(B,A)Y$ ***is of special interest with respect to invertibility of the operator A under some classes of operators.***

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1. Introduction

The study of operator equations $R(A,B)X = 0$ and $C(A,B)X = 0$ has been considered by a number of authors among them: Khalagai and Sheth [6] showed that if two operators $A, B \in B(H)$ are such that $A^2B = BA^2$ then $AB = BA$ if $\sigma(A) \cap \sigma(-A) = \emptyset$ or A is normal and $0 \notin W(A)$ or $\{A\}' = \{A^{2m}\}'$ for some positive integer m. Duggal [1] was able to show that if A is dominant and B^* is M-hyponormal such that C(A, B)X = 0 then $C(A^*, B^*)X = 0$ for some operator X. Khalagai and Nyamai [4] showed that if C(A, B)X = 0 implies $C(A^*, B^*)X = 0$ then A and B are normal if X is one-one or has a dense range. Hongke and Chuan [2] showed that subnormal (normal) operator with left (right) quasiaffine inverse is invertible. They also showed that given two operators A, B $\in$ B(H) such that A and B* are dominant operators and X a quasiaffinity then, R(A, B)X = 0 implies $R(A^*, B^*)X = 0$. Khalagai and Otieno [5] showed that if B is a quasiaffine inverse of A then both A and B are quasiaffinities and that a quasi-invertible partial isometry is unitary. Khalagai [3] showed that a left (right) invertible operator with a right (left) quasiaffine inverse is invertible and that quasiaffine inverse of an operator can be the usual inverse of the operator given $\sigma(X) \cap \sigma(-X) = \emptyset$, where X is the implementing quasiaffinity. In this paper, we make a continuation of this study by showing among other results that a quasiaffine inverse can be the usual inverse of the operator under some conditions different from the one stated by Khalagai [3]. We also deduce some corollaries pertaining to these results and show that a quasinormal partial isometry with a dense range is unitary.

2. Notations and Terminology

In this paper, we denote the range of an operator A by Ran A and its closure by $\overline{RanA}$ The kernel of operator A will be denoted by Ker A. The commutator of two operators A and B is defined by [A, B] = AB − BA and two operators A and B are said to commute if [A, B] = 0. $\{A\}' = \{B : [A,B] = 0\}$. The numerical range of an operator A is denoted by W(A) where $W(A) = \{\langle Ax, x\rangle : ||x|| = 1\}$. The spectrum of A will be denoted by $\sigma(A)$ where $\sigma(A) = \{\lambda \in \mathbb{C} : A - \lambda I \quad is \quad not \quad invertible\}$ An operator A$\in$B(H) is said to be a quasiaffinity if it is injective and has a dense range. An operator A is said to be:

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Dominant, if to each $\lambda \in \mathbb{C}$ there corresponds a number $M_\lambda \leqslant 1$ such that for all $x \in$H $||(A-\lambda)^* x|| \leqslant M_\lambda ||(A-\lambda)x||$.

M-Hyponormal, if there exist a constant $M \leqslant M_\lambda$ such that for all $x \in H$ $||(A-\lambda)^* x|| \leqslant M||(A-\lambda)x||$.
Quasinormal, if $AA^*A = A^*AA$.
Partial isometry, if $A = AA^*A$.
Isometric, if $A^*A = I$.
Co-Isometric, if AA^* =I.
Unitary, if $A^*A = AA^* = I$.

3. Results

Theorem 3.1.

Let A , B $\in$ B(H). If X and Y are quasiaffinities such that AXB = X and BYA = Y.

Then we have

(i) $[A, XY] = 0$
(ii) $[B, YX] = 0$

Proof
(i) From AXB = X we have by post multiplying each side by Y that AXBY = XY. This implies AXBYA = XYA. Since BYA = Y, we have XYA = AXBYA = AXY. Hence the result, XYA = AXY.

(ii) From BYA = Y we have by post multiplying each side by X, BYAX = YX. This implies BYAXB = YXB. Since AXB = X, YXB = BYAXB = BYX. Hence the result, YXB = BYX.

Remark 3.2. *We note from the Theorem 3.1 above that if B is quasiaffine inverse of A then both operators commute with a product of the implementing quasiaffinities in a given order.*

Corollary 3.3. *Let A , B $\in$ B(H) be such that AXB = X and $A^*XB^* = X$. Then we have*
(i) $[A, XX^*] = 0$.
(ii)$[B, X^*X] = 0$.

Proof
Given AXB = X implies $A^*XB^* = X$, by taking adjoint both sides we have $BX^*A = X^*$. Thus, in the Theorem 3.1 above, we replace Y with X^* hence the proof carries through.

Corollary 3.4. *Let A , B $\in$ B(H) be such that AXB = X where A and B * are dominant and X is a quasiaffinity. Then B is a quasiaffine inverse of A.*

Proof
Since A and B^* are dominant then AXB = X implies $A^*XB^* = X$. Taking adjoint gives $BX^*A = X^*$ where X^* is also a quasiaffinity. Hence B is a quasiaffine inverse of A.

Remark 3.5. *We note that in Theorem 3 (ii) of Khalagai[3] the condition*

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$\sigma(X) \cap \sigma(-X) = \emptyset$ *can be replaced with a number of other conditions, which are less stringent by using the following results due to Khalagai and Sheth [6].*

Theorem A. *(Khalagai and Sheth [6]) Let* $A, B \in B(H)$ *be such that* $[B, A^2] = 0$. *Then* $[B, A] = 0$ *under any of the following conditions.*
(i) $\sigma(A) \cap \sigma(-A) = \emptyset$.
(ii) A is normal and $0 \notin W(A)$
(iii) $\{A\}^{'} = \{A^{2m}\}^{'}$ *for some positive integer m.*

Theorem 3.6. *Let A , B* $\in$ *B(H) be such that* $AXB = X$ *implies* $A^*XB^* = X$ *where X is self - adjoint. Then* $B = A^{-1}$ *under any of the following conditions:*
(i) 0 $\notin$ *W(X).*
(ii) $\{X\}^{'} = \{X^{2m}\}^{'}$ *for some positive integer m and X is one one or has a dense range.*

Proof
Given that $AXB = X$ *implies* $A^*XB^* = X$, *we have by Corollary 3.3 above*
$[A, XX^*] = 0 = [B, X^*X]$.

Since X is self-adjoint, then $X = X^*$ *and* $X^*X = X^2$. *This implies* $[A, X^2] = 0 = [B, X^2]$. *By Theorem A above, each of conditions (i) and (ii) implies [A, X] = 0 and [B, X] = 0.*

Consequently, $[A^*, X] = 0$ *and* $[B^*, X] = 0$. *Now from* $AXB = X$, *we have* $ABX = X$ *implying* $(AB - I)X = 0$. *Since 0* $\notin$ *W(X) then X has a dense range. Thus,* $AB - I = 0$. *That is AB = I. Also, from* $A^*XB^* = X$ *taking adjoint, we have BXA = X.*

If [A, X] = 0, then BAX = X. That is $(BA - I)X = 0$. *Since X has a dense range then* $BA - I = 0 \Leftrightarrow BA = I$. *For part (ii) since X has a dense range or is one-one, we also have that* $(AB - I)X = 0$ *and* $(BA - I)X = 0$. *That is:*

AB = I and BA = I. Hence B =A^{-1}.

Remark 3.7. *It follows that if X is a quasiaffinity in Theorem 3.6 above then the quasi-affine inverse B of A is the same as the inverse of A under any of the two conditions.*

Corollary 3.8. *[Khalagai[3] Theorem 3(ii)] Let A , B* $\in$ *B(H) be such that A and* B^* *are dominant satisfying the equation* $AXB = X$ *and X is self- adjoint and a quasiaffinity. Then B is the quasi-affine inverse of A which is equal to* A^{-1} *under any of the following conditions:*

(i) 0 $\notin$ *W(X).*
(ii) $\{X\}^{'} = \{X^{2m}\}^{'}$ *for some positive integer m and X is oneone or has a dense range.*

Proof

We first note that if A and B^* *are dominant operators then* $AXB = X$ *implies*
$A^*\ X\ B^* = X$ *and by tracing the proof of Theorem 3.6 above the results are obtained.*

Corollary 3.9. *Let A, B* $\in$ *B(H). If R(A, B)X = 0 implies* $R(A^*, B^*)X = 0$ *with X being a quasiaffinity then:*

(i) A is co-isometric if and only if B is isometric.
(ii) A is isometric if and only if B is co-isometric.

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Proof

To proof (i) If $AXB = X$ *and* $A^*XB^* = X$ *then* $X = AXB = AA^*XB^*B$. *That is* $X = AA^*XB^*B$. *If B is isometric then* $B^*B = I$ *and* $X = AA^*XB^*B = AA^*X$. *This implies* $X - AA^*X$ *and* $(I - AA^*)X = 0$. *Since X has a dense range then* $I - AA^* = 0$ *and* $AA^* = I$. *Similarly, we have* $X = AA^*XB^*B$. *Substituting*$AA^* = I$ *we have* $X = XB^*B$ *and* $X - XB^*B = 0$. *This implies* $X(I - B^*B) = 0$. *Since X is injective then* $I - B^*B = 0$ *implying* $B^*B = I$.

To Proof (ii), We rearrange the equations $AXB = X$ *and* $A^*XB^* = X$ *so that* $X = A^*XB^* = A^*AXBB^*$ *implying* $X = A^*AXBB^*$. *If B is co-isometric then* $BB^* = I$ *and* $X = A^*AXBB^* = A^*AX$. *This implies* $X - A^*AX = 0$ *and* $(I - A^*A)X = 0$. *Since X has a dense range then* $I - A^*A = 0 \Leftrightarrow A^*A = I$. *Also,*$X = A^*AXBB^*$. *Substituting* $A^*A = I$ *we obtain* $X = XBB^* \Leftrightarrow X - XB^*B = 0$. *This implies* $X(I - BB^*) = 0$. *Since X is injective then* $I - BB^* = 0$ *implying* $BB^* = I$.

Corollary 3.10. *Let A, B ∈ B(H). If R(A, B)X = 0 implies* $R(A^*, B^*)X = 0$ *with X being a quasiaffinity then A is unitary if and only if B is unitary.*

Proof
If A is unitary then its isometric and co-isometric. From Corollary 3.9, B is co-isometric and isometric thus unitary. Similarly, if B is unitary, it is-isometric and co - isometric thus by Corollary 3.9, A is co-isometric and isometric.

Corollary 3.11. *Let A, B ∈ B(H) with A and* B^* *dominant. If R(A, B)X = 0 where X is a quasiaffinity, then A is unitary if and only if B is unitary.*

Proof
If A and B^* *are dominant then R(A, B)X = 0 implies* $R(A^*, B^*)X = 0$. *From Corollary 3.10 the results follow.*

Theorem 3.12

Let A , B ∈ B(H) be such that

R(A, B)X = 0 or R(B, A)Y = 0. If there exist an operator $B_1 \in B(H)$ *such that:*

(i) R(A, B_1*)X = 0 where A and X are oneto-one then* $B_1 = B$
(ii) R(B_1*, A)Y = 0 where A and Y have a dense range then* $B_1 = B$.

Proof

(i) If R(A, B)X = 0 and $AXB_1 = X$ *then* $AXB = AXB_1 = X$.
Thus, $AXB_1 - AXB = 0$. *If A and X are oneto-one then,* $A(XB_1 - XB) = 0$ *implies* $X(B_1 - B) = 0$ *and*
$B_1 - B = 0$. *Thus,* $B_1 = B$.
(ii) If R(B, A)Y = 0 and R(B_1*, A)Y = 0 then* $BYA = B_1YA = Y$. *Thus,*
$BYA - B_1YA = 0$ *and* $(BY - B_1Y)A = 0$. *If A and Y have a dense range then*
$(B - B_1)Y = 0$ *and* $(B - B_1) = 0$. *Thus* $B_1 = B$.

Theorem B. *(Khalagai and Otieno [5] Theorem 1) Let A ∈ B(H). If B is a quasiaffine inverse of A then both A and B are quasiaffinities.*

Corollary 3.13. *Let A , B ∈ B(H) be such that B is quasiaffine inverse of A where X and Y are the implementing quasiaffinities. If there exist operator* $B_1 \in B(H)$ *such that R(A,* B_1*)X = 0 and R(*B_1*, A)Y = 0 then* $B_1 = B$.

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Proof

From Theorem B above both A and B are quasiaffinities. This implies A and B are one-toone and have dense range. By tracing Theorem 3.12 proof follows through.

Remark 3.14 *We note from Theorem 3.12 and Corollary 3.13 above that a quasiaffine inverse is indeed unique.*

Theorem 3.15. *Let A, B, X ∈ B(H) satisfy R(A, B)X = 0. If X has dense range and A is quasinormal partial isometry then A is unitary.*

Proof

If A is quasinormal partial isometry then $A = AA^*A = A^*AA$. *If* $X = AXB$ *then*

$$X = AXB = AA^*AXB = A^*AAXB = AA^*X.$$

This means $X = AA^*X$ *and* $(I - AA^*)X = 0$. *If X has a dense range then* $I - AA^* = 0$ *implying* $AA^* = I$. *Similarly,* $X = A^*AAXB = A^*AX$ *implying* $X = A^*AX$ *and*

$(I - A^*A)X = 0$. *If X has a dense range then* $I - A^*A = 0$ *and* $A^*A = I$. *Thus,*
$A^*A = AA^* = I$.

Corollary 3.16. *Let A, B, X ∈ B(H) satisfy R(A, B)X = 0. If X is a quasiaffinity and A is quasinormal partial isometry then A is unitary.*

Proof

If X is a qausiaffinity then X has a dense range. By tracing Theorem 3.15 the results follows.

Corollary 3.17. *Let A, B ∈ B(H). If a quasiaffine inverse B of A is quasinormal partial isometry then it is unitary.*

Proof

If B is a quasiaffine inverse of A then there exist quasiaffinities X and Y such that

R(A, B)X = 0 and R(B, A)Y = 0. If B is a quasinormal partial isometry and
R(B, A)Y = 0 then by Corollary 3.16 the proof carries through.

Remark 3.18. *Khalagai and Otieno [5] showed that if a partial isometry is a quasiaffinity then it is unitary. In Corollary 3.19, we extend the result to a quasinormal partial isometry with a dense range and notice that it is indeed invertible.*

Corollary 3.19. *Let A ∈ B(H) be a quasinormal partial isometry. If A has a dense range then A is unitary.*

Proof

If A is a quasinormal partial isometry then $A = AA^*A = A^*\ AA$. *This means*
$A - AA^*A = 0$ *and* $(I - AA^*)A = 0$. *Since A has a dense range then,*$I - AA^* = 0$. *That is* $AA^* = I$.
Also,$A = A^*AA$ *and*$A - A^*AA = 0$. *This implies* $(I - A^*A)A = 0$. *Since A has a dense range then* $I - A^*A = 0$ *and* $A^*A = I$. *Thus,* $A^*A = AA^* = I$.

Theorem 3.20. *Let A, B ∈ B(H) such that R(A, B)X = 0 for a quasiaffinity X. If:*
(i) A isometric operator then $C(A^*, B)X = 0$.
(ii) $C(A^*, B)X = 0$ *then A is co-isometric.*

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Proof
(i) If R(A, B)X = 0 and given that A is isometric then $XB = A^*AXB = A^*X$*. This implies* $A^*X = XB$ *hence* $C(A^*, B)X = 0$.
(ii) If $A^*X = XB$ *then multiplying both sides by A from the left we have* $AA^*X = AXB = X$*. Thus,* $AA^*X = X \Leftrightarrow AA^*X - X = 0 \Leftrightarrow (AA^* - I)X = 0$*. Since X has a dense range then* $AA^* - I = 0 \Leftrightarrow AA^* = I$.

Corollary 3.21. *Let A, B∈ B(H) such that R(A, B)X = 0 where X is one-to-one. If:*
(i) B is co-isometric operator then $C(A, B^*)X = 0$.
(ii) $C(A, B^*)X = 0$ *then B is isometric.*

Proof
Following the steps of Theorem 3.20, we multiply AXB = X by B^* *from the right we have* $AXBB^* = XB^*$*. Since B is co-isometric then* $BB^* = I$ *and* $AX = XB^*$ *also multiplying* $AX = XB^*$ *by B from the right we obtain* $AXB = XB^*B$*. Since* $AXB = X$ *then* $XB^*B = AXB = X$*. Thus* $XB^*B = X$ *and* $X - XB^*B = 0$*. Since X is oneto-one, then* $X(I - B^*B) = 0$ *implies* $I - B^*B = 0$ *and* $B^*B = I$

Theorem 3.22.

Let A, B ∈ B(H) be such that C(A,

B)X = 0 for some operator X. If:

(i) A is a one-to-one partial isometry then $R(A^*, B)X = 0$.
(ii) B is a partial isometry with a dense range then $R(A, B^*)X = 0$.

Proof.
If A is a partial isometry then $A = AA^*A$ *and* $C(A, B)X = 0$*. It follows that*

$AX = AA^*AX = AA^*XB$*. Thus,* $AX = AA^*XB$ *and* $AX - AA^*XB = 0$*. This implies* $A(X - A^*XB) = 0$ *and since A is one-to-one then,* $X - A^*XB = 0$ *and* $A^*XB = X$*. That is* $R(A^*, B)X = 0$*. Similarly, If B is a partial isometry and* $C(A, B)X = 0$ *then it follows that* $XB = XBB^*B = AXB^*B$*. Thus,* $XB = AXB^*B$ *.This implies* $(X - AXB^*)B = 0$*. Since B has a dense range then,* $X - AXB^* = 0$ *and* $AXB^* = X$*. That is* $R(A, B^*)X = 0$.

Corollary 3.23. *Let A, B ∈ B(H) be such that* A^* *is dominant and B is M-hyponormal operator satisfying* $C(A^*, B^*)X = 0$ *for some operator X. If:*

(i) A is a one-to-one partial isometry then $R(A^*, B)X = 0$.
(ii) B is a partial isometry with a dense range then $R(A, B^*)X = 0$.

Proof.
If A^* *is dominant and B is M-hyponormal operator then* $C(A^*, B^*)X = 0$ *imply*
*C(A, B)X = 0. Thus, by **Theorem 3.22** the proof follows through.*

Theorem 3.24.

Let A, B, X∈ B(H) be such that C(A, B)X = 0 imply

$C(A^*, B^*)X = 0$*. If X is a quasiaffinity then A and B are unitary operators under each of the following conditions:*

(i) A is a one-to-one partial isometry.
(ii) B is a partial isometry with a dense range.

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Proof

To proof (i): If C(A, B)X = 0 then $X = A^*XB$ *and* $C(A^*, B^*)X = 0$. *Thus,* $X = A^*XB = XB^*B$ *implying* $X = XB^*B$ *and* $X - XB^*B = 0$ *and* $X(I - B^*B) = 0$. *Since X is one-to-one then* $I - B^*B = 0$ *implies* $B^*B = I$. *Since B is normal then* $I = B^*B = BB^*$. *Similarly,* $X = A^*XB = A^*AX$ *implying* $X = A^*AX$ *and* $X - A^*AX = 0$. *Thus,* $(I - A^*A)X = 0$. *Since X has a dense range then* $I - A^*A = 0$ *and* $A^*A = I$. *Since C(A, B)X = 0 implies* $C(A^*, B^*)X = 0$ *and X has a dense range then A is normal implying* $A^*A = AA^* = I$.

To proof (ii): If B is a partial isometry with a dense range and C(A, B)X = 0 then $X = AXB^*$. *Since C(A, B)X = 0 imply* $C(A^*, B^*)X = 0$ *then* $X = AXB^* = XBB^*$ *and* $X = XBB^*$ *Thus* $X - XBB^* = 0$ *and* $X(I - BB^*) = 0$. *Since X is one-to-one then* $I - BB^* = 0$ *and* $BB^* = I$. *B is normal implying* $I = BB^* = B^*B$. *Similarly,*

$X = AXB^* = AA^*X$ *implying* $X = AA^*X$ *and* $X - AA^*X = 0$. *Thus,* $(I - AA^*)X = 0$. *Since X has a dense range then* $I - AA^* = 0$ *and* $AA^* = I$. *Since A is normal then* $A^*A = AA^* = I$.

Corollary 3.25.

Let A, B ∈ B(H) be such that A is dominant and B^* *is*

M-hyponormal satisfying

C(A, B)X = 0. If X is a quasiaffinity then A and B are unitary operators under each of the following conditions:

(i) A is a one-to-one partial isometry.
(ii)B is a partial isometry with a dense range.

Proof

If A is dominant and B^* *is M-hyponormal operator then C(A, B)X = 0 implies*
$C(A^*, B^*)X = 0$. *By tracing Theorem 3.24 the proof carries through.*

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